

# Strongly Balanced 4-Kite Designs Nested into *OQ*-Systems

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## ABSTRACT

In this paper we determine the spectrum for *octagon quadrangle systems* [*OQS*] which can be partitioned into two *strongly balanced 4-kite designs*.

**Keywords:** Graphs; Designs; 4-Kite

## 1. Introduction

Let  $K = (X, \mathcal{E})$  be a graph defined on the vertex set  $X$ . Let  $G$  be a subgraph of  $K$ . A *G-decomposition* of  $K$  is a pair  $\Sigma = (X, \mathcal{B})$ , where  $\mathcal{B}$  is a partition of the edge set of  $K$  into subsets isomorphic to  $G$ . If  $K_v$  is the complete undirected graph defined on the vertex set  $X$ , a *G-decomposition*  $\Sigma = (X, \mathcal{B})$  of  $K_v$  is called a *G-design* of order  $v$  and the classes of the partition  $\mathcal{B}$  are said to be the blocks of  $\Sigma$ . A *G-design* is called *balanced* if for each vertex  $x \in X$ , the number of blocks of  $\Sigma$  containing  $x$  is a constant. Observe that if  $G$  is a regular graph then a *G-design* is always balanced, hence the notion of a balanced *G-design* becomes meaningful only for a non-regular graph  $G$ .

Let  $G$  be a graph and let  $A_1, A_2, \dots, A_h$  be the orbits of the automorphism group of  $G$  on its vertex-set. Let  $\Sigma = (X, \mathcal{B})$  be a *G-design*. We define the degree  $d_{A_i}(x)$  of a vertex  $x \in X$  as the number of blocks of  $\Sigma$  containing  $x$  as an element of  $A_i$ . We say that  $\Sigma = (X, \mathcal{B})$  is a *strongly balanced G-design* if, for every  $i = 1, 2, \dots, h$ , there exists a constant  $C_i$  such that  $d_{A_i}(x) = C_i$ , for every  $x \in X$  [1-3]. Clearly, since for each vertex  $x \in V$  the

relation  $d(x) = \sum_{i=1}^h d_{A_i}(x)$  holds, we have that “a *strongly balanced G-design* is always a *balanced G-design*”. We say that a *G-design* is *simply balanced* if it is balanced, but not strongly balanced.

A cycle of length 4 with a pendant edge, *i.e.* the graph  $C_4 + e$ , is called a *4-kite* and is denoted by  $(a, b, c, d) - e$  or  $(c, b, a, d) - e$ , where  $\{a, b\}$ ,  $\{b, c\}$ ,  $\{c, d\}$ ,  $\{d, a\}$ ,  $\{d, e\}$  are the edges of the 4-kite. Note that a cycle of length 3 with a pendant edge is called a 3-kite or just a

kite. In the case when  $G$  is a 4-kite, a *G-design* is called a *4-kite-design* or also a *4-kite-system*. It is known that a 4-kite design of order  $v$  exists when  $v \equiv 0$  or  $1 \pmod{5}$ . Further research on 4-kite designs can be found in [2]. We will call the vertices  $a$  and  $c$  of the 4-kite  $(a, b, c, d) - e$  the *lateral vertices*,  $b$  the *middle vertex*,  $d$  the *center vertex* and  $e$  the *terminal vertex* [4-6]. Some balanced *G-designs*, when  $G$  is a path, have been studied in [1,3]. Strongly balanced *G-designs* were first introduced in [1], in which the spectrum of simply balanced and strongly balanced  $P_5$  and  $P_6$ -designs have been determined, where  $P_k$  denotes a path with  $k$  vertices.

An *octagon quadrangle* is a graph denoted by  $[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$  and formed by an 8-cycle  $(x_1, x_1, x_2, \dots, x_8)$  with the two additional edges  $\{x_1, x_4\}$  and  $\{x_5, x_8\}$ . An *octagon quadrangle system* [*OQS*] is a *G-design*, where  $G$  is an octagon quadrangle. *OQS*( $v$ )s have been defined and studied in [4,7-9]. In these papers, the main idea was to follow the research about *hexagon triangle systems* and all the others already introduced in the literature, where we can find many authors who have studied in many ways *polygon triangle systems* using triangulations of polygons [5,10,11]. With the study of octagon quadrangles the authors have considered quadrangulations of polygons with new ideas for the research [12,13].

In what follows, if  $\Sigma = (X, \mathcal{B})$  is an *OQS*( $v$ ) in which the family of all  $K_1(Q)$  contained in the blocks of  $\mathcal{B}$  forms a 4-kite design  $\mathcal{K} = (X, \mathcal{C})$ , we will say that  $\Sigma$  is *nesting*  $\mathcal{K}$  or also that  $\mathcal{K}$  is *nested* in  $\Sigma$ . Similar problems, including colorings, can be found also in [14,15].

In this paper, starting from the remark that an *octagon*

quadrangle  $Q = [(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$ , can be partitioned into two 4-kites

$$K_1(Q) = (x_1, x_2, x_3, x_4) - (x_5),$$

$$K_2(Q) = (x_5, x_6, x_7, x_8) - (x_1),$$

the authors study OQSs which can be partitioned into two strongly balanced  $(C_4 + e)$ -designs, determining their spectrum.

### 2. Necessary Existence Conditions

If  $\Sigma = (X, \mathcal{B})$  is strongly balanced 4-kite design, its vertices describe four orbits in the automorphism group of a block, which is a graph  $C_4 + e$ . We will indicate by  $C$  the number of blocks containing any vertex as a center of the 4-kite block, by  $T$  the number of blocks containing any vertex as a terminal, by  $L$  and  $M$  the number of blocks containing any vertex as lateral or median, respectively.

In this section, we determine necessary conditions for the existence of strongly balanced 4-kite designs (order  $v$ , index  $\mu$ ) and for the existence of OQS (order  $v$ , index  $\lambda$ ) nesting strongly balanced 4-kite designs. These conditions are preliminary for conclusive Theorems of Section 3.

**Theorem 2.1.** *If  $\Sigma = (X, \mathcal{B})$  is a strongly balanced 4-kite design of order  $v$  and index  $\mu$ , then:*

- 1)  $C = T = M = \mu \cdot \frac{v-1}{10}$ ;
- 2)  $L = \mu \cdot \frac{v-1}{5}$ ;
- 3)  $\mu \cdot (v-1) \equiv 0, \text{mod} 10$ .

**Proof.** If  $\Sigma = (X, \mathcal{B})$  is a strongly balanced 4-kite design of order  $v$  and index  $\mu$ , following the terminology described above and considering that each vertex occupies  $C$  times the central position in the blocks, necessarily:

$$|\mathcal{B}| = v \cdot C,$$

from which  $C = \mu \cdot \frac{v-1}{10}$ .

$$B_1 = [(0), h, 4h+1, (h+1), (6h+1), 8h+1, 7h+1, (2h+1)],$$

$$B_2 = [(0), h-1, 4h+1, (h+2), (6h+1), 8h, 7h+1, (2h+2)],$$

$$B_3 = [(0), h-2, 4h+1, (h+3), (6h+1), 8h-1, 7h+1, (2h+3)],$$

$$\vdots$$

$$B_i = [(0), h-i-1, 4h+1, (h+i), (6h+1), 8h-(i-2), 7h+1, (2h+i)],$$

$$\vdots$$

$$B_{h-1} = [(0), 2, 4h+1, (2h-1), (6h+1), 7h+3, 7h+1, (3h-1)],$$

$$B_h = [(0), 1, 4h+1, (2h), (6h+1), 7h+2, 7h+1, (3h)].$$

The same considerations can be done to calculate the parameters  $T, M$ , which have the same value of  $C$ . For the last parameter  $L$ , we can consider that:

$$3C + T + 2M + 2L = \mu \cdot (v-1),$$

hence

$$L = \mu \cdot \frac{v-1}{5}.$$

Thus, 1) and 2) are verified and from them 3) holds.  $\square$

**Theorem 2.2.** *Let  $\Sigma = (X, \mathcal{B})$  be an OQS of order  $v$  and index  $\lambda$ , nesting a strongly balanced 4-kite design  $\mathcal{K} = (X, \mathcal{C})$  of index  $\mu$ . Then:*

- 1)  $\lambda = 2 \cdot \mu$ ;
- 2) if  $\mu = 1$  and  $\lambda = 2$ , then  $v \equiv 1, \text{mod} 10$ .

**Proof.** 1) Since:

$$|\mathcal{B}| = \frac{\lambda \cdot v \cdot (v-1)}{20}, |\mathcal{C}| = \frac{\mu \cdot v \cdot (v-1)}{10}$$

and necessarily  $|\mathcal{B}| = |\mathcal{C}|$ , it follows  $\lambda = 2 \cdot \mu$ . 2) From 3) of Theorem 2.1, if  $\mu = 1$  then  $v \equiv 1, \text{mod} 10$ .  $\square$

### 3. Main Existence Theorems

In what follows, if  $B = [(a), b, c, (d), (\alpha), \beta, \gamma, (\delta)]$  is a block of an OQ-system  $\Sigma$  defined in  $Z_v$ , then the translates of  $B$  are all the blocks of type

$$B_j = [(a+j), b+j, c+j, (d+j), (\alpha+j), \beta+j, \gamma+j, (\delta+j)],$$

for every  $j \in Z_v$ .  $B$  is called a base block of  $\Sigma$ .

**Theorem 3.1.** *There exists an OQS, of order  $v$  and index two, nesting a strongly balanced 4-kite design of index one if and only if:*

$$v \equiv 1, \text{mod} 10, v \geq 11.$$

**Proof.**  $\Rightarrow$  Let  $\Sigma = (Z_v, \mathcal{B})$  be an OQS of order  $v$  and index  $\lambda$ , nesting a 4-kite design  $\mathcal{K} = (Z_v, \mathcal{C})$  of order  $v$  and index one. From Theorem 2.2 it is  $\lambda = 2$  and

$$v \equiv 1, \text{mod} 10, v \geq 11.$$

$\Leftarrow$  Consider the following octagon quadrangles:

Consider the system  $\Sigma = (X, \mathcal{B})$ , defined in  $X = Z_\nu$ , having  $B_1, \dots, B_i, \dots, B_h$  as *base blocks*. This means that the blocks  $B_1, B_2, \dots, B_i, \dots, B_h$  belong to  $\mathcal{B}$  and with all their *translates*.

Observe that, for  $h=1$ , the correspondent system defined in  $Z_{11}$  has for blocks all the translates of the following base block:

$$B = [(0), 1, 5, (2), (7), 9, 8, (3)].$$

In every case, it is possible to verify that  $\Sigma$  is an *OQS* of order  $\nu = 10h + 1$  and index  $\lambda = 2$ . Further, if we partition every block

$$Q = [(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)],$$

into the two 4-kites:

$$K_1(Q) = (x_1, x_2, x_3, x_4) - (x_5),$$

$$K_2(Q) = (x_5, x_6, x_7, x_8) - (x_1),$$

we can verify that the collection of all the upper 4-kites forms a 4-kite-design  $\mathcal{K}_1 = (Z_\nu, \mathcal{C}_1)$  of index one. Observe also that the collection of all the lower 4-kites forms a 4-kite-design  $\mathcal{K}_2 = (Z_\nu, \mathcal{C}_2)$  of index one.

We can verify that both the systems  $\mathcal{K}_1, \mathcal{K}_2$  are strongly balanced. In fact, for them it is  $C = T = M = h$ , and  $L = 2h$ .

To verify this, it is enough to consider that the system  $\Sigma$  is constructed by *base blocks* and *difference method*.

This proves that  $\Sigma$  is an *OQS* of order  $\nu = 10h + 1$ ,  $h \geq 1$ , where the two 4-kites designs nested in it have both index one and are both strongly balanced.  $\square$

At last, it follows a result about the existence of strongly balanced 4-kite designs, whose spectrum is unknown.

**Theorem 3.2.** *For every  $\nu \equiv 1, \text{mod } 10, \nu \geq 11$ , there exists strongly balanced 4-kite designs of index one.*

#### 4. Conclusive Remarks and Problems

Theorem 3.1 gives completely the spectrum of *OQS* which can be partitioned into two strongly balanced 4-kite designs. For a given  $\nu$  belongs to the spectrum, Theorem 3.1 gives also the method to construct an *OQS* of order  $\nu$  with the said properties.

For example, if  $\nu = 21$ , the translates of the two base blocks:

$$B_1 = [(0), 2, 9, (3), (13), 17, 15, (7)],$$

$$B_2 = [(0), 1, 9, (4), (13), 16, 15, (6)],$$

constructed on  $Z_{21}$ , define an *OQS* of order  $\nu = 21$  and index  $\lambda = 2$ . We can observe that  $B_1$  can be partitioned into the two 4-kites

$$K_{1,1}(Q) = (0, 2, 9, 3) - 13, K_{1,2}(Q) = (13, 17, 15, 7) - 0,$$

and  $B_2$  into the two 4-kites

$$K_{2,1}(Q) = (0, 1, 9, 4) - 13,$$

$$K_{2,2}(Q) = (13, 16, 15, 6) - 0.$$

We can verify that the translates of  $K_{1,1}$  and  $K_{2,1}$  define a strongly balanced 4-kite designs of order  $\nu = 21$  and index  $\mu = 1$ . Further, a system of the same type and parameters is defined by the translates of  $K_{1,2}$  and  $K_{2,2}$ .

The Theorem 3.1 permits also to find values of  $\nu$  for which there exist strongly balanced 4-kite designs, whose spectrum is still unknown. Thus, the statement of Theorem 3.2 can be the starting point for its determination.

In conclusion, we observe that from Theorem 3.1 follows the more general:

**Theorem 4.1.** *There exists an OQS, of order  $\nu$  and index  $\lambda$ , nesting a strongly balanced 4-kite design of order  $\nu$  and of index  $\mu = \lambda/2$ , if and only if:*

$$\nu \equiv 1, \text{mod } 10, \nu \geq 11.$$

**Proof.** From Theorem 2.2, it is necessarily  $\lambda = 2\mu$ . So, from Theorem 3.1, by a repetition of blocks, the statement follows.  $\square$

We can also point out that, after the determination of the spectrum, found in this paper, it is possible to study other problems about octagon quadrangle systems. It is possible to study the *intersection problem* among them, about which there exist an important literature, following the technique introduced in [16,17]. Also, it should be interesting to examine the *conjecture of Berge* for linear hypergraphs, in the case in which these are *OQSs*, following the ideas seen in [18,19].

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