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Study for System of Nonlinear Differential Equations with Riemann-Liouville Fractional Derivative

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ABSTRACT

In this work, we study existence theorem of the initial value problem for the system of fractional differential equations $D^{\alpha}\bar{x}(t)=A\bar{x}(t), t^{1-\alpha}\bar{x}(t)\big|_{x=0}=\bar{b}$, where D^{α} denotes standard Riemann-Liouville fractional derivative, $0<\alpha<1$, $\bar{x}(t)=\left[x_1(t),x_2(t),\cdots,x_n(t)\right]^{\mathrm{T}}$, $\bar{b}=\left[b_1,b_2,\cdots,b_n\right]^{\mathrm{T}}$ and A is a square matrix. At the same time, power-type estimate for them has been given.

Keywords: Riemann-Liouville Fractional Derivative; Weighted Cauchy-Type Problem; Fractional Differential Equations

1. Introduction

Let M_n denote the $n\times n$ matrix over real fields R or complex fields C. For $h>0$,

$$C_r^0([0,h]):=\left\{f\in C^0((0,h]):\lim_{t\to 0^+} t^r f(t)\text{ exists and is finite}\right\},$$

here $C^0((0,h])$ is the usual space of continuous functions on $(0,h]$, which is a Banach space with the norm

$$\|f\|_r:=\max_{0\le t\le h} t^r\left|f(t)\right|.$$

The space $C_{1-\alpha}^{\alpha}([0,h])$ is defined by

$$C_{1-\alpha}^{\alpha}([0,h]):=\left\{f\in C_{1-\alpha}([0,h]):\text{there exists } c\in R \text{ and } f^*\in C_{1-\alpha}^0([0,h])\ s.t.\ f(t)=ct^{\alpha-1}+I^{\alpha}f^*(t)\right\}.$$

(see [1]).

The existence of solution of initial value problems for fractional order differential equations have been studied in many literatures such as [1-4]. In this paper, we present the analysis of the system of fractional differential equations

$$\begin{cases}D^{\alpha}\bar{x}(t)=A\bar{x}(t),\\ t^{1-\alpha}\bar{x}(t)\big|_{t=0}=\bar{b},\end{cases}\qquad (*)$$

where D^{α} denotes standard Riemann-Liouville fractional derivative, where

$$\bar{x}(t)=\left[x_1(t),x_2(t),\cdots,x_n(t)\right]^{\mathrm{T}},$$

$$D^{\alpha}\bar{x}(t)=\left[D^{\alpha}x_1(t),D^{\alpha}x_2(t),\cdots,D^{\alpha}x_n(t)\right]^{\mathrm{T}},$$

$$1/2<\alpha<1,$$

$\bar{b}=\left[b_1,b_2,\cdots,b_n\right]^{\mathrm{T}}$ and A is a square.

To prove the main result, we begin with some definitions and lemmas. For details, see [1-5].

Definition 1.1 Let f be a continuous function defined on $[a,b]$ and $n-1\le\alpha<n, n\in N$. Then the expression

$$D_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x\frac{f(t)}{(x-t)^{\alpha-n+1}}\mathrm{d}t,\ x>a$$

is called left-sided fractional derivatives of order α.

Definition 1.2 Let f be a continuous function defined on $[a,b]$ and $\alpha>0$. Then the expression

$$I_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)}{(x-t)^{-\alpha+1}}\mathrm{d}t,\ x>a$$

is called left-sided fractional integral of order α.

Lemma 1.3 Given $A\in M_n$ with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ in any prescribed order, there is a unitary

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matrix $U \in M_n$ such that $U^* AU = T = [t_{ij}]$ is upper triangular with diagonal entries $t_{ii} = \lambda_i$, $i = 1, \cdots, n.$ That is, every square matrix A is unitarily equivalent to triangular matrix whose entries are the eigenvalues of A in a prescribed order. Further more, if $A \in M_n(R)$ and if all the eigenvalues of A are real, then U may be chosen to be real and orthogonal.

Lemma 1.4 Assume that $f \in C^0(R_0^+) \cap \mathrm{Loc}^1(R_0^+)$ with fractional derivative of order $0 < \alpha < 1$ that belongs to $C^0(R_0^+) \cap \mathrm{Loc}^1(R_0^+)$. Then

$$I^\alpha D^\alpha f(x) = f(x) + Cx^{\alpha-1}$$

for some $c \in R.$ When the function $f \in C^0(R^+),$ then $c = 0,$ where

$$R_0^{\ +} = \{x \in R, x > 0\} \text{ and } R^+ = \{x \in R, x \geq 0\}.$$

Lemma 1.5 (Schauder's fixed theorem) Assume Ω is a relative subset of a convex set K in a normed space X. Let $A : \bar{\Omega} \to K$ be a compact map with $0 \in \Omega$. Then either

(A_1) A has a fixed point in $\bar{\Omega}$, or

(A_2) there is a $x \in \partial\Omega$ and a $\lambda < 1$ such that $x = \lambda Ax.$

Now, let's us give some hypotheses:

H1: $f(t,x)$ is continuous on $R^+ \times R$ and is such that

$$\left|f(t,x)\right| \leq t^\mu \varphi(t) \mathrm{e}^{-\sigma t} \left|x\right|^m, \mu \geq 0, m > 1, \sigma > 0, \quad (1)$$

where $\varphi(t)$ is a continuous function on $R^+.$

H2: $f(t,x)$ is continuous on $R^+ \times R$ and is such that

$$\left|f(t,x)\right| \leq t^\mu \varphi(t) \left|x\right|^m, \mu \geq 0, m > 1, \quad (2)$$

where $\varphi(t)$ is a continuous function on $R^+.$

Lemma 1.6 Let $1/2 < \alpha < 1.$ If we assume that $0 < q < 1/1 - \alpha$, then the initial value problem

$$\begin{cases} D^\alpha x(t) = x^q(t) + y(t), \\ t^{1-\alpha} x(t)\big|_{t=0} = b, \end{cases} \quad (3)$$

where

$$y(t) \in C_{1-\alpha}^0([0,h]) \cap L^1((0,h)),$$

$$x^q(t) \in C_{1-\alpha}^0([0,h]) \cap L^1((0,h)),$$

has at least a solution $x(t) \in C_{1-\alpha}^0([0,h]) \cap L^1((0,h))$ for $h > 0$ sufficiently small.

Proof. If

$$x^q(t) \in C_{1-\alpha}^0([0,h]) \cap L^1((0,h)),$$

then $q(\alpha - 1) > -1$, by Lemma 1.4, We are therefore reduced the initial problem to the nonlinear integral equation

$$x(t) = bt^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1} x^q(s)\mathrm{d}s + \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s\right). \quad (4)$$

The existence of a solution to Problem (3) can be formulated as a fixed point equation $Tx = x,$ where

$$(Tx)(t) = bt^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1} x^q(s)\mathrm{d}s + \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s\right) \quad (5)$$

in the space $C_{1-\alpha}^0([0,h]) \cap L^1((0,h))$.

Define

$$S = \left\{x \in C_{1-\alpha}^0([0,h]) : \left\|x - bt^{\alpha-1}\right\|_{1-\alpha} \leq r + \frac{1}{\alpha} h^{2\alpha-1} \left\|y\right\|_{1-\alpha}\right\}.$$

Clearly, it is closed, convex and nonempty.

Step I. We shall prove that we note that $TS \subseteq S.$

We note that

$$\begin{aligned}
\left\|Tx - bt^{\alpha-1}\right\|_{1-\alpha} &= \max_{t\in[0,h]} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \cdot \left\{\left|\int_0^t (t-s)^{\alpha-1} x^q(s)\mathrm{d}s + \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s\right|\right\} \\
&\leq \max_{t\in[0,h]} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \left\{\left|\int_0^t (t-s)^{\alpha-1} s^{q(\alpha-1)} s^{q(1-\alpha)} x^q(s)\mathrm{d}s\right| + \left|\int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha} y(s)\mathrm{d}s\right|\right\} \\
&= \max_{t\in[0,h]} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \left\{\left|\int_0^t (t-s)^{\alpha-1} s^{q(\alpha-1)} s^{q(1-\alpha)} x^q(s)\mathrm{d}s\right|\right\} + \max_{t\in[0,h]} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \left\{\left|\int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha} y(s)\mathrm{d}s\right|\right\} \\
&\leq \frac{\Gamma(q(\alpha-1)+1)}{\Gamma(q(\alpha-1)+1+\alpha)\Gamma(\alpha)} h^{q(\alpha-1)+1} \left\|x\right\|_{1-\alpha}^q + \frac{1}{\alpha} h^{2\alpha-1} \left\|y\right\|_{1-\alpha}.
\end{aligned}$$

Since $\left\|x\right\|_{1-\alpha} \leq r + |b| + \frac{1}{\alpha} h^{2\alpha-1} \left\|y\right\|_{1-\alpha}$, it will be sufficient to impose

$$\left\|x - bt^{\alpha-1}\right\|_{1-\alpha} \leq \text{const.} h^{q(\alpha-1)+1} \left(r + |b| + \frac{1}{\alpha} h^{2\alpha-1} \left\|y\right\|_{1-\alpha}\right)^q \leq r.$$

In view of the assumption $q(\alpha - 1) + 1 > 0,$ the second estimate is satisfied if say $r = |b|$ and h is chosen sufficiently small.

Step II. We shall prove that the operator T is compact. To prove the compactness of

$$T: C^0_{1-\alpha}([0,h]) \to C^0_{1-\alpha}([0,h])$$

defined by (5), it will be sufficient to argue on the operator

$$T_*: C^0([0,h]) \to C^0([0,h])$$

defined in this way:

$$(T_* x)(t) = t^{1-\alpha} T\left(t^{\alpha-1} x(t)\right).$$

We have $T_* x = b + T^* x$ where the operator

$$(T^* x)(t) = \frac{t^{1-\alpha}}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1} s^{q(\alpha-1)} x^q(s)\,ds + \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} y(s)\,ds\Big).$$

Turn out to be compact from classical sufficient conditions, since $q(\alpha-1) > -1, \alpha-1 > -1$. By Lemma 1.5, we have that Problem (3) has least a solution.

The proof is complete.

Lemma 1.7 Suppose that $f(t,x)$ satisfies H1, $\mu-(m-1)(1-\alpha)>0$ and $\alpha > 1/2$. If $\|\varphi\|_q < L$ for some $q > 1/\alpha$, then the problem

$$\begin{cases} D^\alpha x(t) = f(t,x), \\ t^{1-\alpha} x(t)\big|_{t=0} = b, \end{cases} \tag{6}$$

exists a positive constant C such that $|x(t)| \le C t^{\alpha-1}$, $t > 0$.

Lemma 1.8 Let $x \in C^0_{1-\alpha}([0,h])$ with $\alpha > 1/2$. Suppose further that $\mu-(m-1)(1-\alpha)>0$. Then Problem (6) and its associated integral equation

$$x(t) = b t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s))\,ds \tag{7}$$

are equivalent.

Lemma 1.9 Assume that $\alpha > 1/2$. $f(t,x)$ satisfies H2, and $\|\varphi\|_q < K$ for some $q > 1/2$. Suppose further that $\mu + 1/p < m(1-\alpha)$, then there exists $C > 0$ and $0 < \delta < 1-\alpha$ such that any solution of (6) exists globally and satisfies

$$|x(t)| \le C t^{-\delta}, t \ge a > 0. \tag{8}$$

2. Main Results

Theorem 2.1 Let $A \in M_n$ then initial problem (*) has a solution $\overline{x}(t) \in R^n$, where

$$\overline{x}(t) = \left[x_1(t), x_2(t), \cdots, x_n(t)\right]^{\mathrm{T}},$$

$$x_i(t) \in C^0_{1-\alpha}([0,h]) \cap L^1((0,h))$$

for all $i = 1, 2, \cdots, n$ and sufficiently small $h > 0$.

Proof. Given $A \in M_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ by Lemma 1.3, there is a unitary matrix $U \in M_n$ such that

$$U^* A U = T = \left[t_{ij}\right]$$

is upper triangular with diagonal entries $t_{ii} = \lambda_i, i = 1, \cdots, n.$

Let $\overline{y}(t) = U^* \overline{x}(t)$, we have

$$D^\alpha \overline{y}(t) = U^* D^\alpha \overline{x}(t) = U^* A \overline{x}(t) = U^* A U \overline{y}(t) = T\overline{y}(t).$$

At the same time, the initial problem (*) changed into

$$\begin{cases} D^\alpha \overline{y}(t) = T\overline{y}(t), \\ t^{1-\alpha} \overline{y}(t)\big|_{t=0} = U^* \overline{b}. \end{cases} \tag{**}$$

Now, let's consider the problem (**).

Clearly, the problem (**) is equivalent to the following n problems

$$\begin{cases} D^\alpha y_i(t) = \sum_{j=i}^n t_{ij} y_j(t), \\ t^{1-\alpha} y_i(t)\big|_{t=0} = b_i, \end{cases}$$

for $i, j = 1, 2, \cdots, n.$ where b_i is the i th entries of the vector $U^* \overline{b}$.

Consider the weighed Cauchy-type problem

$$\begin{cases} D^\alpha y_n(t) = t_{nn} y_n(t), \\ t^{1-\alpha} y_n(t)\big|_{t=0} = b_n. \end{cases}$$

In Lemma 1.6, take $q = 1, y(t) = 0.$ Then by lemma 1.6, $\exists h > 0$, s.t. the above problem has at least a solution

$$y_n(t) \in C^0_{1-\alpha}([0,h]) \cap L^1((0,h)).$$

Consider the following weighed Cauchy-type problem

$$\begin{cases} D^\alpha y_{n-1}(t) = t_{n-1,n-1} y_{n-1}(t) + t_{n-1,n} y_n(t), \\ t^{1-\alpha} y_{n-1}(t)\big|_{t=0} = b_{n-1}. \end{cases}$$

In Lemma 1.6, take $q = 1, y(t) = t_{n-1,n} y_n(t).$ Then by Lemma 1.6, $\exists h > 0$, s.t. the above problem has at least a solution $y_n(t) \in C^0_{1-\alpha}([0,h]) \cap L^1((0,h))$.

Similarly, there has at least a solution in

$$C^0_{1-\alpha}([0,h]) \cap L^1((0,h))$$

for the rest n-2 initial problem in (**), denote by $y_{n-2}(t), y_{n-3}(t), \cdots, y_1(t)$ respectively. And therefore, there has at least a solution

$$\overline{y}(t) = \left[y_1(t), y_2(t), \cdots, y_n(t)\right]^{\mathrm{T}}$$

of the problem (**). Let $\overline{x}(t) = U\overline{y}(t)$, it is required for us.

The proof is completed.

Since the problem (**) is equivalent to the following n problems

$$\begin{cases} D^{\alpha} y_i(t) = \sum_{j=i}^{n} t_{ij} y_j(t), \\ t^{1-\alpha} y_i(t)\big|_{t=0} = b_i, \end{cases} \quad (9)$$

for $i, j = 1, 2, \cdots, n.$ where b_i is the i th entries of the vector $U^{*}\overline{b}$. Next, we shall discuss these equations in (9).

Theorem 2.2 Assume that the right hand of these equations in (9) satisfied H1, $\mu - (m-1)(1-\alpha) > 0$, $\alpha > 1/2$ and $\|\varphi\|_q < L$ for some $q > 1/\alpha$, If the solution of the problems (**) denoted by

$$\overline{x}(t) = \left[x_1(t), x_2(t), \cdots, x_n(t)\right]^{\mathrm{T}},$$

then there exists some constant $C > 0$ such that $|x_i(t)| \le \|U\|_{\infty} Ct^{\alpha-1}, t > 0$ for all $i = 1, 2, \cdots, n.$

Proof. Similar to the proof of Theorem 2.1, now consider the following weighted Cauchy-type problem

$$\begin{cases} D^{\alpha} y_n(t) = t_{nn} y_n(t), \\ t^{1-\alpha} y_n(t)\big|_{t=0} = b_n. \end{cases}$$

Then by Lemma 1.7, there exists some constant $C_n > 0$ such that $|y_n(t)| \le C_n t^{\alpha-1}, t > 0.$

Consider the following problem

$$\begin{cases} D^{\alpha} y_{n-1}(t) = t_{n-1,n-1} y_{n-1}(t) + t_{n-1,n} y_n(t), \\ t^{1-\alpha} y_{n-1}(t)\big|_{t=0} = b_{n-1}. \end{cases}$$

Then by Lemma 1.7, there exists some constant $C_{n-1} > 0$ such that $|y_{n-1}(t)| \le C_{n-1} t^{\alpha-1}, t > 0.$

Similarly, there exist some positive constants $C_{n-2}, C_{n-3}, \cdots, C_1$ such that

$$|y_i(t)| \le C_i t^{\alpha-1}, t > 0.$$

for all $i = n-2, n-3, \cdots, 1.$

Let $\overline{x}(t) = U\overline{y}(t), C = \max_{1 \le i \le n}\{C_i\}$. Then we have

$$|x_i(t)| \le \|U\|_{\infty} Ct^{\alpha-1}, t > 0,$$

for all $i = 1, 2, \cdots, n.$

The proof is completed.

Theorem 2.3 Assume that $\alpha > 1/2$, the right-hand of these equations in (9) satisfied H2, and $\|\varphi\|_q < K$

For some $q > 1/2$. Suppose further that

$$\mu + 1/p < m(1-\alpha).$$

If denote solution of the problems (**) $\overline{x}(t)$ by

$$\overline{x}(t) = \left[x_1(t), x_2(t), \cdots, x_n(t)\right]^{\mathrm{T}}.$$

Then there exists some constant $C > 0$ and $0 < \delta < 1 - \alpha$, such that

$$|x_i(t)| \le \|U\|_{\infty} Ct^{-\delta}, t \ge a > 0,$$

for all $i = 1, 2, \cdots, n.$

Using Lemmas 1.3 and 1.9, the proof is similar to Theorem 2.2. Therefore, it is omitted.

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