

An Adaptive Least-Squares Mixed Finite Element Method for Fourth Order Parabolic Problems

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ABSTRACT

A least-squares mixed finite element (LSMFE) method for the numerical solution of fourth order parabolic problems analyzed and developed in this paper. The Ciarlet-Raviart mixed finite element space is used to approximate. The *a posteriori* error estimator which is needed in the adaptive refinement algorithm is proposed. The local evaluation of the least-squares functional serves as a *a posteriori* error estimator. The posteriori errors are effectively estimated. The convergence of the adaptive least-squares mixed finite element method is proved.

Keywords: Adaptive Method; Least-Squares Mixed Finite Element Method; Fourth Order Parabolic Problems; Least-Squares Functional; *A Posteriori* Error

1. Introduction

A general theory of the least-squares method has been developed by A. K. Aziz, R. B. Kellogg and A. B. Stephens in [1]. The most important advantage leads to a symmetric positive definite problem. In the least-squares mixed finite element approach, a least-squares residual minimization is introduced. This method has an advantage which is not subject to the LBB [1] condition. The mixed finite element methods of least-squares type have been the object of many studies recently (see, e.g. Stokes Equation [2], Elliptic Problem [3], Newtonian Fluid Flow Problem [4], Transmission Problems [5], Sobolev Equations [6], Parabolic Problems [7] *et al.*). The adaptive least-squares mixed finite element method have been studied in recent several years (see, e.g. the linear elasticity [8]), but the research of adaptive method about fourth order parabolic problems is not common.

Adaptive methods are now widely used in the scientific computation. In this paper, we are interested in the adaptive least-squares mixed finite element method for fourth order parabolic problems, fourth order parabolic problems are fundamental partial differential equations. It occurs in various areas of applied mathematics and science. Our emphasis in this paper is on the performance of an adaptive refinement strategy based on the *a*

posteriori error estimator inherent in the least-squares formulation by the local evaluation of the functional. During the last 15 - 20 years a big amount of work has been devoted to a *a posteriori* error estimation problem, *i.e.*, computing reliable bounds on the error of given numerical approximation to the solution of partial differential equations using only numerical solution and the given data. In order to operate the *a posteriori* error estimator should be neither under nor overestimate the error. The *a posteriori* error is effectively estimated, and proved the convergence of the adaptive least-squares mixed finite element method in this paper.

An outline of the paper is as follows. The least-squares formulation of fourth order parabolic problems is described in Section 2. It includes continuous and coercivity properties of the least-squares variational formulation. Appropriate spaces for the finite element approximation and a generalization of the coercivity shown in Section 2 to the discrete form is discussed in Section 3. In Section 4, *a posteriori* error estimators which are needed in an adaptive refinement algorithm are composed with the least-squares functional, and posteriori errors are effectively estimated. The convergence of the adaptive least-squares mixed finite element method is shown in Section 5. Finally, we summarize our findings and present conclusions in Section 6. In this paper, we define C to be a generic positive constant.

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2. A Least-Squares Formulation of Fourth Order Parabolic Problems

We start from the equations of fourth order parabolic problems in the form:

$$u_t - \Delta u + \Delta^2 u = 0, \text{ in } \Omega \times (0, T) \tag{1}$$

$$u = 0, \text{ on } \partial\Omega \times (0, T) \tag{2}$$

$$\frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega \times (0, T) \tag{3}$$

$$u(x, 0) = u_0(x), x \in \Omega \tag{4}$$

where $\Omega \subset R^n$ is a bounded domain, with boundary $\partial\Omega, \forall T > 0$. We shall consider an adaptive least-squares mixed finite element method for (1)-(4).

Now we set $\Delta u = \sigma$, then, we have:

$$u_t - \sigma + \Delta \sigma = 0, \text{ in } \Omega \times (0, T) \tag{5}$$

$$\Delta u - \sigma = 0, \text{ in } \Omega \times (0, T) \tag{6}$$

$$u = 0, \text{ on } \partial\Omega \times (0, T) \tag{7}$$

$$\frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega \times (0, T) \tag{8}$$

$$u(x, 0) = u_0(x), x \in \Omega \tag{9}$$

We introduce the Sobolev spaces:

$$H^1(\Omega) = \{p \in L^2(\Omega) : \nabla p \in L^2(\Omega)\},$$

$$H(\Omega) = H_0^1(\Omega),$$

$$H_0^m(\Omega) = \{v \in H^m(\Omega) : D^\alpha v|_{\partial\Omega} = 0, |\alpha| < m\}.$$

Now, let us define the least-squares problem: find $(\sigma, u) \in H^1(\Omega) \times H(\Omega)$ [9] such that

$$J(\sigma, u) = \inf_{q \in H^1(\Omega), v \in H(\Omega)} J(q, v), \tag{10}$$

where

$$J(q, v) = (v_t - q + \Delta q, v_t - q + \Delta q)_{0,\Omega} + (\Delta v - q, \Delta v - q)_{0,\Omega}. \tag{11}$$

We introduce the least-squares functional:

$$F(\sigma, u) = \|u_t - \sigma + \Delta \sigma\|_{0,\Omega}^2 + \|\Delta u - \sigma\|_{0,\Omega}^2.$$

Taking variations in (10) with respect to q and v , the weak statement becomes: find $(\sigma, u) \in H^1(\Omega) \times H(\Omega)$ such that

$$B(\sigma, u; q, v) = 0, (\forall v \in H(\Omega), \forall q \in H^1(\Omega)) \tag{12}$$

where

$$B(\sigma, u; q, v) = (u_t - \sigma + \Delta \sigma, v_t - q + \Delta q)_{0,\Omega} + (\Delta u - \sigma, \Delta v - q)_{0,\Omega}. \tag{13}$$

Theorem 2.1. The bilinear form $B(\cdot, \cdot; \cdot, \cdot)$ is continuous and coercive. In other words, there exist positive constants α and β , such that

$$B(\sigma, u; q, v) \leq \beta \left(\|u_t\|_{0,\Omega}^2 + \|\sigma\|_{0,\Omega}^2 + \|\Delta \sigma\|_{0,\Omega}^2 + \|\Delta u\|_{0,\Omega}^2 \right)^{\frac{1}{2}} \left(\|v_t\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2 + \|\Delta q\|_{0,\Omega}^2 + \|\Delta v\|_{0,\Omega}^2 \right)^{\frac{1}{2}},$$

$$B(q, v; q, v) \geq \alpha \left(\|v_t\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2 + \|\Delta q\|_{0,\Omega}^2 + \|\Delta v\|_{0,\Omega}^2 \right),$$

holds for all $(\sigma, u), (q, v) \in H^1(\Omega) \times H(\Omega)$.

Proof: 1) For the upper bound we have:

$$\begin{aligned} B(q, v; q, v) &= (v_t - q + \Delta q, v_t - q + \Delta q)_{0,\Omega} + (\Delta v - q, \Delta v - q)_{0,\Omega} = \|v_t - q + \Delta q\|_{0,\Omega}^2 + \|\Delta v - q\|_{0,\Omega}^2 \\ &\leq C \left(\|v_t\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2 + \|\Delta q\|_{0,\Omega}^2 + \|\Delta v\|_{0,\Omega}^2 \right). \end{aligned}$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 2.1.

2) For the lower bound.

$$\begin{aligned} B(q, v; q, v) &= (v_t - q + \Delta q, v_t - q + \Delta q)_{0,\Omega} + (\Delta v - q, \Delta v - q)_{0,\Omega} \\ &= (v_t, v_t)_{0,\Omega} + (q, q)_{0,\Omega} + (\Delta q, \Delta q)_{0,\Omega} + 2(v_t, \Delta q)_{0,\Omega} - 2(v_t, q)_{0,\Omega} - 2(q, \Delta q)_{0,\Omega} + (q, q)_{0,\Omega} + (\Delta v, \Delta v)_{0,\Omega} - 2(q, \Delta v)_{0,\Omega} \\ &= (v_t, v_t)_{0,\Omega} + (q, q)_{0,\Omega} + (\Delta q, \Delta q)_{0,\Omega} + 2(v_t, \Delta q)_{0,\Omega} + (q, q)_{0,\Omega} + (\Delta v, \Delta v)_{0,\Omega} - 2(q, \Delta q + v_t + \Delta v)_{0,\Omega} \\ &\geq (v_t, v_t)_{0,\Omega} + (q, q)_{0,\Omega} + (\Delta q, \Delta q)_{0,\Omega} + 2(\varepsilon v_t, \Delta q)_{0,\Omega} + (q, q)_{0,\Omega} + (\Delta v, \Delta v)_{0,\Omega} - 2(q, \Delta q + v_t + \Delta v)_{0,\Omega} \\ &\geq \|v_t\|_{0,\Omega}^2 + 2\|q\|_{0,\Omega}^2 + \|\Delta q\|_{0,\Omega}^2 + \|\Delta v\|_{0,\Omega}^2 - \varepsilon \left(\varepsilon_1 \|v_t\|_{0,\Omega}^2 + \frac{\|\Delta q\|_{0,\Omega}^2}{\varepsilon_1} \right) - \frac{\|q\|_{0,\Omega}^2}{\varepsilon_2} - \varepsilon_2 \left(\|v_t\|_{0,\Omega}^2 + \|\Delta q\|_{0,\Omega}^2 + \|\Delta v\|_{0,\Omega}^2 \right) \\ &= (1 - \varepsilon \varepsilon_1 - \varepsilon_2) \|v_t\|_{0,\Omega}^2 + \left(2 - \frac{1}{\varepsilon_2} \right) \|q\|_{0,\Omega}^2 + \left(1 - \frac{\varepsilon}{\varepsilon_1} - \varepsilon_2 \right) \|\Delta q\|_{0,\Omega}^2 + (1 - \varepsilon_2) \|\Delta v\|_{0,\Omega}^2, \end{aligned}$$

so we can select the positive constants $\varepsilon < 1, \varepsilon_1, \varepsilon_2$ satisfying

$$(1 - \varepsilon\varepsilon_1 - \varepsilon_2) > 0, \left(2 - \frac{1}{\varepsilon_2}\right) > 0, \left(1 - \frac{\varepsilon}{\varepsilon_1} - \varepsilon_2\right) > 0,$$

we have

$$B(q, v; q, v) \geq \alpha \left(\|v_t\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2 + \|\Delta q\|_{0,\Omega}^2 + \|\Delta v\|_{0,\Omega}^2 \right).$$

The proof of Theorem 2.1 is therefore completed.

Theorem 2.2. The Equations (5)-(9) has a unique solution, and the solution is $(\sigma, u) \in H^1(\Omega) \times H(\Omega)$.

Proof: From Theorem 2.1, we know that the bilinear form $B(\cdot, \cdot; \cdot, \cdot)$ is coercive and bounded on $H^1(\Omega) \times H(\Omega)$. Then the result follows from Lax-Milgram theorem.

3. Finite Element Approximation

In principle, the LSMFE approach simply consists of minimizing (12) in finite-dimensional subspaces $H_h(\Omega) \subset H^1(\Omega)$ and $M_h(\Omega) \subset H(\Omega)$. Suitable spaces are based on a triangulation \mathcal{T}_h of Ω and consist of piecewise polynomials with sufficient continuity conditions.

Now we consider the Ciarlet-Raviart mixed finite element form. Let $H_h(\Omega) = H^1(\Omega) \times C^0(\bar{\Omega})$ and

$$B_h(\sigma_h, u_h; q, v) \leq \beta_h \left(\sum_{T \in \mathcal{T}_h} \left(\|u_{ht}\|_{0,T}^2 + \|\sigma_h\|_{0,T}^2 + \|\Delta \sigma_h\|_{0,T}^2 + \|\Delta u_h\|_{0,T}^2 \right) \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \left(\|v_t\|_{0,T}^2 + \|q\|_{0,T}^2 + \|\Delta q\|_{0,T}^2 + \|\Delta v\|_{0,T}^2 \right) \right)^{\frac{1}{2}},$$

$$B_h(q, v; q, v) \geq \alpha_h \sum_{T \in \mathcal{T}_h} \left(\|v_t\|_{0,T}^2 + \|q\|_{0,T}^2 + \|\Delta q\|_{0,T}^2 + \|\Delta v\|_{0,T}^2 \right),$$

which holds for all $(q, v) \in H_h(\Omega) \times M_h(\Omega)$, $(\sigma_h, u_h) \in H_h(\Omega) \times M_h(\Omega)$. The proof is the same as the Theorem 2.1, we omit the proof.

4. Posteriori Error Estimation

One of the main motivations for using least-squares finite element approaches is the fact that the element-wise evaluation of the functional serves as an *a posteriori* error estimator.

A posteriori estimate attempt to provide quantitatively accurate measures of the discretization error through the so-called a posteriori error estimators which are derived by using the information obtained during the solution process. In recent years, the use of a *a posteriori* error estimators has become an efficient tool for assessing and controlling computational errors in adaptive computations [10].

Now we define the least-squares functional:

$$F_h(\sigma_h, u_h) = \sum_{T \in \mathcal{T}_h} \left(\|u_{ht} - \sigma_h + \Delta \sigma_h\|_{0,T}^2 + \|\Delta u_h - \sigma_h\|_{0,T}^2 \right). \quad (15)$$

$M_h(\Omega) = H_h(\Omega) \cap H_0^1(\Omega)$, let \mathcal{T}_h be a class quasi-uniform regular partition of Ω .

The least-squares functional:

$$F_h(\sigma, u) = \sum_{T \in \mathcal{T}_h} \left(\|u_t - \sigma + \Delta \sigma\|_{0,T}^2 + \|\Delta u - \sigma\|_{0,T}^2 \right). \quad (14)$$

Minimizing the functional (14) is equivalent to the following variational problem: find $\sigma_h \in H_h$ and $u_h \in M_h$ such that

$$B_h(\sigma_h, u_h; q, v) = 0,$$

holds for all $(q, v) \in H_h(\Omega) \times M_h(\Omega)$.

The discrete bilinear form $B(\cdot, \cdot; \cdot, \cdot)$ is defined as follows:

$$B_h(\sigma_h, u_h; q, v) = \sum_{T \in \mathcal{T}_h} \left((u_{ht} - \sigma_h + \Delta \sigma_h, v_t - q + \Delta q)_{0,T} + (\Delta u_h - \sigma_h, \Delta v - q)_{0,T} \right),$$

which holds for all $(q, v) \in H_h(\Omega) \times M_h(\Omega)$, $(\sigma_h, u_h) \in H_h(\Omega) \times M_h(\Omega)$.

Theorem 3.1. The bilinear $B_h(\cdot, \cdot; \cdot, \cdot)$ is continuous and coercive, *i.e.*, there exist positive constants α_h and β_h such that

We have

$$F_h(\sigma - \sigma_h, u - u_h) = \sum_{T \in \mathcal{T}_h} \left(\|u_t - u_{ht} - (\sigma - \sigma_h) + \Delta(\sigma - \sigma_h)\|_{0,T}^2 + \|\Delta(u - u_h) - (\sigma - \sigma_h)\|_{0,T}^2 \right),$$

so we define the posteriori estimator as following:

$$F_h(\sigma - \sigma_h, u - u_h) =: \sum_{T \in \mathcal{T}_h} \eta^2. \quad (16)$$

Theorem 4.1. The least-squares functional constitutes an *a posteriori* error estimator. In other words, for

$$\eta^2 = \|u_t - u_{ht} - (\sigma - \sigma_h) + \Delta(\sigma - \sigma_h)\|_{0,T}^2 + \|\Delta(u - u_h) - (\sigma - \sigma_h)\|_{0,T}^2,$$

there exist positive constants α_T and β_T such that

$$\sum_{T \in \mathcal{T}_h} \eta^2 \leq \beta_T \sum_{T \in \mathcal{T}_h} \left(\|u_t - u_{ht}\|_{0,T}^2 + \|\sigma - \sigma_h\|_{0,T}^2 + \|\Delta(\sigma - \sigma_h)\|_{0,T}^2 + \|\Delta(u - u_h)\|_{0,T}^2 \right),$$

$$\sum_{T \in \mathcal{T}_h} \eta^2 \geq \alpha_T \sum_{T \in \mathcal{T}_h} \left(\|u_t - u_{ht}\|_{0,T}^2 + \|\sigma - \sigma_h\|_{0,T}^2 + \|\Delta(\sigma - \sigma_h)\|_{0,T}^2 + \|\Delta(u - u_h)\|_{0,T}^2 \right).$$

Proof: We know

$$\sum_{T \in \mathcal{T}_h} \eta^2 = \mathcal{F}_h(\sigma - \sigma_h, u - u_h) = \sum_{T \in \mathcal{T}_h} \left(\|u_t - u_{ht} - (\sigma - \sigma_h) + \Delta(\sigma - \sigma_h)\|_{0,T}^2 + \|\Delta(u - u_h) - (\sigma - \sigma_h)\|_{0,T}^2 \right)$$

$$= B_h(\sigma - \sigma_h, u - u_h; \sigma - \sigma_h, u - u_h).$$

From Theorem 3.1, we have:

$$B_h(\sigma - \sigma_h, u - u_h; \sigma - \sigma_h, u - u_h) \leq \beta_h \sum_{T \in \mathcal{T}_h} \left(\|u_t - u_{ht}\|_{0,T}^2 + \|\sigma - \sigma_h\|_{0,T}^2 + \|\Delta(\sigma - \sigma_h)\|_{0,T}^2 + \|\Delta(u - u_h)\|_{0,T}^2 \right),$$

$$B_h(\sigma - \sigma_h, u - u_h; \sigma - \sigma_h, u - u_h) \geq \alpha_h \sum_{T \in \mathcal{T}_h} \left(\|u_t - u_{ht}\|_{0,T}^2 + \|\sigma - \sigma_h\|_{0,T}^2 + \|\Delta(\sigma - \sigma_h)\|_{0,T}^2 + \|\Delta(u - u_h)\|_{0,T}^2 \right).$$

The positive constants $\alpha_T = C\alpha_h$ and $\beta_T = C\beta_h$, this completes the proof.

Remark: The mesh is adapted and based on a *posteriori* error estimate of the fourth order elliptic problems. Based on the computed *a posteriori* error estimator η , we use a mesh optimization procedure to compute the size of elements in the new mesh. Adaptive refinement strategies consist in refining those triangles with the largest values of η .

5. Convergence Analysis of Adaptive Least-Squares Mixed Finite Element Method

We now briefly introduce the main idea of adaptive least-squares mixed finite element methods through local refinement. Given an initial triangulation \mathcal{T}_0 , we shall generate a sequence of nested conforming triangulations \mathcal{T}_k using the following loop (*):

(*) SOLVE \rightarrow ESTIMATE \rightarrow MARK \rightarrow REFINE.

More precisely to get \mathcal{T}_{k+1} from \mathcal{T}_k we first solve (5)-(9) to get u_k on \mathcal{T}_k . The error is estimated using u_k and to mark a set of \mathcal{T}_k that are to be refined. Triangles are refined in such a way that the triangulation is still shape regular and conforming.

The a posteriori error estimator is essential part of the ESTIMATE step. The a posteriori error estimator is usually split into local error indicators and they are then employed to make local modifications by dividing the elements whose error indicator is large and possibly coarsening the elements whose error indicator is small.

The convergence of local refinement algorithms based on the repetition of loop (*) is established by the error reduction type result. Let \mathcal{T}_H be a shape regular triangulation of Ω and \mathcal{T}_h is a refinement of \mathcal{T}_H such that $H_H(\Omega) \subset H_h(\Omega)$. Let σ_H and σ_h be the finite

element approximation of σ in $H_H(\Omega)$ and $H_h(\Omega)$, respectively. We shall use the following results in the proof of the convergence [11]:

$$\|\sigma - \sigma_h\|_0^2 = \|\sigma - \sigma_H\|_0^2 - \|\sigma_H - \sigma_h\|_0^2. \tag{17}$$

Let \mathcal{T}_0 be an initial shape regular triangulation, let σ_k be a solution of (5)-(9) in the k -th loop. We have the following theorem:

Theorem 5.1. Let σ_k be a solution obtained in the k -th loop in the algorithm, then there exists a constants $\delta \in (0,1)$ depending the shape regularity of \mathcal{T}_0 such that:

$$\|\sigma - \sigma_k\|_0^2 \leq C\delta^k,$$

and thus the algorithm will terminate in finite steps.

Proof: In the MARK step we select $\theta \in (0,1)$ such that

$$\eta(\sigma, \sigma_H) \leq \theta \eta(\sigma_h, \sigma_H).$$

From Theorem 4.1, we obtain the following inequality:

$$\|\sigma - \sigma_H\|_0^2 \leq C\eta(\sigma, \sigma_H)^2,$$

$$\|\sigma_H - \sigma_h\|_0^2 \geq C\eta(\sigma_h, \sigma_H)^2.$$

By (17) we have:

$$\|\sigma - \sigma_h\|_0^2 \leq (1 - C\theta)\|\sigma - \sigma_H\|_0^2,$$

so there exists a constant $\delta \in (0,1)$ such that

$$\|\sigma - \sigma_{k+1}\|_0^2 \leq \delta \|\sigma - \sigma_k\|_0^2,$$

we let $e_k = \|\sigma - \sigma_k\|_0^2$, we then get

$$e_{k+1} \leq \delta e_k,$$

which by recursion implies

$$e_{k+1} \leq \delta^{k+1} e_0.$$

So the adaptive least-squares mixed finite element method is converged.

6. Summary and Conclusions

We describe an adaptive least-squares mixed finite element procedure for solving the fourth order parabolic problems in this paper, and the procedure uses a least-squares mixed finite element formulation and adaptive refinement based on a *posteriori* error estimate. The methods were applied to study the continuous and coercivity of the fourth order parabolic problems.

In this paper, we applied relatively standard *a posteriori* error estimation techniques to adaptively solve the fourth order parabolic problems and shown the convergence of the adaptive least-squares mixed finite element method.

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