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Existence and Uniqueness of Solutions to Impulsive Fractional Integro-Differential Equations with Nonlocal Conditions

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ABSTRACT

In this article, by using Schaefer fixed point theorem, we establish sufficient conditions for the existence and uniqueness of solutions for a class of impulsive integro-differential equations with nonlocal conditions involving the Caputo fractional derivative.

Keywords: Caputo Fractional Derivative; Impulses; Nonlocal Conditions; Existence; Uniqueness; Fixed Point

1. Introduction

Fractional differential equations appear naturally in a number of fields such as physics, engineering, biophysics, blood flow phenomena, aerodynamics, electron-analytical chemistry, biology, control theory, etc., An excellent account in the study of fractional differential equations can be found in [1-11] and references therein. Undergoing abrupt changes at certain moment of times like earthquake, harvesting, shock etc, these perturbations can be well-approximated as instantaneous change of state or impulses. Furthermore, these processes are modeled by impulsive differential equations. In 1960, Milman and Myshkis introduced impulsive differential equations in their papers [12]. Based on their work, several monographs have been published by many authors like Semoilenko and Perestyuk [13], Lak-shmikantham *et al*. [14], Bainov and Semoinov [15,16], Bainov and Covachev [17] and Benchohra *et al*. [18]. Impulsive fractional differential equations represent a real framework for mathematical modelling to real world problems. Significant progress has been made in the theory of impulsive fractional differential equations [19-21].

We consider a class of impulsive fractional integro-differential equations with nonlocal conditions of the form

$$
\begin{aligned}
&{}^cD^\alpha y(t) = f\left(t, y(t), \int_0^t h(t,r)\, y(r)\,\mathrm{d}r\right),\\
&t \in J = [0,T], t \neq t_k, k = 1,2,\cdots,m,
\end{aligned} \tag{1.1}
$$

$$
\Delta y(t)\big|_{t=t_k} = I_k\left(y\left(t_k^-\right)\right), \tag{1.2}
$$

$$
y(0) + g\left(y(t)\right) = y_0. \tag{1.3}
$$

Where ${}^cD^\alpha$ is the Caputo fractional derivative, the function $f(t,\cdot,\cdot): J \times R^2 \to R$ is continuous and the function $h(t,r): D \to R, D = \{(t,r) \in J \times J : 0 \le r \le t \le T\}$ is continuous, $h_0 = \max\{h(t,r) : (t,r) \in D\}$;

$$
\begin{aligned}
&I_k : R \to R, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T,\\
&\Delta y(t)\big|_{t=t_k} = y\left(t_k^+\right) - y\left(t_k^-\right),\\
&y\left(t_k^+\right) = \lim_{h\to 0^+} y(t_k + h)
\end{aligned}
$$

and $y\left(t_k^-\right) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at t_k, and $g : PC(J,R) \to R$ is a continuous function, $y_0 \in R$.

Nonlocal conditions were initiated by Byszewski [22] who proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [23,24], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(y(t))$ may be given by

$$
g(y(t)) = \sum_{i=1}^{p} c_i y(\tau_i),
$$

where $c_i, i = 1,2,\cdots,p$ are given constants and

$0<\tau_1<\tau_2<\cdots<\tau_p<T$.

In this article, our aim is to show sufficient conditions for the existence and uniqueness of solutions of solutions to impulsive fractional integro-differential equations with nonlocal conditions.

2. Preliminaries

In this section, we introduce some notations, definitions and preliminary facts which are used throughout this paper. By $C(J,R)$ we denote the Banach space of all continuous functions from J into R with the norm

$$\|y\|=\sup\{|y(t)|:t\in J\}.$$

Definition 2.1 [5,8]: The fractional (arbitrary) order integral of the function $h\in L^1([a,b],R_+)$ of order $\alpha\in R_+=[0,+\infty)$ is defined by

$$I_a^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\,\mathrm{d}s,$$

where Γ is the gamma function, when $a=0, I_a^\alpha h(t)=I^\alpha h(t)$.

Definition 2.2 [5,8]: For a function h given on the interval $[a,b]$, Riemann-Liouville fractional-order derivative of order α of h, is defined by

$$D_a^\alpha h(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_a^t (t-s)^{n-\alpha-1}h(s)\,\mathrm{d}s,$$

here $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α, when $a=0, D_a^\alpha h(t)=D^\alpha h(t)$.

Definition 2.3 [14]: For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of order α of h, is defined by

$${}^cD_a^\alpha h(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)\,\mathrm{d}s,$$

where $n=[\alpha]+1$.

Lemma 2.4 [25]: (Schaefer's fixed point theorem). Let X be a Banach space and $F:X\to X$ be a completely continuous operator. If the set $\mathrm{E}=\{y\in X: y=\lambda F(y), 0<\lambda<1\}$ is bounded, then F has at least a fixed point in *X*.

3. Existence of Solutions

Consider the set of functions

$$\begin{aligned}&PC(J,R)\\&=\{y(t):J\to R; y(t)\in C((t_k,t_{k+1}],R), k=0,1,\cdots,m\\&\quad\text{and there existy } (t_k^-) \text{ and } y(t_k^+), k=1,2,\cdots,m\\&\quad\text{with } y(t_k^-)=y(t_k)\}.\end{aligned}$$

Definition 3.1: A function $y(t)\in PC(J,R)$ whose α-derivative exists on J is said to be a solution of (1.1)-(1.3), if y satisfies the equation

$${}^cD^\alpha y(t)=f\left(t,y(t),\int_0^t h(t,r)y(r)\,\mathrm{d}r\right),$$

on J' and satisfies the conditions

$$\Delta y(t)\big|_{t=t_k}=I_k\left(y(t_k^-)\right), k=1,2,\cdots,m,$$
$$y(0)+g(y(t))=y_0$$

where $J'=[0,T]/\{t_1,t_2,\cdots,t_m\}$.

To prove the existence of solutions to (1.1)-(1.3), we need the following auxiliary lemmas.

Lemma 3.2: Let $\alpha>0$, then the equation

$${}^cD^\alpha h(t)=0$$

has solutions

$$h(t)=c_0+c_1t+\cdots+c_{n-1}t^{n-1},$$
$$(c_i\in R, i=1,2,\cdots,n-1, n=[\alpha]+1).$$

Lemma 3.3: Let $\alpha>0$, then

$$I^\alpha\,{}^cD^\alpha h(t)=h(t)+c_0+c_1t+\cdots+c_{n-1}t^{n-1},$$

for some $c_i\in R, i=1,2,\cdots,n-1, n=[\alpha]+1$.

As a consequence of Lemma 3.2 and Lemma 3.3, we have the following result

Lemma 3.4: Let $0<\alpha<1$, and let $h:J\to R$ be continuous. A function y is a solution of the fractional integral equation

$$y(t)=\begin{cases} y_0-g(y(t))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}h(s)\,\mathrm{d}s,\\ \text{if } t\in[0,t_1],\\ y_0-g(y(t))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{i=1}^k\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h(s)\,\mathrm{d}s\\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^t (t-s)^{\alpha-1}h(s)\,\mathrm{d}s\\ +\displaystyle\sum_{i=1}^k I_i\left(y(t_i^-)\right), \text{if } t\in[t_k,t_{k+1}], (k=1,2,\cdots,m),\end{cases} \tag{3.1}$$

if and only if $y(t)$ is a solution of the fractional nonlocal BVP

$${}^cD^\alpha y(t)=h(t), t\in J', \tag{3.2}$$

$$\Delta y(t)\big|_{t=t_k}=I_k\left(y(t_k^-)\right), k=1,2,\cdots,m, \tag{3.3}$$

$$y(0)+g(y(t))=y_0. \tag{3.4}$$

Proof Assume $y(t)$ satisfies (3.2)-(3.4).
If $t\in[0,t_1]$ then ${}^cD^\alpha y(t)=h(t)$.

Lemma 3.3 implies

$$y(t)=y_0-g(y(t))+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\mathrm{d}s.$$

If $t\in[t_1,t_2]$, by Lemma 3.3, it follows that

$$y(t)=y(t_1^+)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}h(s)\mathrm{d}s$$
$$=\Delta y(t)\big|_{t=t_1}+y(t_1^-)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}h(s)\mathrm{d}s$$
$$=I_1(y(t_1^-))+y_0-g(y)+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}h(s)\mathrm{d}s.$$

If $t\in[t_2,t_3]$, then from Lemma 3.3 we get

$$y(t)=y(t_2^+)+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t(t-s)^{\alpha-1}h(s)\mathrm{d}s$$
$$=\Delta y\big|_{t=t_2}+y(t_2^-)+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t(t-s)^{\alpha-1}h(s)\mathrm{d}s$$
$$=I_2(y(t_2^-))+I_1(y(t_1^-))+y_0-g(y(t))+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}h(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t(t-s)^{\alpha-1}h(s)\mathrm{d}s.$$

If $t\in[t_k,t_{k+1}]$, then again from $t\in[t_2,t_3]$ we have (3.1).

Conversely, assume that y satisfies the impulsive fractional integral equation (3.1). If $t\in[0,t_1]$, then $y(0)+g(y(t))=y_0$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we get ${}^cD^\alpha y(t)=h(t)$.

If $t\in[t_k,t_{k+1}],k=1,2,\cdots,m$ and using the fact that ${}^cD^\alpha C=0$, where C is a constant, we conclude that ${}^cD^\alpha y(t)=h(t)$.

Also, we can easily show that

$$\Delta y\big|_{t=t_k}=I_k(y(t_k^-)),k=1,2,\cdots,m.$$

Theorem: Assume that:

(H_1) There exists a constant $M>0$ such that $|f(t,u,v)|\le M$ for each $t\in J$ and each $u,v\in R$;

(H_2) There exists a constant $l_k>0$ such that $|I_k(u)|\le l_k$, for each $u\in R$ and $k=1,2,\cdots,m$;

(H_3) There exists a constant $l>0$ such that $|g(u)|\le l$, for each $u\in PC(J,R)$, then the problem (1.1)-(1.3) has at least one solution on J.

Proof Consider the operator $F:PC(J,R)\to PC(J,R)$ defined by

$$F(y(t))=\begin{cases}y_0-g(y(t))+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\mathrm{d}s, \text{ if } t\in[0,t_1],\\ y_0-g(y(t))+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\mathrm{d}s\\ \quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\mathrm{d}s+\sum_{i=1}^k I_i(y(t_i^-)), \text{if } t\in[t_k,t_{k+1}],(k=1,2,\cdots,m),\end{cases}$$

Clearly, the fixed points of the operator F are solution of the problem (1.1)-(1.3).

We shall use Schaefer's fixed point theorem to prove that F has a fixed point. The proof will be given in several steps.

Step 1: F is continuous.

Let $\{y_n(t)\}$ be a sequence such that $y_n\to y$ in $PC(J,R)$. Then for each

$$t\in J_0=[0,t_1],\left|F(y_n(t))-F(y(t))\right|$$
$$\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f\left(s,y_n(s),\int_0^s h(s,r)y_n(r)\mathrm{d}r\right)-f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s.$$

Since f is continuous function, we have $|F(y_n(t))-F(y(t))|\to 0,$ as $n\to\infty$.

For each $t\in J_k=[t_k,t_{k+1}]$,

$$\left|F(y_n(t))-F(y(t))\right|\le\frac{1}{\Gamma(\alpha)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\times\left|f\left(s,y_n(s),\int_0^s h(s,r)y_n(r)\mathrm{d}r\right)-f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t_i-s)^{\alpha-1}\times\left|f\left(s,y_n(s),\int_0^s h(s,r)y_n(r)\mathrm{d}r\right)-f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s+\sum_{i=1}^k\left|I_i(y_n(t_i^-))-I_i(y(t_i^-))\right|.$$

Since f and $I_i, i=1,2,\cdots,m$ are continuous functions, we have $\left|F\left(y_n(t)\right)-F\left(y(t)\right)\right|\to 0,$ as $n\to\infty$.

Therefore, F is continuous.

Step 2: F maps bounded sets into bounded sets in $PC(J,R)$.

Indeed, it is enough to show that for any $\eta^*>0$, there exists a positive constant ℓ such that for each $y\in B_{\eta^*}=\left\{y\in PC(J,R):\|y\|_\infty\le\eta^*\right\}$, we have $\|F(y)\|\le\ell$. By (H$_1$), (H$_2$) and (H$_3$), for each $t\in[0,t_1]$, we have

$$\left|F\left(y(t)\right)\right|\le|y_0|+\left|g\left(y(t)\right)\right|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s$$
$$\le|y_0|+l+\frac{M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathrm{d}s\le|y_0|+l+\frac{M}{\Gamma(\alpha+1)}t^\alpha$$
$$\le|y_0|+l+\frac{MT^\alpha}{\Gamma(\alpha+1)}.$$

For $t\in[t_k,t_{k+1}],(k=1,2,\cdots,m)$, we have

$$\left|F\left(y(t)\right)\right|\le|y_0|+\left|g\left(y(t)\right)\right|+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s+\sum_{i=1}^{k}\left|I_i\left(y\left(t_i^-\right)\right)\right|$$
$$\le|y_0|+l+\frac{M}{\Gamma(\alpha+1)}\sum_{i=1}^{k}t_i^\alpha+\frac{M}{\Gamma(\alpha+1)}t^\alpha+\sum_{i=1}^{k}l_i\le|y_0|+l+\frac{(k+1)MT^\alpha}{\Gamma(\alpha+1)}+\sum_{i=1}^{k}l_i.$$

Let

$$\ell=\max\left\{|y_0|+l+\frac{MT^\alpha}{\Gamma(\alpha+1)},|y_0|+l+\frac{(k+1)MT^\alpha}{\Gamma(\alpha+1)}+\sum_{i=1}^{k}l_i\right\},$$
$k=1,2,\cdots,m,$

then $\left\|F\left(y(t)\right)\right\|\le\ell.$

Step 3: F maps bounded sets into equicontinuous sets of $PC(J,R)$.

Let $\tau_1,\tau_2\in J,\tau_1<\tau_2$, B_{η^*} be a bounded set of $PC(J,R)$ as in Step 2, and let $y\in B_{\eta^*}$. For $\tau_1,\tau_2\in[0,t_1]$, we have

$$\left|F\left(y(\tau_2)\right)-F\left(y(\tau_1)\right)\right|$$
$$=\frac{1}{\Gamma(\alpha)}\left|\int_0^{\tau_2}(\tau_2-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\mathrm{d}s-\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\mathrm{d}s\right|$$
$$\le\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}\left|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right|\times\left|f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left|(\tau_2-s)^{\alpha-1}\right|\times\left|f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s$$
$$\le\frac{M}{\Gamma(\alpha)}\int_0^{\tau_2}\left|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right|\mathrm{d}s+\frac{M}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left|(\tau_2-s)^{\alpha-1}\right|\mathrm{d}s\le\frac{M}{\Gamma(\alpha+1)}\left|2(\tau_2-\tau_1)^\alpha+\tau_2^\alpha-\tau_1^\alpha\right|\mathrm{d}s.$$

For $\tau_1,\tau_2\in[t_k,t_{k+1}],(k=1,2,\cdots,m)$, we have

$$\left|F\left(y(\tau_2)\right)-F\left(y(\tau_1)\right)\right|\le+\sum_{0<t_k<\tau_2-\tau_1}\left|I_k\left(y\left(t_k^-\right)\right)\right|$$
$$+\frac{1}{\Gamma(\alpha)}\left|\int_0^{\tau_2}(\tau_2-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\mathrm{d}s-\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\mathrm{d}s\right|$$
$$\le\sum_{0<t_k<\tau_2-\tau_1}\left|I_k\left(y\left(t_k^-\right)\right)\right|+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}\left|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right|\times\left|f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left|(\tau_2-s)^{\alpha-1}\right|\times\left|f\left(s,y(s),\int_0^s h(s,r)y(r)\mathrm{d}r\right)\right|\mathrm{d}s$$
$$\le\sum_{0<t_k<\tau_2-\tau_1}\left|I_k\left(y\left(t_k^-\right)\right)\right|+\frac{M}{\Gamma(\alpha)}\int_0^{\tau_2}\left|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right|\mathrm{d}s+\frac{M}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left|(\tau_2-s)^{\alpha-1}\right|\mathrm{d}s$$
$$\le\sum_{0<t_k<\tau_2-\tau_1}\left|I_k\left(y\left(t_k^-\right)\right)\right|+\frac{M}{\Gamma(\alpha+1)}\left|2(\tau_2-\tau_1)^\alpha+\tau_2^\alpha-\tau_1^\alpha\right|\mathrm{d}s.$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzel'a-Ascoli theorem, we can conclude that $F: PC(J,R) \to PC(J,R)$ is completely continuous.

As a consequence of Lemma 2.4 (Schaefer's fixed point theorem), we deduce that F has a fixed point which is a solution of the problem (1.1)-(1.3).

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