Scientific
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Laplace Transform Analytical Restructure

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ABSTRACT

In this paper, the Laplace transform definition is implemented without resorting to Adomian decomposition nor Homotopy perturbation methods. We show that the said transform can be simply calculated by differentiation of the original function. Various analytic consequent results are given. The simplicity and efficacy of the method are illustrated through many examples with shown Maple graphs, and transform tables are provided. Finally, a new infinite series representation related to Laplace transforms of trigonometric functions is proposed.

Keywords: Laplace Transform; Natural Transform; Sumudu Transform

1. Introduction

Integral transforms methods have been used to a great advantage in solving differential equations. Limitations of Fourier series technique, were overcome by the extensive coverage of the Fourier transform to functions $f(t)$, which need not be periodic [1]. The complex variable in the Fourier transform is substituted by a single variable s to obtain the well known Laplace transform [1-8], a favorite tool in solving initial value problems (IVPs). The integral equation defined by Léonard Euler was first named as Laplace by Spitzer in 1878. However the very first Laplace transform applications were established by Bateman in 1910 to solve Rutherford's radioactive decay, and Bernstein in 1920 with theta functions. For a real function $f(t)$ with variable $t \in [0, \infty)$, the Laplace transform, designated by the operator, $\mathscr{L}$, giving rise to a function in s, $F(s)$, in the right half complex plane, is defined by,

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt; \ \operatorname{Re}(s) > 0.$$

While we completely focus on the Laplace transform, in this paper, many of the ideas herein stem from recent work on the Sumudu transform, and studies and observations connecting the Laplace transform with the Sumudu transform through the Laplace-Sumudu Duality (LSD) for $t \in [0, \infty)$, and the Bilateral Laplace Sumudu Duality (BLSD) for $t \in \mathbb{R}$ [9-16]. Indeed, considering the

s-multiplied two-sided Laplace transform,

$$s\int_{-\infty}^\infty e^{-st} f(t) dt = \int_{-\infty}^\infty s e^{-st} f(t) dt;$$
$$\operatorname{Re}(s) \in (-\infty, \infty).$$

by making the parameter change s with $1/u$ in the equation above we get the two-sided Sumudu transform,

$$\mathbb{S}[f(t)]$$
$$= \frac{1}{u}\int_{-\infty}^\infty e^{-t/u}(t) dt = \int_{-\infty}^\infty \frac{e^{-t/u}}{u} f(t) dt$$
$$= \int_0^\infty e^{-t} f(ut) dt;$$
$$u \in (-\tau_1, \tau_2).$$

Here, the constants τ_1 and τ_2 may be finite (or) infinite, and are based on the exponential boundedness nature required on $f(t)$ in the domain set $A =$

$$\left\{ f(t) \middle| \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

While Analyses about the properties of the Sumudu transform, transform tables, and many of its physical applications can be found in [9-12,14,16], investigations, applications, and transform tables stemming from the Natural transform can be found in [17-21]. This new integral transform combines both (one sided) Sumudu and (one sided) Laplace transforms by,

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$$\mathbb{N}^+\left[f\left(t\right)\right]=\int_0^\infty \mathrm{e}^{-st}f\left(ut\right)\mathrm{d}t;\ \mathrm{Re}\left(s\right)>0, u\in\left(0,\tau_2\right).$$

Obviously, taking, $u \equiv 1$, in the Natural transform, leads to the Laplace transform, and taking, $s \equiv 1$, results in one sided Sumudu transform. We note that while the Natural can be bilateral like both Sumudu and Bilateral Laplace, when the variable t is chosen positive in the definition, both $\mathrm{Re}\left(s\right)$ and the variable, u, must be positive as well, as as this ought to correct a related one sided Natural Transform defintion misprint appearing in our papers [17,18].

The gist and essence of this work is solving the Laplace integral equation once by differention, and by integration by parts. Divided into two major sections, this paper in Section 2 explains the various multiple shift properties connected with the Laplace transform by just differentiating the original function. The new infinite series representation of trigonometric functions related with the Laplace transform is proved in Section 3. In consequence of our formulations and derivations, three tables are provided at the end of the Section 3 ended with concluding remarks and directions for some future work. The tables are respectively covering derivatives periods for the function $E \sin \omega t$ (in **Table 1**), 21 trignometric series expansions entries (in **Table 2**), and 16 main Laplace transform properties, as generated by Proposition 3 (in **Table 3**). Examples 1, 2, and 3 in the body of the text of Section 2, as well as Example 6 and Entry 17 of transform **Table 2** in Section 3, are respectively afforded Maple graphs (see **Figures 1-5**), showing both the time function invoked in the corresponding example, and its resulting Laplace trasform.

2. Laplace Transforms by Function Differentiation

As stated earlier in the introduction, an ultimate goal of ours, among others, is calculating the Laplace transform of $f\left(t\right)$ by simple differentiation rather than usual integration. We show that we can do this, without resorting to the Adomian nor homotopy methods, **ADM**, and

HPM, as was done in [22,23]. Along with the Laplace series definition below, some elementary properties are proved.

Definition *The Laplace transform (henceforth designated as $F(s)$) of the exponential order and sectionwise continuous function $f\left(t\right)\in\mathbb{R}$, is defined by,*

$$\mathscr{L}\left[f\left(t\right)\right]=-\left[\mathrm{e}^{-st}\sum_{n=0}^\infty \frac{1}{s^{n+1}}\frac{\mathrm{d}^n f\left(t\right)}{\mathrm{d}t^n}\right]_0^\infty;\ \ \mathrm{Re}\left(s\right)>0. \quad (1)$$

Remark *We observe that, from the traditional Laplace transform, taking $u=f\left(t\right)$, so that*

$$u^{(0)}=f\left(t\right), u^{(1)}=\frac{\mathrm{d}f\left(t\right)}{\mathrm{d}t},\cdots, u^{(n)}=\frac{\mathrm{d}^n f\left(t\right)}{\mathrm{d}t^n}\ \ and$$

$$\mathrm{d}v=\mathrm{e}^{-st}\mathrm{d}t\text{ , so that}$$

$$v_{(0)}=-\frac{\mathrm{e}^{-st}}{s}, v_{(1)}=\frac{\mathrm{e}^{-st}}{s^2},\cdots, v_{(n)}=\frac{\left(-1\right)^{n+1}\mathrm{e}^{-st}}{s^{n+1}}. \text{ Now using}$$

$u^{(n)}$ *and* $v_{(n)}$ *in Bernoulli's integration by parts,*

$$\int_0^\infty u\,\mathrm{d}v=\left[\sum_{n=0}^\infty\left(-1\right)^n u^n v_n\right]_0^\infty.$$

and noting $\left(-1\right)^{2n+1}=-1$ for $n\geq 0$ Equation (1) follows.

Can't one choose $u=\mathrm{e}^{-st}$ and $\mathrm{d}v=f\left(t\right)\mathrm{d}t$ for solving the Laplace integral equation by parts? The detailed answer with analysis is given in Section 3. For simplicity, we use hereafter $\dfrac{\mathrm{d}^n f\left(t\right)}{\mathrm{d}t^n}=f^{(n)}\left(t\right)$.

Multiple Shifts and Periodicity Results

Theorem 1 *The Laplace transform of i-th derivative of $f\left(t\right)$, with respect to t is defined by,*

$$\mathscr{L}\left[\frac{\mathrm{d}^i f\left(t\right)}{\mathrm{d}t^i}\right]=-\left[\mathrm{e}^{-st}\sum_{n=0}^\infty \frac{f^{(n)}\left(t\right)}{s^{n-i+1}}\right]_0^\infty-\sum_{k=0}^{i-1}s^{i-(k+1)}f^{(k)}\left(0\right).$$

Proof. The LHS of above equation is $s^i\mathscr{L}\left[f\left(t\right)\right]-\sum_{k=0}^{i-1}s^{i-(k+1)}f^k\left(0\right)$, substituting Equation (1) for $\mathscr{L}\left[f\left(t\right)\right]$, the proof is completed.

Table 1. Period function calculations.

$f\left(t\right)$	$E\sin\omega t$	$t\to 0$	$t\to T=\dfrac{\pi}{\omega}$	$f\left(0\right)-f\left(T\right)$
$f\left(t\right)$	$E\sin\omega t$	-	-	-
$f^{(1)}\left(t\right)$	$E\omega\cos\omega t$	$E\omega$	$-\mathrm{e}^{-\frac{s\pi}{\omega}}E\omega$	$E\omega\left(1+\mathrm{e}^{-\frac{s\pi}{\omega}}\right)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(2n)}\left(t\right);\left(n\geq 0\right)$	$\left(-1\right)^n E\omega^{2n}\times\sin\omega t$	-	-	-
$f^{(2n+1)}\left(t\right);\left(n\geq 0\right)$	$\left(-1\right)^n E\omega^{2n+1}\times\cos\omega t$	$\left(-1\right)^n E\times\omega^{2n+1}$	$\left(-1\right)^{n+1}\mathrm{e}^{-\frac{s\pi}{\omega}}\times E\omega^{2n+1}$	$\left(-1\right)^n E\omega^{2n+1}\times\left(1+\mathrm{e}^{-\frac{s\pi}{\omega}}\right)$

Table 2. New infinite series representation of trigonometric functions.

S. No	$f(t)$	$-\sum_{n=0}^{\infty} f_{(n+1)}(t)\big|_{t\to 0} \int_0^{\infty} \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
1	e^t	$-\sum_{n=0}^{\infty}\int_0^{\infty} \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
2	e^{-t}	$\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty} \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
3	$\sin t$	$\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty} \dfrac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
4	$\cos t$	$\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
5	$\sin^2 t$	$\dfrac{1}{2}+\sum_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{2^{2n+3}}\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
6	$\cos^2 t$	$\dfrac{1}{2}+\sum_{n=0}^{\infty}\dfrac{(-1)^{n}}{2^{2n+3}}\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
7	$\sin^3 t$	$\sum_{n=0}^{\infty}\dfrac{(-1)^{n}}{4}\left[3-\dfrac{1}{3^{2n+1}}\right]\int_0^{\infty} \dfrac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
8	$\cos^3 t$	$\sum_{n=0}^{\infty}\dfrac{(-1)^{n}}{4}\left[3+\dfrac{1}{3^{2(n+1)}}\right]\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
9	$\sin^4 t$	$\dfrac{3}{8}+\sum_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{2^{2n+3}}\left[1-\dfrac{1}{2^{2(n+2)}}\right]\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
10	$\cos^4 t$	$\dfrac{3}{8}+\sum_{n=0}^{\infty}\dfrac{(-1)^{n}}{2^{2n+3}}\left[1+\dfrac{1}{2^{2(n+2)}}\right]\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
11	$\sin^5 t$	$\sum_{n=0}^{\infty}\dfrac{(-1)^{n}}{8}\left[5-\dfrac{5}{6\times 3^{2n}}+\dfrac{1}{10\times 5^{2n}}\right]\int_0^{\infty} \dfrac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
12	$\cos^5 t$	$\sum_{n=0}^{\infty}\dfrac{(-1)^{n}}{16}\left[10+\dfrac{5}{3^{2(n+1)}}+\dfrac{1}{5^{2(n+1)}}\right]\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
13	$\sin t\cos t$	$\sum_{n=0}^{\infty}\dfrac{(-1)^{n}}{2^{2(n+1)}}\int_0^{\infty} \dfrac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
14	$\sin^2 t\cos^2 t$	$\dfrac{1}{8}+\sum_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{2\times 4^{2n+3}}\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
15	$\sin^3 t\cos^3 t$	$\sum_{n=0}^{\infty}\dfrac{(-1)^{n}}{32}\left[\dfrac{3}{2^{2n+1}}-\dfrac{1}{6^{2n+1}}\right]\int_0^{\infty} \dfrac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
16	$\sin^4 t\cos^4 t$	$\dfrac{3}{128}+\sum_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{32\times 4^{2(n+1)}}\left[1-\dfrac{1}{4^{n+2}}\right]\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
17	$\sin^5 t\cos^5 t$	$\sum_{n=0}^{\infty}\dfrac{(-1)^{n}}{512}\left[\dfrac{5}{2^{2n}}-\dfrac{5}{6^{2n+1}}+\dfrac{1}{10^{2n+1}}\right]\int_0^{\infty} \dfrac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
18	$\sinh t$	$-\sum_{n=0}^{\infty}\int_0^{\infty} \dfrac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
19	$\cosh t$	$-\sum_{n=0}^{\infty}\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
20	$\sinh^2 t$	$-\dfrac{1}{2}-\sum_{n=0}^{\infty}\dfrac{1}{4\times 2^{2n+1}}\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$
21	$\cosh^2 t$	$\dfrac{1}{2}-\sum_{n=0}^{\infty}\dfrac{1}{4\times 2^{2n+1}}\int_0^{\infty} \dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}} J_0\left(2\sqrt{vt}\right)\mathrm{d}v$

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Table 3. Laplace transform properties with respect to the Proposition 3.

S. No	$f(t)$	$\mathscr{L}\big[f(t)\big]$
1	$\dfrac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}$	$\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}s^{n+i}f_{(n+1)}(t)\right]_0^{\infty}-\sum\limits_{k=0}^{i-1}s^{i-(k+1)}f^{(k)}(0)$
2	$\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}}f(\tau)(\mathrm{d}\tau)^i$	$\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}s^{n-i}f_{(n+1)}(t)\right]_0^{\infty}$
3	$t^m f(t)$	$(-1)^m\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}\prod\limits_{j=0}^{m-1}(n-j)s^{n-m}f_{(n+1)}(t)\right]_0^{\infty}$
4	$\dfrac{f(t)}{t^m}$	$(-1)^m\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}\prod\limits_{j=1}^{m}\dfrac{s^{n+m}f_{(n+1)}(t)}{(n+j)}\right]_0^{\infty}$
5	$t^m\dfrac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}$	$(-1)^m\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}\prod\limits_{j=0}^{m-1}(n+i-j)s^{n-m+i}f_{(n+1)}(t)\right]_0^{\infty}+(-1)^{m+1}\sum\limits_{k=0}^{i-1}\prod\limits_{j=1}^{m}\big[i-(k+j)\big]s^{i-(k+m+1)}f^{(k)}(0)$
6	$\dfrac{1}{t^m}\dfrac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}$	$(-1)^m\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}\prod\limits_{j=1}^{m}\dfrac{s^{n+m+i}f_{(n+1)}(t)}{(n+i+j)}\right]_0^{\infty}+(-1)^{m+1}\sum\limits_{k=0}^{i-1}\prod\limits_{j=0}^{m-1}\dfrac{s^{i-k+m-1}f^{(k)}(0)}{(i-k+j)}$
7	$t^m\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}}f(\tau)(\mathrm{d}\tau)^i$	$(-1)^m\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}\prod\limits_{j=0}^{m-1}(n-i-j)s^{n-m-i}f_{(n+1)}(t)\right]_0^{\infty}$
8	$\dfrac{1}{t^m}\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}}f(\tau)(\mathrm{d}\tau)^i$	$(-1)^m\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}\prod\limits_{j=1}^{m}\dfrac{s^{n+m-i}f_{(n+1)}(t)}{(n-i+j)}\right]_0^{\infty}$
9	$\dfrac{\mathrm{d}^i}{\mathrm{d}t^i}t^m f(t)$	$(-1)^m\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}\prod\limits_{j=0}^{m-1}(n-j)s^{n+i-m}f_{(n+1)}(t)\right]_0^{\infty}-\sum\limits_{k=0}^{i-1}s^{i-(k+1)}t^m f^{(k)}(0)$
10	$\dfrac{\mathrm{d}^i}{\mathrm{d}t^i}\dfrac{f(t)}{t^m}$	$(-1)^m\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}\prod\limits_{j=1}^{m}\dfrac{s^{n+i+m}f_{(n+1)}(t)}{(n+j)}\right]_0^{\infty}-\sum\limits_{k=0}^{i-1}s^{i-(k+1)}\dfrac{f^{(k)}(0)}{t^m}$
11	$\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}}\tau^m f(\tau)(\mathrm{d}\tau)^i$	$(-1)^m\left[\sum\limits_{n=0}^{\infty}\prod\limits_{j=0}^{m-1}(n-j)s^{n-i-m}f_{(n+1)}(t)\right]_0^{\infty}$
12	$\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}}\dfrac{f(\tau)}{\tau^m}(\mathrm{d}\tau)^i$	$(-1)^m\left[\sum\limits_{n=0}^{\infty}\prod\limits_{j=1}^{m}\dfrac{s^{n-i+m}f_{(n+1)}(t)}{(n+j)}\right]_0^{\infty}$
13	Periodic function	$\dfrac{1}{1-\mathrm{e}^{-sT}}\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}s^n f_{(n+1)}(t)\right]_T^0$
14	$\int_0^t f(t-\xi)g(\xi)\mathrm{d}\xi$	$\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}s^{2n}f_{(n+1)}(t)\times g_{(n+1)}(t)\right]_0^{\infty}$
15	*Initial Value*	$\lim\limits_{t\to0}f(t)=\lim\limits_{s\to\infty}\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}s^{n+1}f_{(n+1)}(t)\right]_0^{\infty}$
16	*Final Value*	$\lim\limits_{t\to\infty}f(t)=\lim\limits_{s\to0}\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}s^{n+1}f_{(n+1)}(t)\right]_0^{\infty}$

Theorem 2 *The Laplace transform of i-th antiderivative of f(t), in the domain* $[0,t]$ *with respect to* t*, is given by,*

$$\mathscr{L}\left[\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}}f(\tau)(\mathrm{d}\tau)^i\right]=-\left[\mathrm{e}^{-st}\sum\limits_{n=0}^{\infty}\dfrac{f^{(n)}(t)}{s^{n+i+1}}\right]_0^{\infty}.$$

Proof. Applying Equation (1) in $\dfrac{\mathscr{L}\big[f(t)\big]}{s^i}$ and performing the usual computations, yields the RHS of the equation above and proves out theorem.

Theorem 3 *For* $m\geq1$*, the Laplace transform of the function* $t^m f(t)$*, is given by,*
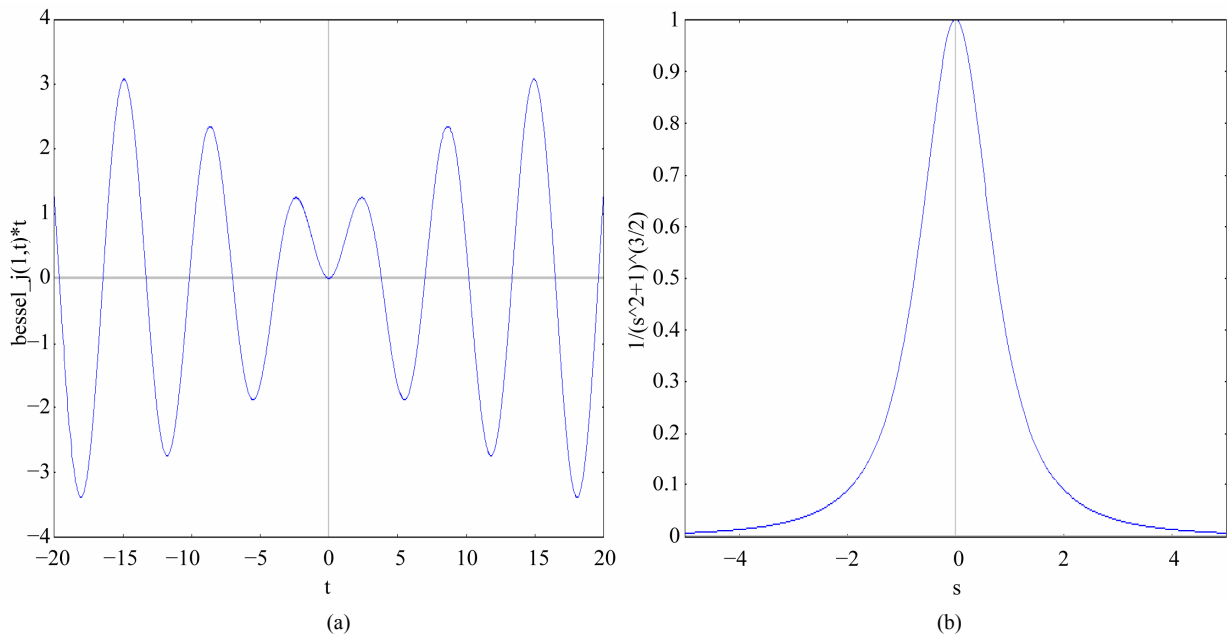
(a)

(b)

Figure 1. Graph of example 1. (a) $f(t) = tJ_1(t)$ **; (b)** $F(s) = \dfrac{1}{\left(s^2+1\right)^{\frac{3}{2}}}$ **.**
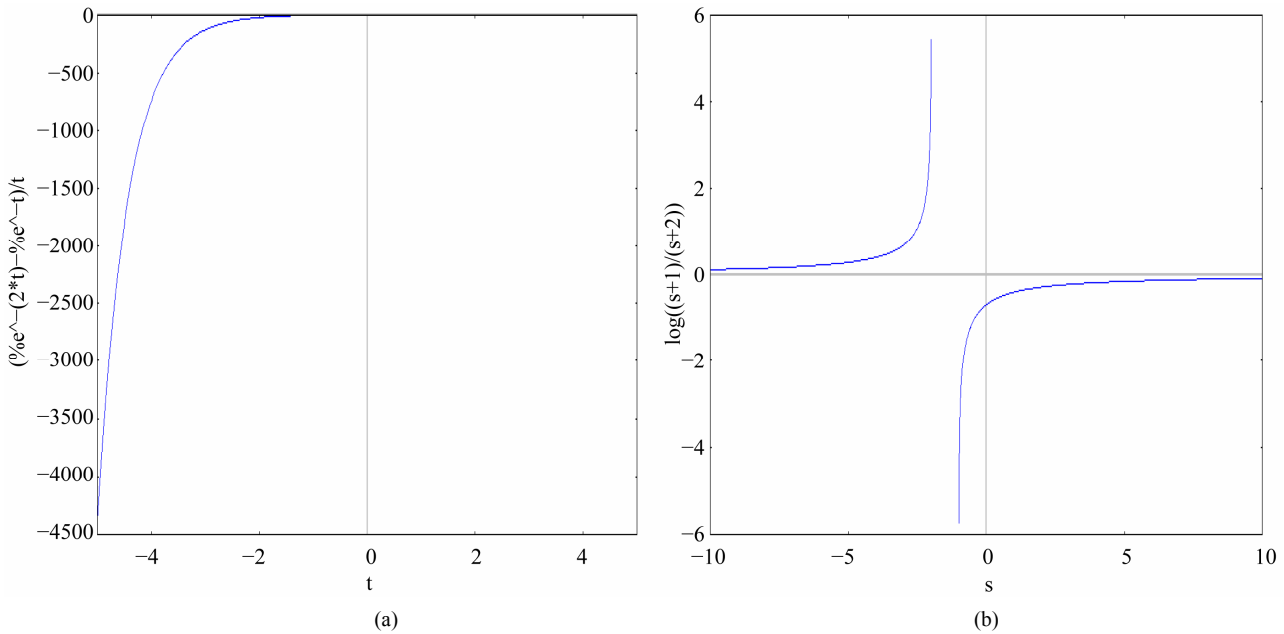

(a)

(b)

Figure 2. Graph of example 2 with $a = 1$, $b = 2$. (a) $f(t) = \dfrac{e^{-2t} - e^{-t}}{t}$ **; (b)** $F(s) = \log\left(\dfrac{s+1}{s+2}\right)$ **.**

$$\mathscr{L}\left[t^m f(t)\right] = -\left[e^{-st} \sum_{n=0}^{\infty} \prod_{j=1}^{m} \frac{(n+j) f^{(n)}(t)}{s^{n+m+1}} \right]_0^{\infty}.$$

Proof. From the theory of Laplace transform

$$\mathscr{L}\left[t^m f(t)\right] = (-1)^m \frac{d^m}{ds^m} \mathscr{L}\left[f(t)\right], \text{ when } \mathscr{L}\left[f(t)\right]$$

is given by Equation (1),

$$\mathscr{L}\left[t^m f(t)\right] = (-1)^m \frac{d^m}{ds^m} \left\{ -\left[e^{-st} \sum_{n=0}^{\infty} \frac{f^{(n)}(t)}{s^{n+1}} \right]_0^{\infty} \right\}. \quad (2)$$

When $m = 1$ in Equation (2),

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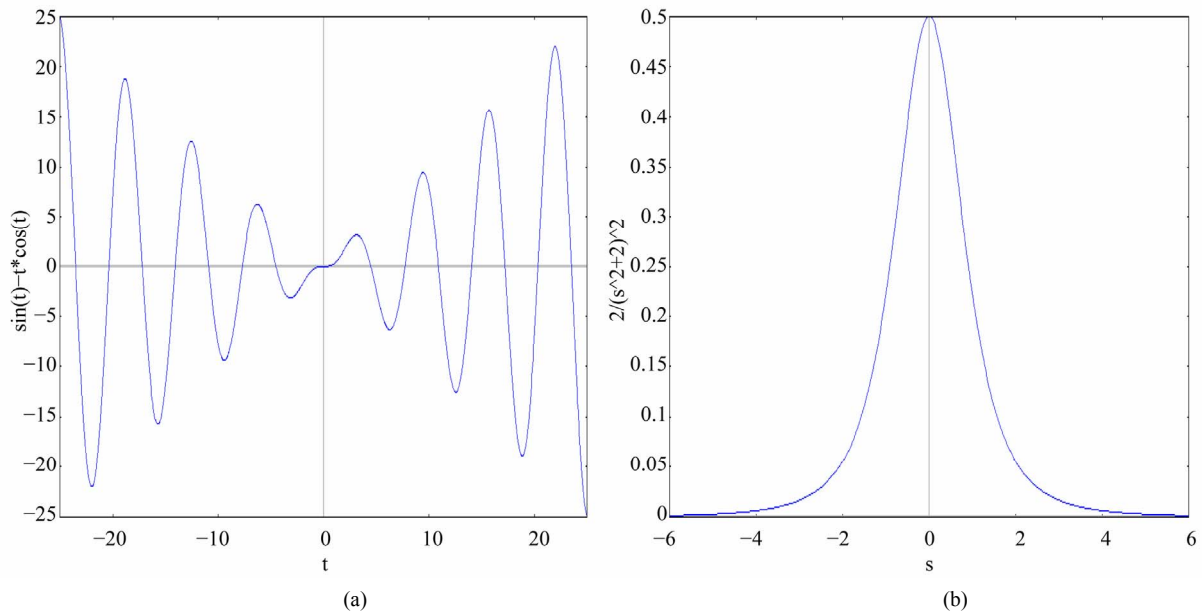
Figure 3. Graph of example 3. (a) $f(t) = \int_0^t \tau \sin\tau d\tau$ **; (b)** $F(s) = \dfrac{2}{\left(s^2+1\right)^2}$ **.**
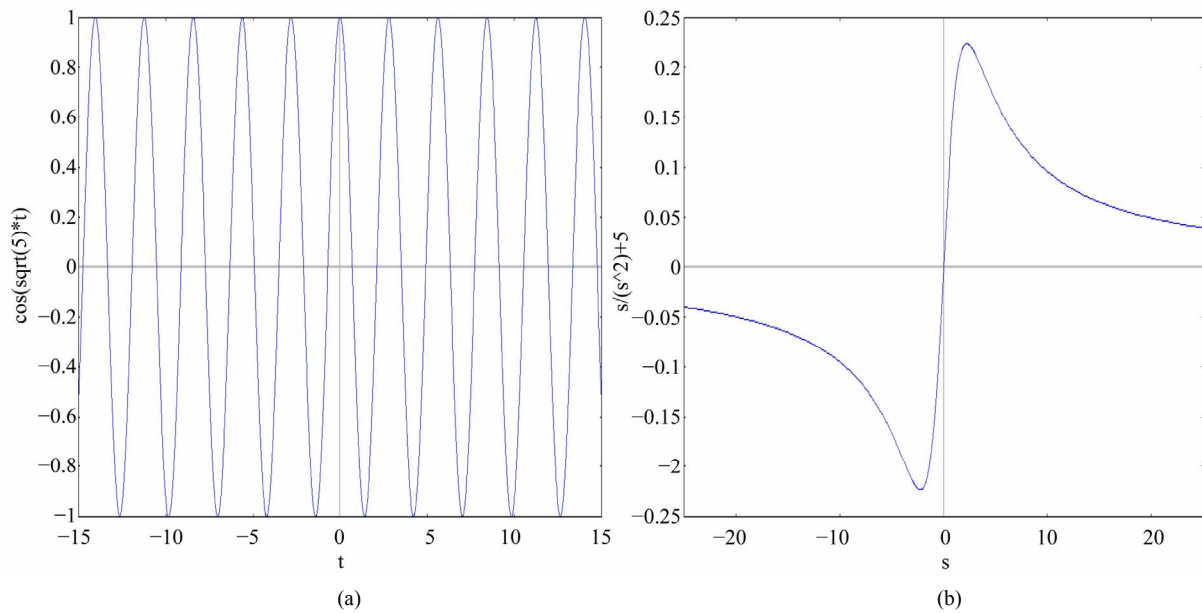
Figure 4. Graph of example 6 with $a = 5$. (a) $f(t) = \cos\sqrt{5}t$ **; (b)** $F(s) = \dfrac{s}{\left(s^2+5\right)}$ **.**

$$\mathscr{L}\left[tf(t)\right] = -\frac{d}{ds}\left\{-\left[e^{-st}\sum_{n=0}^{\infty}\frac{f^{(n)}(t)}{s^{n+1}}\right]_0^{\infty}\right\}$$

$$= -\left[e^{-st}\sum_{n=0}^{\infty}\frac{(n+1)f^{(n)}(t)}{s^{n+2}}\right]_0^{\infty}.$$

(3)

When $m = 2$ in Equation (2),

$$\mathscr{L}\left[t^2 f(t)\right] = -\frac{d^2}{ds^2}\left[e^{-st}\sum_{n=0}^{\infty}\frac{f^{(n)}(t)}{s^{n+1}}\right]_0^{\infty}$$

$$= -\left[e^{-st}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)f^{(n)}(t)}{s^{n+3}}\right]_0^{\infty}.$$

(4)

Finally for the non-negative integer m, after simplifi-
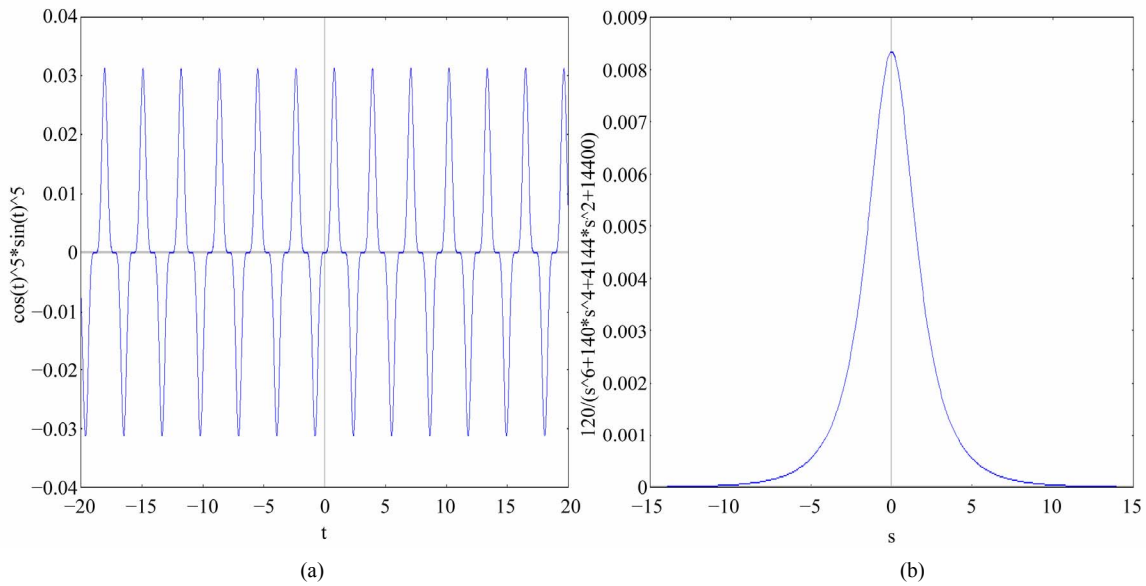
Figure 5. Graph of entry 17 of Table 2 and its Laplace transform. (a) $f(t) = \sin^5 t \cos^5 t$; **(b)**

$$F(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{512}\left(\frac{5}{2^{2n}} - \frac{5}{6^{2n+1}} + \frac{1}{10^{2n+1}}\right)s^{2n}.$$

cation,

$$(-1)^m \frac{\mathrm{d}^m}{\mathrm{d}s^m}\left(\frac{\mathrm{e}^{-st}}{s^{n+1}}\right)$$

$$= \frac{(n+1)(n+2)\cdots(n+m)}{s^{n+m+1}}$$

$$= \prod_{j=1}^{m} \frac{(n+j)}{s^{n+m+1}}.$$

which yields the result of Theorem 3.

Theorem 4 *The Laplace transform of the function* $\dfrac{f(t)}{t^m}$, *for* $m \geq 1$, *is,*

$$\mathscr{L}\left[\frac{f(t)}{t^m}\right] = -\left[\mathrm{e}^{-st}\sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{f^{(n)}(t)}{(n-j)s^{n-m+1}}\right]_0^{\infty}.$$

Proof. Substituting Equation (1) for $F(\eta)$ in

$$\mathscr{L}\left[\frac{f(t)}{t^m}\right] = \underbrace{\int_s^{\infty}\cdots\int_s^{\infty}}_{m\ \text{times}} F(\eta)\,(\mathrm{d}\eta)^m \text{ and after the usual}$$

computations, Theorem 4 follows.

Example 1 *As an application of Theorem* 3, *the Laplace transform of* $tJ_1(t)$, *where*

$$J_1(t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+1}}{2^{2m+1}m!(m+1)!}, \text{ denotes the first kind}$$

order one Bessel function, is calculated as follows, (*graph shown in* **Figure 1**),

$$f(t) = J_1(t)$$

$$= \frac{t}{2\cdot0!\cdot1!} - \frac{t^3}{2^3\cdot1!\cdot2!} + \frac{t^5}{2^5\cdot2!\cdot3!} - \frac{t^7}{2^7\cdot3!\cdot4!} + \cdots$$

$$f^{(1)}(t) = \frac{1}{2\cdot0!\cdot1!} - \frac{3t^2}{2^3\cdot1!\cdot2!} + \frac{5t^4}{2^5\cdot2!\cdot3!} - \frac{7t^6}{2^7\cdot3!\cdot4!} + \cdots$$

$$f^{(2)}(t) = -\frac{6t}{2^3\cdot1!\cdot2!} + \frac{20t^3}{2^5\cdot2!\cdot3!} - \frac{42t^5}{2^7\cdot3!\cdot4!} + \cdots$$

$$\vdots$$

$$f^{(2n)}(t) = (-1)^n \text{ functions of } t + (-1)^{n+1}\cdots$$

$$f^{(2n+1)}(t) = (-1)^n \frac{(2n+1)!}{2^{2n+1}\cdot n!\cdot(n+1)!} + (-1)^{n+1}$$

functions of $t\cdots$

Now substituting the above derivatives in Equation (3), and after applying both the limits, $f^{(2n)}(t) \equiv 0$ and

$$f^{(2n+1)}(t) \equiv \frac{(-1)^n (2n+1)!}{2^{2n+1}\cdot n!\cdot(n+1)!} \text{ for } (n\geq0),$$

$$\mathscr{L}\left[tJ_1(t)\right] = \frac{2\cdot1!}{2\cdot0!\cdot1!\,s^3} - \frac{4\cdot3!}{2^3\cdot1!\,2!\,s^5} + \cdots$$

$$+ (-1)^n \frac{2(n+1)(2n+1)!}{2^{2n+1}\cdot n!(n+1)!\,s^{2n+3}}$$

$$= \frac{1}{s^3}\left[1 - \frac{4\cdot3!}{2^3\cdot1!\cdot2!\cdot s^2} + \cdots + (-1)^n \frac{2(n+1)(2n+1)!}{2^{2n+1}\cdot n!\cdot(n+1)!\cdot s^{2n}}\right]$$

$$= \frac{1}{s^3}\left[1 + \frac{1}{s^2}\right]^{-\frac{3}{2}} = \frac{1}{\left(s^2+1\right)^{\frac{3}{2}}}.$$

AM

The multiple-shift theorems that follow are useful in treating differential and integral equations with polynomial coefficients.

Example 2 *The Laplace transform of* $\dfrac{\mathrm{e}^{-bt}-\mathrm{e}^{-at}}{t}$ *is calculated by taking,*

$$f(t)=\mathrm{e}^{-bt}-\mathrm{e}^{-at}, f^{(1)}(t)=-b\mathrm{e}^{-bt}+a\mathrm{e}^{-at},$$
$$f^{(2)}(t)=b^2\mathrm{e}^{-bt}-a^2\mathrm{e}^{-at},\cdots,$$
$$f^{(n)}(t)=(-1)^n b^n \mathrm{e}^{-bt}+(-1)^{n+1} a^n \mathrm{e}^{-at};(n\geq 0)$$

(**Figure 2**), *When* $m=1$ *in Theorem* 4,

$$\mathscr{L}\left[\frac{f(t)}{t}\right]=-\left[\mathrm{e}^{-st}\sum_{n=0}^{\infty}\frac{f^{(n)}(t)}{ns^n}\right]_0^{\infty}.$$

So that,

$$\mathscr{L}\left[\frac{\mathrm{e}^{-bt}-\mathrm{e}^{-at}}{t}\right]$$
$$=\frac{-b+a}{s}+\frac{b^2-a^2}{2s^2}+\cdots+\frac{(-1)^n b^n+(-1)^{n+1} a^n}{ns^n}$$
$$=\left[\frac{a}{s}-\cdots+(-1)^{n+1}\frac{a^n}{ns^n}\right]-\left[\frac{b}{s}-\cdots+(-1)^{n+1}\frac{b^n}{ns^n}\right]$$
$$=\log\left(1+\frac{a}{s}\right)-\log\left(1+\frac{b}{s}\right)=\log\left(\frac{s+a}{s+b}\right).$$

Theorem 5 *Let* $i,m\geq 1$, *when the* i-*th derivative of the function* $f(t)$, *with respect to* t *is shifted by* t^m, *then the Laplace transform is given by,*

$$\mathscr{L}\left[t^m\frac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}\right]$$
$$=-\left[\mathrm{e}^{-st}\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{(n-i+j)f^{(n)}(t)}{s^{n+m-i+1}}\right]_0^{\infty}\qquad(5)$$
$$+\sum_{k=0}^{i-1}\prod_{j=1}^{m}(-1)^{m+1}\left[i-(k+j)\right]s^{i-(k+m+1)}f^{(k)}(0).$$

Proof. The proof is simple, we have

$$\mathscr{L}\left[t^m\frac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}\right]=(-1)^m\frac{\mathrm{d}^m}{\mathrm{d}s^m}\mathscr{L}\left[\frac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}\right]\quad\text{where}$$

$\mathscr{L}\left[\dfrac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}\right]$ is given by Theorem 1.

Theorem 6 *For non-negative integers* i *and* m, *when the* i-*th derivative of the function* $f(t)$, *with respect to* t *is shifted with* $\dfrac{1}{t^m}$, *then the Laplace transform is given by,*

$$\mathscr{L}\left[\frac{1}{t^m}\frac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}\right]$$
$$=-\left[\mathrm{e}^{-st}\sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{f^{(n)}(t)}{(n-i-j)s^{n-m-i+1}}\right]_0^{\infty}\qquad(6)$$
$$+\sum_{k=0}^{i-1}\prod_{j=0}^{m-1}\frac{(-1)^{m+1} s^{i-k+m-1} f^{(k)}(0)}{(i-k+j)}.$$

Proof. The LHS of above equation is

$\underbrace{\int_s^{\infty}\cdots\int_s^{\infty}}_{m\text{ times}}\mathscr{L}\left[\dfrac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}\right](\mathrm{d}\eta)^m$ and $\mathscr{L}\left[\dfrac{\mathrm{d}^i f(t)}{\mathrm{d}t^i}\right]$ are given

by Theorem 1, and after proper calculations, the proof is calculated.

Theorem 7 *For* $i,m\geq 1$, *the Laplace transform of the* i-*th antiderivative of the function* $f(t)$ *with respect to* t *in the interval* $[0,t]$, *shifted with* t^m, *is given by,*

$$\mathscr{L}\left[t^m\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}}f(\tau)(\mathrm{d}\tau)^i\right]$$
$$=-\left[\mathrm{e}^{-st}\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{(n+i+j)f^{(n)}(t)}{s^{n+m+i+1}}\right]_0^{\infty}.$$

Proof. Applying Theorem 2 in LHS yields the RHS of the Equation above.

Theorem 8 *The Laplace transform of the* i-*th antiderivative of the function* $f(t)$, *with respect to* t *in the interval* $[0,t]$ *shifted with* $\dfrac{1}{t^m}; m\geq 1$, *is given by,*

$$\mathscr{L}\left[\frac{1}{t^m}\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}}f(\tau)(\mathrm{d}\tau)^i\right]$$
$$=-\left[\mathrm{e}^{-st}\sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{f^{(n)}(t)}{(n+i-j)s^{n-m+i+1}}\right]_0^{\infty}.$$

Proof. Computing the summation in the RHS of the Equation in Theorem 2 with respect to s in the domain $[s,\infty)$, m times, yields the proof.

We now establish the following results,

Theorem 9 *For* $i,m\geq 1$, *the Laplace transform of the* i-*th derivative of* $t^m f(t)$ *with respect to* t *is given by,*

$$\mathscr{L}\left[\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[t^m f(t)\right]\right]=-\left[\mathrm{e}^{-st}\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{(n+j)f^{(n)}(t)}{s^{n-i+m+1}}\right]_0^{\infty}\quad(7)$$
$$-\sum_{k=0}^{i-1}s^{i-(k+1)}t^m f^{(k)}(0).$$

Proof. Substituting Theorem 3 in

$$s^i \mathscr{L}\left[t^m f(t)\right] - \sum_{k=0}^{i-1} s^{i-(k+1)} t^m f^{(k)}(0).$$

Theorem 10 *The Laplace transform of the i-th derivative with respect to t of $\dfrac{f(t)}{t^m}$, for non-negative integers, i and m, is given by,*

$$\mathscr{L}\left[\frac{d^i}{dt^i}\left[\frac{f(t)}{t^m}\right]\right]$$

$$= -\left[e^{-st} \sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{f^{(n)}(t)}{(n-j)s^{n-i-m+1}}\right]_0^\infty \qquad (8)$$

$$-\sum_{k=0}^{i-1}\frac{s^{i-(k+1)}f^{(k)}(0)}{t^m}.$$

Proof. Substituting Theorem 4 in

$$s^i \mathscr{L}\left[\frac{f(t)}{t^m}\right] - \sum_{k=0}^{i-1} s^{i-(k+1)}\frac{f^{(k)}(0)}{t^m}.$$

Example 3 *Consider the function,* $\int_0^t \tau \sin\tau d\tau$, *then taking* $f(t) = \sin t$, *yields the expected derivatives,* $f^{(1)}(t) = \cos t, f^{(2)}(t) = -\sin t, \cdots, f^{(2n)}(t) = (-1)^n \sin t$, *and,* $f^{(2n+1)}(t) = (-1)^n \cos t$; $(n \geq 0)$. *Next, for* $i = m = 1$ *in Theorem* 11,

$$\mathscr{L}\left[\int_0^t \tau f(\tau)d\tau\right] = -\left[e^{-st}\sum_{n=0}^{\infty}\frac{(n+1)f^{(n)}(t)}{s^{n+3}}\right]_0^\infty.$$

Therefore (**Figure 3**),

$$\mathscr{L}\left[\int_0^t \tau \sin\tau d\tau\right] = \frac{2}{s^4} - \frac{4}{s^6} + \cdots + \frac{(-1)^n 2(n+1)}{s^{2(n+2)}}$$

$$= \frac{2}{s^4}\left[1 - \frac{2}{s^2} + \cdots + \frac{(-1)^n(n+1)}{s^{2n}}\right]$$

$$= \frac{2}{\left(s^2+1\right)^2}.$$

Theorem 11 *For non-negative integers i and m, the Laplace transform of the i-th antiderivative with respect to t in the domain $[0,t]$ of $t^m f(t)$, is given by,*

$$\mathscr{L}\left[\underbrace{\int_0^t \cdots \int_0^t}_{i\text{ times}} \tau^m f(\tau)(d\tau)^i\right]$$

$$= -\left[e^{-st}\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{(n+j)f^{(n)}(t)}{s^{n+i+m+1}}\right]_0^\infty.$$

Proof. From the property of Laplace transform, the LHS of above equation is $\dfrac{\mathscr{L}\left[t^m f(t)\right]}{s^i}$ in which Theo-

rem 3 is substituted and simplified.

Theorem 12 *The Laplace transform of the i-th antiderivative with respect to t in the domain $[0,t]$ of $\dfrac{f(t)}{t^m}$, where, $i, m \geq 1$, is given by,*

$$\mathscr{L}\left[\underbrace{\int_0^t \cdots \int_0^t}_{i\text{ times}} \frac{f(\tau)}{\tau^m}(d\tau)^i\right] = -\left[e^{-st}\sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{f^{(n)}(t)}{(n-j)s^{n+i-m+1}}\right]_0^\infty.$$

Proof. The proof is straightforward where we multiplied $\dfrac{1}{s^i}$ to Theorem 4.

Example 4 *Consider the function,* $\int_0^t \dfrac{1-e^{-\tau}}{\tau}d\tau$, *which Laplace transform we can find by taking,* $f(t) = 1 - e^{-t}$, *yielding,* $f^{(1)}(t) = e^{-t}, f^{(2)}(t) = -e^{-t}, \cdots$ *and,* $f^{(n)}(t) = (-1)^{n+1} e^{-t}$. *Now, since from the theorem above, for* $i = m = 1$, *we have,*

$$\mathscr{L}\left[\int_0^t \frac{f(\tau)}{\tau}d\tau\right] = -\left[e^{-st}\sum_{n=0}^{\infty}\frac{f^{(n)}(t)}{ns^{n+1}}\right]_0^\infty,$$

we consequently get,

$$\mathscr{L}\left[\int_0^t \frac{1-e^{-\tau}}{\tau}d\tau\right]$$

$$= \frac{1}{s^2} - \frac{1}{2s^3} + \frac{1}{3s^4} - \cdots + (-1)^{n+1}\frac{1}{ns^{n+1}}$$

$$= \frac{1}{s}\left(\frac{1}{s} - \frac{1}{2s^2} + \cdots + (-1)^{n+1}\frac{1}{ns^n}\right) = \frac{1}{s}\log\left(1 + \frac{1}{s}\right).$$

From Theorem 5 through Theorem 12, there is no restriction on positive integers m and i, which means both can be same (or) different and either of the integer can less than (or) greater than to one another.

The Theorem 5 and the Theorem 9 varies only in the coefficients, that is the order of the derivative, the same holds for Theorem 6 and Theorem 10, again the Theorem 7 and Theorem 11 varies only in the coefficients, that is the order of the anti-derivative, similarly for Theorem 8 and 12. Hence we have the following propositions, respectively.

Proposition 1 *If the function $f(t)$ and its $(i-1)$ derivative with respect to t go to zero as $t \to 0$, then,*

$$\mathscr{L}\left[\frac{d^i}{dt^i}\left[t^m f(t)\right]\right] = \mathscr{L}\left[t^m \frac{d^i}{dt^i}f(t)\right]_{[\text{Coefficients}+i]}. \qquad (9)$$

$$\mathscr{L}\left[\frac{d^i}{dt^i}\left[\frac{f(t)}{t^m}\right]\right] = \mathscr{L}\left[\frac{1}{t^m}\frac{d^i}{dt^i}f(t)\right]_{[\text{Coefficients}+i]}. \qquad (10)$$

Proposition 2

$$\mathscr{L}\left[\underbrace{\int_0^t \cdots \int_0^t}_{i \text{ times}} \tau^m f(\tau)(\mathrm{d}\tau)^i\right]$$

$$= \mathscr{L}\left[t^m \underbrace{\int_0^t \cdots \int_0^t}_{i \text{ times}} f(\tau)(\mathrm{d}\tau)^i\right]_{[\text{Coefficients}-i]}. \quad (11)$$

$$\mathscr{L}\left[\underbrace{\int_0^t \cdots \int_0^t}_{i \text{ times}} \frac{f(\tau)}{\tau^m}(\mathrm{d}\tau)^i\right]$$

$$= \mathscr{L}\left[\frac{1}{t^m} \underbrace{\int_0^t \cdots \int_0^t}_{i \text{ times}} f(\tau)(\mathrm{d}\tau)^i\right]_{[\text{Coefficients}-i]}. \quad (12)$$

The following initial and final value, convolution, and function periodicity related theorems can be easily verified through conventional Laplace transform theory.

Theorem 13 *Let the function,* $f(t)$, *be Laplace transformable, then,*

$$\lim_{t \to 0} f(t) = -\lim_{s \to \infty}\left[\mathrm{e}^{-st} \sum_{n=0}^{\infty} \frac{f^{(n)}(t)}{s^n}\right]_0^{\infty}. \quad (13)$$

$$\lim_{t \to \infty} f(t) = -\lim_{s \to 0}\left[\mathrm{e}^{-st} \sum_{n=0}^{\infty} \frac{f^{(n)}(t)}{s^n}\right]_0^{\infty}. \quad (14)$$

Theorem 14 *The Laplace transform of the convolution of two functions* $f(t)$, *and,* $g(t)$, *is given by,*

$$\mathscr{L}\left[\int_0^t f(t-\xi)g(\xi)\mathrm{d}\xi\right] = \left[\mathrm{e}^{-st} \sum_{n=0}^{\infty} \frac{f^{(n)}(t) \times g^{(n)}(t)}{s^{2(n+1)}}\right]_0^{\infty}.$$

Theorem 15 *The Laplace transform of the periodic function* $f(t)$ *with period* T, *so that* $f(t+T) = f(t)$, *is given by,*

$$\mathscr{L}[f(t)] = \frac{-1}{1-\mathrm{e}^{-sT}}\left[\mathrm{e}^{-st} \sum_{n=0}^{\infty} \frac{f^{(n)}(t)}{s^{n+1}}\right]_0^T.$$

Proof. Writing Equation (1) as,

$$\mathscr{L}[f(t)] = -\left[\mathrm{e}^{-st} \sum_{n=0}^{\infty} \frac{1}{s^{n+1}} \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right]_0^T$$

$$- \left[\mathrm{e}^{-st} \sum_{n=0}^{\infty} \frac{1}{s^{n+1}} \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right]_T^{\infty}$$

Now substituting $t = \eta + T$ in the second infinite series of the above equation so that the limits (T, ∞) changes to $(0, \infty)$ and by having $f^n(\eta + T) = f^n(\eta)$ and after rearranging and evaluating completes the proof.

Example 5 *The full sine-wave rectifier is given by the*

function, $f(t) = E \sin \omega t$, *with the period* $T = \dfrac{\pi}{\omega}$.

Using Theorem 15, *the Laplace transform of the full sine-wave rectifier is calculated by using the entries of column 5 of **Table 1**,*

$$\mathscr{L}[E \sin \omega t]$$

$$= \frac{1+\mathrm{e}^{-\frac{s\pi}{\omega}}}{1-\mathrm{e}^{-\frac{s\pi}{\omega}}}\left[\frac{E\omega}{s^2} - \frac{E\omega^3}{s^4} + \frac{E\omega^5}{s^6} - \frac{E\omega^7}{s^8} + \cdots + (-1)^n \frac{E\omega^{2n+1}}{s^{2(n+1)}}\right]$$

$$= \frac{E\omega}{s^2+\omega^2}\cosh\left(\frac{s\pi}{2\omega}\right).$$

3. Laplace Transforms by Integration by Parts

The Laplace transform of t is calculated by substituting $f(t) = t$ in the Laplace integral transform, now by taking $u = t$ and $\mathrm{d}v = \mathrm{e}^{-st}\mathrm{d}t$ evaluating by parts gives $\dfrac{1}{s^2}$. On the other hand, to calculate the Laplace transform of $\sin t$, we take $u = \sin t$ and $\mathrm{d}v = \mathrm{e}^{-st}\mathrm{d}t$ and after evaluation leads $\dfrac{1}{s^2+1}$. Here we can also take $u = \mathrm{e}^{-st}$ and $\mathrm{d}v = \sin t \mathrm{d}t$ again it gives the same Laplace transform. Hence, in this section, we solve the Laplace integral equation by taking, $u = \mathrm{e}^{-st}$, and $\mathrm{d}v = f(t)\mathrm{d}t$, and integrating by parts. Below, the sub-scripts in say $f_{(n)}(t)$ represents the order of integration n in the variable t, $\left(\underbrace{\int \cdots \int}_{(n) \text{ times}} f(t)(\mathrm{d}t)^n\right)$.

Subject to some constraints we then generally have,

Proposition 3 *The Laplace transform of a Taylor's seriezable trigonometric function,* $f(t)$, *is given by,*

$$\mathscr{L}[f(t)] = \left[\mathrm{e}^{-st} \sum_{n=0}^{\infty} s^n \underbrace{\int \cdots \int}_{(n+1) \text{ times}} f(t)(\mathrm{d}t)^{n+1}\right]_0^{\infty};$$

$$\mathrm{Re}(s) > 0.$$

Proof. Now $u = \mathrm{e}^{-st}$, so that $u^{(0)} = \mathrm{e}^{-st}, u^{(1)} = -s\mathrm{e}^{-st}, \cdots, u^{(n)} = (-1)^n s^n \mathrm{e}^{-st}$,
Next $\mathrm{d}v = f(t)\mathrm{d}t$, leads to

$$v_{(0)} = \int f(t)\mathrm{d}t, v_{(1)} = \iint f(t)(\mathrm{d}t)^2, \cdots, v_{(n)}$$

$$= \underbrace{\int \cdots \int}_{(n+1) \text{ times}} f(t)(\mathrm{d}t)^{n+1}$$

Substituting $u^{(n)}$ and $v_{(n)}$ in the Bernoulli's formula of continuous integration by parts and observing $(-1)^{2n}$ is positive for all $n \geq 0$ gives Proposition 3.

Example 6 *The Laplace transform of* $\cos\sqrt{at}$ *with*

non-negative integer, a, is calculated by simply integrating the function. Now, for $n \geq 0$,

$$f(t) = \cos\sqrt{a}t, \quad f_{(1)}(t) = \frac{\sin\sqrt{a}t}{\sqrt{a}}, \cdots,$$

$$f_{(2n+1)}(t) = \frac{(-1)^n \sin\sqrt{a}t}{a^{\frac{2n+1}{2}}}$$

and, $f_{(2n+2)}(t) = \frac{(-1)^{n+1}\cos\sqrt{a}t}{a^{n+1}}$. *Furthermore, in view of Proposition* 3, *when applying the upper and lower limits in the antiderivatives above, we get.* $f_{(2n+1)}(t) = 0$, *and* $f_{2(n+1)} = \frac{(-1)^{n+1}}{a^{n+1}}$, *whence we get,* (*function and its Laplace transform in **Figure** 4*)

$$\mathscr{L}\left[\cos\sqrt{a}t\right] = -\left[-\frac{s}{a} + \frac{s^3}{a^2} - \cdots + (-1)^{n+1}\frac{s^{2n+1}}{a^{n+1}}\right]$$

$$= \frac{s}{a}\left[1 - \frac{s^2}{a} + \cdots + (-1)^n\frac{s^{2n}}{a^n}\right] = \frac{s}{s^2+a}.$$

We agree that constants and polynomials cannot be Laplace transformed with the Proposition 3, since the continuous integration of constant and polynomials with respect to t does not converge anywhere when $t \to \infty$, and $t \to 0$.

3.1. New Infinite Series Representation for Trig Functions

In the Proposition 3 the limitations of $f(t)$ to be Taylor's seriezable trigonometric function is acceptable only on a theoritical point of view, from the evaluation of Laplace transform of trigonometric functions vice-versa of Definition of Section 2 and Proposition 3. On the other hand, we show under what condition the Proposition 3 exists? Definitely the answer would be by finding the inverse Laplace transform of Proposition 3.

For simplicity's sake, re-writing the Proposition 3, is akin to evaluating the limits and representing, $f_{(n+1)}(t)$, in Proposition 3. The Laplace transform of Taylor's seriezable trigonometric function $f(t)$ is simply defined by,

$$\mathscr{L}\left[f(t)\right] = -\lim_{t\to 0}\sum_{n=0}^{\infty}s^n f_{(n+1)}(t); \;\; \mathrm{Re}(s) > 0.$$

The inverse Laplace transform of Proposition 3 would be same as inverse Laplace transform of the above equation, and hence it is enough to find the inverse Laplace transform of $s^n, n \geq 0$.

For a start up, when $n = 0$ in s^n, the inverse Laplace transform of 1 would be $\delta(t)$ which is Dirac

delta function [2] since, $\mathscr{L}\left[\delta(t)\right] = \int_{0-}^{\infty}\delta(t)\,\mathrm{e}^{-st}\,\mathrm{d}t = 1$.

Again when $n = 1$ in s^n the inverse Laplace transform of s is given by the first derivative of Dirac delta function with respect to t, $\delta^{(1)}(t)$. In particular, readers are invited to consider connected relation to Dirac delta function but in the Sumudu transform context (see Equations (2.19), (2.20), (4.18), and (4.20) in [9]). In general, the inverse Laplace transform of s^n is given by $\delta^{(n)}(t)$, since $\mathscr{L}\left[\delta^{(n)}(t)\right] = \int_{0-}^{\infty}\delta^{(n)}(t)\mathrm{e}^{-st}\mathrm{d}t = s^n$. In all cases upto the *n-th* derivative the initial value theorem is undefined for $\delta(t)$, and $\delta^{(n)}(t)$ leads of course to the study of generalized functions (see [2], and references therein for more details).

We prove the inverse Laplace transform of singular functions that satisfy the Tauberian (initial value) theorem in the following proposition where the trigonometric functions are represented in new infinite series, where coefficients are calculated by integrating the function, [24].

Proposition 4 *The necessary condition for the existence of Proposition* 3 (*and hence the above equation*) *is that, the Taylor seriezable trigonometric function* $f(t) \in \mathbb{R}$ *can be expressed as,*

$$f(t) = -\sum_{n=0}^{\infty}f_{(n+1)}(t)\Big|_{t\to 0}\int_0^{\infty}\frac{\mathrm{d}^n}{\mathrm{d}t^n}J_0\left(2\sqrt{vt}\right)\mathrm{d}v.$$

Proof. Taking inverse Laplace transform of $\mathscr{L}\left[f(t)\right] = -\lim_{t\to 0}\sum_{n=0}^{\infty}s^n f_{(n+1)}(t).$

$$f(t) = -\lim_{t\to 0}\sum_{n=0}^{\infty}\mathscr{L}^{-1}\left(s^n\right)f_{n+1}(t). \tag{15}$$

In [14] the Bilateral Laplace Sumudu Duality (**BLSD**) was established. the inverse Laplace transform of 1 is given by (see Equation (5.10) in [14]),

$$\mathscr{L}\left[\int_0^{\infty}J_0\left(2\sqrt{vt}\right)\mathrm{d}v\right] = \int_0^{\infty}\mathscr{L}\left[J_0\left(2\sqrt{vt}\right)\right]\mathrm{d}v$$

$$= \int_0^{\infty}\frac{1}{s}\mathrm{e}^{-\frac{v}{s}}\mathrm{d}v = 1. \tag{16}$$

Thus $\mathscr{L}^{-1}[1] = \int_0^{\infty}J_0\left(2\sqrt{vt}\right)\mathrm{d}v$, here the $J_0\left(2\sqrt{vt}\right) = \sum_{m=0}^{\infty}\frac{(-1)^m(vt)^m}{(m!)^2}$ is the first kind Bessel's function of order zero. And this particular function will play the major role in the exponential kerneled integral transforms (see Equations (30) through (35) in [20]). And the Laplace transform is taken with respect to t, since v and t are independent, the permissibility of interchange of order of integration is considered in favour. Though the function $\int_0^{\infty}J_0\left(2\sqrt{vt}\right)\mathrm{d}v$ gives no meaning (as be-

comes zero when evaluated) but as per the Laplace (integral) transform point of view this is worth (see Theorem 5.1. Equation (5.8) in [14]). By having,

$$\lim_{t \to 0} J_0\left(2\sqrt{vt}\right) = 1. \qquad (17)$$

The Laplace transform of the first derivative of $J_0\left(2\sqrt{vt}\right)$ with respect to t is $e^{-\frac{v}{s}} - 1$ and with the help of Equations (16) and (17),

$$\mathscr{L}\left[\int_0^\infty \frac{d}{dt} J_0\left(2\sqrt{vt}\right) dv\right]$$
$$= s\mathscr{L}\left[\int_0^\infty J_0\left(2\sqrt{vt}\right) dv\right] - \lim_{t \to 0} \int_0^\infty J_0\left(2\sqrt{vt}\right) dv = s. \qquad (18)$$

Therefore the inverse Laplace transform of s is $\int_0^\infty \frac{d}{dt} J_0\left(2\sqrt{vt}\right) dv$. In general, from the Laplace transform of the n-th derivative of function with respect to t,

$$\mathscr{L}\left[\frac{d^n}{dt^n} J_0\left(2\sqrt{vt}\right)\right]$$
$$= s^n \mathscr{L}\left[J_0\left(2\sqrt{vt}\right)\right] - \sum_{k=0}^{n-1} s^{n-(k+1)} \lim_{t \to 0} \frac{d^k}{dt^k} J_0\left(2\sqrt{vt}\right) \qquad (19)$$
$$= s^{n-1} e^{-\frac{v}{s}} - \sum_{k=0}^{n-1} s^{n-(k+1)} \lim_{t \to 0} \frac{d^k}{dt^k} J_0\left(2\sqrt{vt}\right).$$

But,

$$\lim_{t \to 0} \frac{d^k}{dt^k} J_0\left(2\sqrt{vt}\right); (k \geq 0) = \frac{(-1)^k v^k}{k!}. \qquad (20)$$

Finally from Equations (19) and (20),

$$\mathscr{L}\left[\int_0^\infty \frac{d^n}{dt^n} J_0\left(2\sqrt{vt}\right) dv\right]$$
$$= \int_0^\infty s^{n-1} e^{-\frac{v}{s}} dv - \int_0^\infty \sum_{k=0}^{n-1} \frac{(-1)^k s^{n-(k+1)} v^k}{k!} dv \qquad (21)$$
$$= s^n - \left[\sum_{k=0}^{n-1} \frac{(-1)^k s^{n-(k+1)} v^{k+1}}{k!(k+1)}\right]_0^\infty.$$

Since the second part of right hand side of Equation (21) is zero,

$$\mathscr{L}^{-1}\left(s^n\right) = \int_0^\infty \frac{d^n}{dt^n} J_0\left(2\sqrt{vt}\right) dv; (n \geq 0). \qquad (22)$$

Substituting Equation (22) in Equation (15) for $\mathscr{L}^{-1}\left(s^n\right)$ completes the proof of Proposition 4.

Thus, the function $\int_0^\infty \frac{d^n}{dt^n} J_0\left(2\sqrt{vt}\right) dv$ from the Equation (20) satisfies the initial value theorem (unlike Dirac delta function) which is zero. To concretize ideas, we give the following example.

Example 7 *Consider the function* $\sin^2 t$, *then*

$$f(t) = \sin^2 t, \quad f_{(1)} = \frac{t}{2} - \frac{\sin 2t}{4}, \quad f_{(2)} = \frac{t^2}{2 \cdot 2!} - \frac{\cos 2t}{8},$$

$$f_{(3)} = \frac{t^3}{2 \cdot 3!} + \frac{\sin 2t}{16}, \quad f_{(4)} = \frac{t^4}{2 \cdot 4!} - \frac{\cos 2t}{32}, \cdots,$$

$$f_{(2n+1)}(t) = \frac{t^{2n+1}}{2 \cdot (2n+1)!} + \frac{(-1)^{n+1} \sin 2t}{2^{2(n+1)}}, \text{ and,}$$

$$f_{(2n+2)}(t) = \frac{t^{2(n+1)}}{2 \cdot [2(n+1)]!} + \frac{(-1)^n \cos 2t}{2^{2n+3}}. \text{ Applying Pro-}$$

position 4, as t tends to zero, all the $f_{(2n+1)}(t) \equiv 0$, *and* $f_{(2n+2)}(t) = \frac{(-1)^n}{2^{2n+3}}$, *and since* $\frac{1}{2}$ *is the common factor,* $\forall n \geq 0$, *the function,* $\sin^2 t$, *can be written in the new infinite series as,*

$$\sin^2 t = \frac{1}{2} + \sum_{n=0}^\infty \frac{(-1)^n}{2^{2n+3}} \int_0^\infty \frac{d^{2n+1}}{dt^{2n+1}} J_0\left(2\sqrt{vt}\right) dv.$$

From Equation (22), it is wothy to note that the Laplace transform of the integral of n-th derivative of $J_0\left(2\sqrt{vt}\right)$, with respect to t, in the domain $(0, \infty)$ with respect to v, is simply the s power the order of the derivative of $J_0\left(2\sqrt{vt}\right)$.

Along with that of the function, $\sin^2 t$, in light of this new infinite series Proposition 4, **Table 2** gives all new infinite series expansions of basic trigonometric functions. The extra factor in the infinite series of entries 5, 6, 9, 10, 14, 16, 20 and 21 are common for all $n \geq 0$, while integrating. Furthermore, the following expression is easily derivable from the Bessel's function,

$$\frac{d^n J_0\left(2\sqrt{vt}\right)}{dt^n} = \sum_{m=0}^\infty \frac{(-1)^m v^m t^{m-n}}{m!(m-n)!}.$$

Therefore, the Laplace transform of $\sin^2 t$, can be calculated through,

$$\mathscr{L}\left[\sin^2 t\right]$$
$$= \mathscr{L}\left[\frac{1}{2} + \sum_{n=0}^\infty \frac{(-1)^n}{2^{2n+3}} \int_0^\infty \frac{d^{2n+1}}{dt^{2n+1}} J_0\left(2\sqrt{vt}\right) dv\right]$$
$$= \frac{1}{2s} + \sum_{n=0}^\infty \frac{(-1)^n s^{2n+1}}{2^{2n+3}} = \frac{2}{s(s^2+4)}.$$

Entry 17 of **Table 2** has the following Laplace transform (shown in **Figure 5**),

$$\mathscr{L}\left[\sin^5 t \cos^5 t\right]$$
$$= \sum_{n=0}^\infty \frac{(-1)^n}{512} \left[\frac{5}{2^{2n}} - \frac{5}{6^{2n+1}} + \frac{1}{10^{2n+1}}\right] s^{2n}.$$

3.2. LaplaceTransform Properties in View of Proposition 3

The Laplace transform of multiple shifts functions can readily be derived with the help of Proposition 3. Since the derivation of the various properties are straightforward and similar to the theorems of Section 2.1, we give directly the Laplace transform of shifted functions, based on Proposition 3 in **Table 3** where m and i are non-negative integers.

Example 8 *The Laplace transform of the function,* $t\sin\sqrt{at}$, *is obtained by simply integrating* $\sin\sqrt{at}$,

thus $f(t)=\sin\sqrt{at}$, $f_{(1)}(t)=-\dfrac{\cos\sqrt{at}}{\sqrt{a}}$, $\cdots$,

$$f_{(2n+1)}(t)=\frac{(-1)^{n+1}\cos\sqrt{at}}{\left(\sqrt{a}\right)^{2n+1}},\quad f_{(2n+2)}(t)=\frac{(-1)^{n+1}\sin\sqrt{at}}{\left(a\right)^{n+1}}.$$

When, $m=1$, *in entry 3 of* **Table 3**,

$$\mathscr{L}\left[tf(t)\right]=-\left[e^{-st}\sum_{n=0}^{\infty}ns^{n-1}f_{(n+1)}(t)\right]_{0}^{\infty}.$$

Therefore, when $t\to 0$, and $t\to\infty$, $f_{(2n+2)}(t)\equiv 0$, and $f_{(2n+1)}(t)=\dfrac{(-1)^{n+1}}{\left(\sqrt{a}\right)^{2n+1}};(n\geq 0)$, yielding,

$$\mathscr{L}\left[t\sin\sqrt{at}\right]=\frac{2s}{\left(\sqrt{a}\right)^{3}}-\frac{4s^{3}}{\left(\sqrt{a}\right)^{5}}+\cdots$$

$$+\frac{(-1)^{n}2(n+1)s^{2n+1}}{\left(\sqrt{a}\right)^{2n+3}}=\frac{2s\sqrt{a}}{\left(s^{2}+a\right)^{2}}.$$

Example 9 *The Laplace transform of* $\dfrac{\cos t}{t}$ *is calculated by integrating* $\cos t$*. Now,* $f(t)=\cos t$, $f_{(1)}(t)=\sin t$,

$f_{(2)}(t)=-\cos t,\cdots,f_{(2n+1)}(t)=(-1)^{n}\sin t$, *and,*

$f_{2(n+1)}(t)=(-1)^{n+1}\cos t$*. Now, for,* $t\to 0$, *and,*

$t\to\infty$, $f_{(2n+1)}(t)=0$, *and,* $f_{2(n+1)}(t)=(-1)^{n+1};(n\geq 0)$.

Finally, with $m=1$, *in the entry 4 of* **Table 3**,

$$\mathscr{L}\left[\frac{f(t)}{t}\right]=-\left[e^{-st}\sum_{n=0}^{\infty}\frac{s^{n+1}f_{(n+1)}(t)}{(n+1)}\right]_{0}^{\infty}.$$

So that,

$$\mathscr{L}\left[\frac{\cos t}{t}\right]=-\frac{s^{2}}{2}+\frac{s^{4}}{4}-\frac{s^{6}}{6}+\cdots+\frac{(-1)^{n+1}s^{2(n+1)}}{2(n+1)}$$

$$=-\frac{1}{2}\left[s^{2}-\frac{s^{4}}{2}+\cdots+\frac{(-1)^{n}s^{2(n+1)}}{n+1}\right]=\log\left(\frac{1}{\sqrt{s^{2}+1}}\right).$$

Example 10 *The Laplace transform of,* $\int_{0}^{t}\tau\cos\tau\mathrm{d}\tau$, *is calculated. For,* $f(t)=\cos t$, *after taking limits, we get,* $f_{(2n+1)}(t)=(-1)^{n+1};(n\geq 0)$. *Hence, applying the formula with,* $i=m=1$, *in entry 11 of* **Table 3**,

$$\mathscr{L}\left[\int_{0}^{t}\tau f(\tau)\mathrm{d}\tau\right]=-\left[e^{-st}\sum_{n=0}^{\infty}ns^{n-2}f_{n+1}(t)\right]_{0}^{\infty}.$$

We consequently then have,

$$\mathscr{L}\left[\int_{0}^{t}\tau\cos\tau\mathrm{d}\tau\right]=\sum_{n=0}^{\infty}(-1)^{n+1}(2n+1)s^{2n-1}$$

$$=-\frac{1}{s}+\sum_{n=1}^{\infty}(-1)^{n+1}(2n+1)s^{2n-1}$$

$$=-\frac{1}{s}+\sum_{n=1}^{\infty}(-1)^{n+1}(n+1)s^{2n-1}+\sum_{n=1}^{\infty}(-1)^{n+1}ns^{2n-1}$$

$$=-\frac{1}{s}\left[\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)s^{2n}\right]+s\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)s^{2n}$$

$$=\left(s-\frac{1}{s}\right)\left[1-2s^{2}+3s^{4}-\cdots+(-1)^{n}(n+1)s^{2n}\right]$$

$$=\left(\frac{s^{2}-1}{s}\right)\left[1+s^{2}\right]^{-2}=\frac{s^{2}-1}{s\left(s^{2}+1\right)^{2}}.$$

It is important to note that with respect to the entries 5 and 9 (and entries 6 and 10) of **Table 3**, the proposition 1 Equation (9) (and Equation (10)) holds true. Similarlly with respect to the entries 7 and 11 (and entries 8 and 12) of **Table 3**, the proposition 2 Equation (11) (and Equation (12)) remains the same.

3.3. Concluding Remarks and Future Work

As far as the Section 2 is concerned, when the function is Laplace transformed by differentiation, then the inverse Laplace is automatically an integration process. Having worked with various examples, our proposed methods lead to exact solutions. A remaining open query is that of defining the inverse for the Laplace transform by using similar tools and processes as in Proposition 3. But in view of the concept of Section 2 above, Laplace and inverse Laplace transform are the respective reciprocal processes of differentiation and integration of the function.

If so, then with the Proposition 3, the inverse Laplace transform will be the process of differentiating. For example consider the function $\cos t$, its Laplace transform by the Proposition 3 is given by $\sum_{n=0}^{\infty}(-1)^{n}s^{2n+1}$ which gives $\dfrac{s}{s^{2}+1}$. Hence for finding the original function, when equating the coefficients of identical powers of s with Proposition 3, we get $f_{(2n+1)}(t)=(-1)^{n+1}$. As the sub-scripts denote the order of integration. Now

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by differentiating $f_{(2n+1)}(t) = (-1)^{n+1}, (2n+1)$ times, one should get the infinite series of the function, $\cos t$ as entry 4 of **Table 2**.

As part of some future works in this regard, we aspire to pursing working schemes of this paper, and establishing more comprehensive tables as was done for the Sumudu transform in [10], and for the Natural transform in [17]. With this said, it is perhaps research worthy, in the near future, to put all considered aspects in the framework and applications of the theory of reproducing kernels [25].

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