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Solving *nth*-Order Integro-Differential Equations Using the Combined Laplace Transform-Adomian Decomposition Method

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ABSTRACT

In this paper, the Combined Laplace Transform-Adomian Decomposition Method is used to solve *nth*-order integro-differential equations. The results show that the method is very simple and effective.

Keywords: Integro-Differential Equations; Laplace Transform Method; Adomian Decomposition Method

1. Introduction

In the recent literature there is a growing interest to solve integro-differential equations. The reader is referred to [1-3] for an overview of the recent work in this area. In the beginning of the 1980's, Adomian [4-7] proposed a new and fruitful method (so-called the Adomian decomposition method) for solving linear and nonlinear (algebraic, differential, partial differential, integral, etc.) equations. It has been shown that this method yields a rapid convergence of the solutions series to linear and nonlinear deterministic and stochastic equations. The main objective of this work is to use the Combined Laplace Transform-Adomian Decomposition Method (CLT-ADM) in solving the *nth*-order integro-differential equations.

Let us consider the general functional equation

$$y - Ny = f, \tag{1.1}$$

where N is a nonlinear operator, f is a known function, and we are seeking the solution y satisfying (1.1). We assume that for every f, Equation (1.1) has one and only one solution.

The Adomian's technique consists of approximating the solution of (1.1) as an infinite series

$$y = \sum_{n=0}^{\infty} y_n, \tag{1.2}$$

and decomposing the nonlinear operator N as

$$Ny = \sum_{n=0}^{\infty} A_n, \tag{1.3}$$

where A_n are polynomials (called Adomian polynomials) of $y_0, y_1, \cdots, y_n$ [4-7] given by

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, n = 0,1,2,\cdots.$$

The proofs of the convergence of the series $\sum_{n=0}^{\infty} y_n$ and $\sum_{n=0}^{\infty} A_n$ are given in [6,8-12]. Substituting (1.2) and (1.3) into (1.1) yields

$$\sum_{n=0}^{\infty} y_n - \sum_{n=0}^{\infty} A_n = f.$$

Thus, we can identify

$$y_0 = f,$$
$$y_{n+1} = A_n\left(y_0, y_1, \cdots, y_n\right), n = 0,1,2,\cdots.$$

Thus all components of y can be calculated once the A_n are given. We then define the n-terms approximant to the solution y by $\varnothing_n[y] = \sum_{i=0}^{n-1} y_i$ with $\lim_{n\to\infty}\varnothing_n[y] = y.$

2. General *nth*-Order Integro-Differential Equations

Let us consider the general *nth*-order integro-differential equations of the type [1,2]:

$$y^{(n)}(x)+f(x)y(x)+\int_a^b k(x,t)y^{(m)}(t)\mathrm{d}t=g(x), \tag{2.1}$$
$$a<x<b$$

with initial conditions

$$y(a)=\alpha_0, y'(a)=\alpha_1, y''(a)=\alpha_2,\cdots,y^{(n-1)}(a)=\alpha_{n-1},$$

where $\alpha_i, i=0,1,\cdots,n-1$ are real constants, m and n are integers and $m<n$. In Equation (2.1) the functions $f(x), g(x)$ and the kernel $k(x,t)$ are given real-valued functions, and $y(x)$ is the solution to be determined. We assume that Equation (2.1) has the unique solution.

To solve the general *nth*-order integro-differential Equation (2.1) using, the Laplace transform method, we recall that the Laplace transforms of the derivatives of $y(x)$ are defined by

$$\mathcal{L}\left\{y^{(n)}(x)\right\}(s)=s^n\mathcal{L}\{y(x)\}(s)-s^{n-1}y(0)-s^{n-2}y'^{(0)}-\cdots-y^{(n-1)}(0).$$

Applying the Laplace transform $\mathcal{L}$ to both sides of (2.1) and taking into account the fact that the convolution theorem for Laplace transform [13,14] gives:

$$\begin{aligned}&s^n\mathcal{L}\{y(x)\}(s)-s^{n-1}y(0)-s^{n-2}y'(0)-\cdots-y^{(n-1)}(0)\\&=\mathcal{L}\{g(x)\}(s)-\mathcal{L}\{f(x)*y(x)\}(s)\\&\quad-\mathcal{L}\left\{\int_a^b k(x,t)y^{(m)}(t)\mathrm{d}t\right\}(s)\\&=\mathcal{L}\{g(x)\}(s)-\mathcal{L}\{f(x)\}(s)\mathcal{L}\{y(x)\}(s)\\&\quad-\int_a^b\mathcal{L}\{k(x,t)\}(s)y^{(m)}(t)\mathrm{d}t.\end{aligned}$$

This can be reduced to

$$\begin{aligned}&\mathcal{L}\{y(x)\}(s)\\&=\frac{s^{n-1}y(0)+s^{n-2}y'(0)+\cdots+y^{(n-1)}(0)}{s^n+\mathcal{L}\{f(x)\}(s)}+\frac{\mathcal{L}\{g(x)\}(s)}{s^n+\mathcal{L}\{f(x)\}(s)}\\&\quad-\frac{1}{s^n+\mathcal{L}\{f(x)\}(s)}\int_a^b\mathcal{L}\{k(x,t)\}(s)y^{(m)}(t)\mathrm{d}t.\end{aligned} \tag{2.2}$$

Substituting (1.2) into (2.2) leads to

$$\begin{aligned}&\mathcal{L}\left\{\sum_{n=0}^{\infty}y_n(x)\right\}(s)\\&=\frac{s^{n-1}y(0)+s^{n-2}y'(0)+\cdots+y^{(n-1)}(0)}{s^n+\mathcal{L}\{f(x)\}(s)}+\frac{\mathcal{L}\{g(x)\}(s)}{s^n+\mathcal{L}\{f(x)\}(s)}\\&\quad-\frac{1}{s^n+\mathcal{L}\{f(x)\}(s)}\int_a^b\mathcal{L}\{k(x,t)\}(s)\sum_{n=0}^{\infty}y_n^{(m)}(t)\mathrm{d}t.\end{aligned}$$

The Adomian decomposition method presents the recursive relation

$$\begin{aligned}&\mathcal{L}\{y_0(x)\}(s)\\&=\frac{s^{n-1}y(0)+s^{n-2}y'(0)+\cdots+y^{(n-1)}(0)}{s^n+\mathcal{L}\{f(x)\}(s)}+\frac{\mathcal{L}\{g(x)\}(s)}{s^n+\mathcal{L}\{f(x)\}(s)}\\&\mathcal{L}\{y_{n+1}(x)\}(s)\\&=-\frac{1}{s^n+\mathcal{L}\{f(x)\}(s)}\int_a^b\mathcal{L}\{k(x,t)\}(s)y_n^{(m)}(t)\mathrm{d}t,\\&n=0,1,2,\cdots.\end{aligned} \tag{2.3}$$

A necessary condition for (2.3) to comply is that

$$\lim_{s\to\infty}\frac{1}{s^n+\mathcal{L}\{f(x)\}(s)}=0.$$

Applying the inverse Laplace transform to both sides of the first part of (2.3) gives $y_0(x)$, and using the recursive relation (2.3) gives the components of $y_n(x), n\geq 0$. We then define the n-terms approximant to the solution $y(x)$ by $\phi_n[y(x)]=\sum_{i=0}^{n-1}y_i(x)$ with $\lim_{n\to\infty}\phi_n[y(x)]=y(x)$. In this paper, the obtained series solution converges to the exact solution.

2.1. A Test of Convergence

The convergence of the method is established by Theorem 3.1 in [9]. In fact, on each interval the inequality $\|y_{i+1}\|_2<\alpha\|y_i\|_2$ is required to hold for $i=0,1,\cdots,n$, where $0<\alpha<1$ is a constant and n is the maximum order of the approximant used in the computation. Of course, this is only a necessary condition for convergence, because it would be necessary to compute $\|y_i\|_2$ for every $i=0,1,\cdots,n$ in order to conclude that the series is convergent.

2.2. Definition

Let $\phi_n(x), n=1,2,\cdots$ be the successive approximations to the solution $y(x)$ of a problem. If the positive constants K, p exist such that

$$K=\lim_{n\to\infty}\frac{|\phi_{n+1}(x_i)-y(x_i)|}{|\phi_n(x_i)-y(x_i)|^p},$$

then we call p the (estimated) Local Order of Convergence (EOC) at the point x_i. The constant K is called Convergence Factor at x_i.

3. Applications

In this section, the CLT-ADM for solving *nth*-order integro-differential equations is illustrated in the three examples given below. To show the high accuracy of the solution results from applying the present method to our problem (2.1) compared with the exact solution, the maximum error is defined as:

$$E_n = \left\| y_{\text{Exact}}(x) - \phi_n(x) \right\|_\infty ,$$

where $n = 1, 2, \cdots$ represents the number of iterations. Moreover, we give a comparison among the CLT-ADM, Homotopy perturbation method (HPM) [1] and the variational iteration method (VIM) [2]. The computations associated with the examples were performed using Maple 13 package.

Example 1

Solve the second-order integro-differential equation by using the CLT-ADM [1,2]:

$$\begin{cases} y''(x) = \mathrm{e}^x - x + \int_0^1 xty(t)\,\mathrm{d}t, \\ y(0) = 1, y'(0) = 1 \end{cases} \tag{3.1}$$

As mentioned above, taking Laplace transform of both sides of (3.1) gives

$$\mathcal{L}\{y''(x)\}(s) = \mathcal{L}\{\mathrm{e}^x - x\}(s) + \mathcal{L}\left\{\int_0^1 xty(t)\,\mathrm{d}t\right\}(s)$$

so that

$$s^2Y(s) - sy(0) - y'(0) = \frac{1}{s-1} - \frac{1}{s^2} + \frac{1}{s^2}\int_0^1 ty(t)\,\mathrm{d}t$$

or equivalently

$$Y(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^2(s-1)} + \frac{1}{s^4}\int_0^1 ty(t)\,\mathrm{d}t$$

where $\mathcal{L}\{y(x)\}(s) = Y(s)$. Substituting the series assumption for $Y(s)$ as given above in (1.2), and using the recursive relation (2.3) we obtain

$$\begin{aligned} &Y_0(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^2(s-1)}, \\ &\mathcal{L}\{y_{n+1}(x)\}(s) = \frac{1}{s^4}\int_0^1 ty_n(t)\,\mathrm{d}t, n = 0, 1, 2, \cdots. \end{aligned} \tag{3.2}$$

Taking the inverse Laplace transform of both sides of the first part of (3.2) gives $y_0(x)$, and using the recursive relation (3.2) gives

$$y_0(x) = \mathrm{e}^x - \frac{1}{3!}x^3,$$

$$\vdots$$

$$y_n(x) = \frac{29}{3! \cdot 30^n}x^3, n = 1, 2, \cdots.$$

Thus the series solution is given by

$$\phi_n(x) = \sum_{i=0}^{n-1} y_i(x) = \mathrm{e}^x - \frac{1}{3! \cdot 30^{n-1}}x^3, n = 1, 2, \cdots$$

$$y(x) = \lim_{n\to\infty}\phi_n(x) = \lim_{n\to\infty}\left(\mathrm{e}^x - \frac{1}{3! \cdot 30^{n-1}}x^3\right) = \mathrm{e}^x$$

that converges to the exact solution $y_{\text{Exact}}(x) = \mathrm{e}^x$. In **Table 1**, the maximum errors and the EOC are presented for $x = 0.2(0.2)1$. Comparing it with the HPM and VIM results given in [1,2], we notice that the result obtained by the present method is very superior (lower error combined with less number of iterations) to that obtained by HPM and VIM. From **Table 1**, it can be deduced that, the error decreased monotonically with the increment of the integer n.

Example 2

Solve the third-order integro-differential equation by using the CLT-ADM [1,2]:

$$\begin{cases} y'''(x) = \sin x - x - \int_0^{\pi/2} xty'(t)\,\mathrm{d}t, \\ y(0) = 1, y'(0) = 0, y''(0) = -1 \end{cases} \tag{3.3}$$

As early mentioned, taking Laplace transform of both sides of (3.3) gives

$$\mathcal{L}\{y'''(x)\}(s) = \mathcal{L}\{\sin x - x\}(s) - \mathcal{L}\left\{\int_0^{\pi/2} xty'(t)\,\mathrm{d}t\right\}(s)$$

so that

$$\begin{aligned} &s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\ &= \frac{1}{s^2+1} - \frac{1}{s^2} - \frac{1}{s^2}\int_0^{\pi/2} ty'(t)\,\mathrm{d}t \end{aligned}$$

or equivalently

$$Y(s) = \frac{1}{s} - \frac{1}{s^3} - \frac{1}{s^5} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5}\int_0^{\pi/2} ty'(t)\,\mathrm{d}t$$

where $\mathcal{L}\{y(x)\}(s) = Y(s)$. Substituting the series assumption for $Y(s)$ as given above in (1.2), and using the recursive relation (2.3) we obtain

$$\begin{aligned} &Y_0(s) = \frac{1}{s} - \frac{1}{s^3} - \frac{1}{s^5} + \frac{1}{s^3(s^2+1)}, \\ &\mathcal{L}\{y_{n+1}(x)\}(s) = -\frac{1}{s^5}\int_0^{\pi/2} ty_n'(t)\,\mathrm{d}t, n = 0, 1, 2, \cdots. \end{aligned} \tag{3.4}$$

Table 1. Maximum error and EOC for Example 1.

x	E_3	E_6	E_8	EOC
0.2	1.4815E−06	5.4870E−11	6.0966E−14	0.99999
0.4	1.1852E−05	4.3896E−10	4.8773E−13	1.00000
0.6	4.0000E−05	1.4815E−09	1.6461E−12	0.99999
0.8	9.4815E−05	3.5117E−09	3.9018E−12	1.00000
1.0	1.8519E−04	6.8587E−09	7.6208E−12	1.00000

According to the requirements of our test, $\frac{\|y_{i+1}\|_2}{\|y_i\|_2} < 1$ for all $i = 0, 1, 2, \cdots, n$.

Taking the inverse Laplace transform of both sides of the first part of (3.4) gives $y_0(x)$, and using the recursive relation (3.4) gives

$$y_0(x) = \cos x - \frac{1}{4!}x^4,$$
$$\vdots$$
$$y_n(x) = \frac{\left(\pi^5 + 960\right)\pi^{5(n-1)}}{4!\cdot 960^n}x^4, n = 1,2,\cdots.$$

The series solution is therefore given by

$$\phi_n(x) = \sum_{i=0}^{n-1} y_i(x) = \cos x + \frac{(-1)^n \pi^{5(n-1)}}{4!\cdot 960^{n-1}}x^4, n = 1,2,\cdots$$

$$y(x) = \lim_{n\to\infty}\phi_n(x) = \lim_{n\to\infty}\left(\cos x + \frac{(-1)^n \pi^{5(n-1)}}{4!\cdot 960^{n-1}}\right) = \cos x$$

that converges to the exact solution $y_{\text{Exact}}(x) = \cos x$. In **Table 2**, the maximum errors and the EOC are shown for $x = 0.2(0.2)1$. Comparing it with the HPM and VIM results given in [1,2], we notice that the result obtained by the present method is very superior (lower error combined with less number of iterations) to that obtained by HPM and VIM. From **Table 2**, it can be concluded that, the error decreased monotically with the increment of the integer n.

Example 3

Solve the eighth-order integro-differential equation by using the CLT-ADM [1,2]:

$$\begin{cases} y^{(8)}(x) = -8\mathrm{e}^x + x^2 + y(x) + \int_0^1 x^2 y'(t)\,\mathrm{d}t, \\ y(0) = 1, y'(0) = 0, y''(0) = -1, y'''(0) = -2, \\ y^{(4)}(0) = -3, y^{(5)}(0) = -4, y^{(6)}(0) = -5, y^{(7)}(0) = -6. \end{cases} \quad (3.5)$$

As previously mentioned, taking Laplace transform of both sides of (3.5) gives

$$\mathcal{L}\left\{y^{(8)}(x)\right\}(s) = \mathcal{L}\left\{-8\mathrm{e}^x + x^2 + y(x)\right\}(s) + \mathcal{L}\left\{\int_0^1 x^2 y'(t)\,\mathrm{d}t\right\}(s)$$

Table 2. Maximum error and EOC for Example 2.

x	E_3	E_6	E_8	EOC
0.2	6.7743E−06	2.1943E−07	2.2297E−08	0.99999
0.4	1.0839E−04	3.5109E−06	3.5676E−07	0.99999
0.6	5.4872E−04	1.7774E−05	1.8061E−06	1.00000
0.8	1.7342E−03	5.6175E−05	5.7082E−06	1.00000
1.0	4.2339E−03	1.3714E−04	1.3936E−05	0.99999

According to the requirements of our test, $\frac{\|y_{i+1}\|_2}{\|y_i\|_2} < 1$ for all $i = 0,1,2,\cdots,n$.

so that

$$s^8Y(s) - s^7y(0) - s^6y'(0) - s^5y''(0) - s^4y'''(0) - s^3y^{(4)}(0) - s^2y^{(5)}(0) - sy^{(6)}(0) - y^{(7)}(0) = \frac{-8}{s-1} + \frac{2}{s^3} + Y(s) + \frac{2}{s^3}\int_0^1 y'(t)\,\mathrm{d}t$$

or equivalently

$$Y(s) = \frac{s^7}{s^8-1} - \frac{s^5}{s^8-1} - \frac{2s^4}{s^8-1} - \frac{3s^3}{s^8-1} - \frac{4s^2}{s^8-1} - \frac{5s}{s^8-1} - \frac{6}{s^8-1} + \frac{2}{s^3\left(s^8-1\right)} - \frac{8}{(s-1)\left(s^8-1\right)} + \frac{2}{s^3\left(s^8-1\right)}\int_0^1 y'(t)\,\mathrm{d}t$$

where $\mathcal{L}\{y(x)\}(s) = Y(s)$. Substituting the series assumption for $Y(s)$ as given above in (1.2), and using the recursive relation (2.3) we obtain

$$Y_0(s) = \frac{s^7}{s^8-1} - \frac{s^5}{s^8-1} - \frac{2s^4}{s^8-1} - \frac{3s^3}{s^8-1} - \frac{4s^2}{s^8-1} - \frac{5s}{s^8-1} - \frac{6}{s^8-1} + \frac{2}{s^3\left(s^8-1\right)} - \frac{8}{(s-1)\left(s^8-1\right)} \quad (3.6)$$

$$\mathcal{L}\{y_{n+1}(x)\}(s) = \frac{2}{s^3\left(s^8-1\right)}\int_0^1 y_n'(t)\,\mathrm{d}t, n = 0,1,2,\cdots.$$

Taking the inverse Laplace transform of both sides of the first part of (3.6) gives $y_0(x)$, and using the recursive relation (3.6) gives

$$y_0(x) = \frac{5}{4}\mathrm{e}^x + \frac{1}{4}\mathrm{e}^{-x} - x\mathrm{e}^x - x^2 - \frac{1}{2}\cos x + \sin\left(\frac{\sqrt{2}}{2}x\right)\sinh\left(\frac{\sqrt{2}}{2}x\right),$$

$$y_1(x) = x^2 + \frac{1}{2}\cos x - \frac{1}{2}\cosh x + \sin\left(\frac{\sqrt{2}}{2}x\right)\sinh\left(\frac{\sqrt{2}}{2}x\right),$$

$$y_2(x) = 0.5512\times10^{-6}x^2 + 0.2756\times10^{-6}\cos x - 0.2756\times10^{-6}\cosh x - 0.5512\times10^{-6}\sin\left(\frac{\sqrt{2}}{2}x\right)\sinh\left(\frac{\sqrt{2}}{2}x\right),$$

$$y_3(x) = 0.3038\times10^{-12}x^2 + 0.1519\times10^{-12}\cos x - 0.1519\times10^{-12}\cosh x - 0.3038\times10^{-12}\sin\left(\frac{\sqrt{2}}{2}x\right)\sinh\left(\frac{\sqrt{2}}{2}x\right),$$

$$y_4(x) = 0.1674\times10^{-18}x^2 + 0.8371\times10^{-19}\cos x - 0.8371\times10^{-19}\cosh x - 0.1674\times10^{-18}\sin\left(\frac{\sqrt{2}}{2}x\right)\sinh\left(\frac{\sqrt{2}}{2}x\right),$$

and so on for other components. Consequently, the series solution is given by

Table 3. Maximum error and EOC for Example 3.

x	E_1	E_2	E_3
0.2	5.6437E−14	0.3111E−19	0.99999
0.4	5.7792E−11	0.3185E−16	0.99999
0.6	3.3326E−09	0.1837E−14	0.99999
0.8	5.9179E−08	0.3262E−13	0.99999
1.0	5.5115E−07	0.3038E−12	1.00000

According to the requirements of our test, $\frac{\|y_{i+1}\|_2}{\|y_i\|_2} < 1$ for all $i = 0,1,2,\cdots,n$.

$$y(x) = \lim_{n\to\infty} \phi_n(x)$$
$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} y_i(x) = \frac{5}{4}e^x + \frac{1}{4}e^{-x} - xe^x - \frac{1}{2}\cosh x = (1-x)e^x$$

that converges to the exact solution $y_{\text{Exact}}(x) = (1-x)e^x$. In **Table 3**, the maximum errors and the EOC are given for $x = 0.2(0.2)1$. Comparing it with the VIM results given in [2], we realize that the result obtained by the present method is very superior (lower error combined with less number of iterations) to that obtained by VIM. From **Table 3**, it can be deduced that, the error decreased monotonically with the increment of the integer n.

4. Conclusion

The CLT-ADM has been applied for solving *nth*-order integro-differential equations. Comparison of the results obtained by the present method with that obtained by HPM and VIM reveals that the present method is superior because of the lower error and less number of needed iteration. It has been shown that error is monotonically reduced with the increment of the integer *n*.

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