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Fisher's Fiducial Inference for Parameters of Uniform Distribution

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ABSTRACT

Fisher's Fiducial Inference for the parameters of a totality uniformly distributed on $[\alpha,\beta]$ is discussed. The corresponding fiducial distributions are derived. The maximum fiducial estimators, fiducial median estimators and fiducial expect estimators of α and β are got. The problems about the fiducial interval, fiducial region and hypothesis testing are discussed. An example which showed that Neyman-Pearson's confidence interval has some place to be improved is illustrated. An idea about deriving fiducial distribution is proposed.

Keywords: Fiducial Inference; Uniformly Distribution; Parameters; Fiducial Interval; Hypothesis Testing

1. Introduction

In 1930 Fisher proposed an inference method based on the idea of fiducial probability [1,2]. Fisher's fiducial inference has been much applied in practice. The fiducial argument stands out somewhat of an enigma in classical statistics. The enigma mentioned above need statistical scholar to solve.

Fisher's fiducial inference for the parameters of a totality $\mathcal{U}_{[\alpha,\beta]}$ is discussed. The corresponding fiducial distributions are derived. The maximum fiducial estimators, the fiducial median estimators and the fiducial expect estimators of α and β are got. The problems about the fiducial interval, fiducial region and hypothesis testing are discussed.

The example below shows that Neyman-Pearson's confidence interval has some place to be improved. Let $X_1,X_2,\cdots,X_n$ be i.i.d., $X_j\sim\mathcal{U}_{[\theta-0.5,\theta+0.5]}$ for each *j*. $1-\alpha=0.95, n=12$. By [3] p. 16 Corollary 3.2 the density function of $\left(\min\{X_i\}-\theta,\max\{X_i\}-\theta\right)$ is

$$f(y,z)=\begin{cases} n(n-1)(z-y)^{n-2}, \text{if } 0\le y\le z\le 1 \\ 0, \quad \text{otherwise}\end{cases}$$

Appling pivotal function

$$\frac{1}{2}\left(\min\{X_i\}+\max\{X_i\}\right)-\theta$$

And using its density

$$f(u)=\begin{cases} n\left[(1+2u)^{n-1}+(1-2u)^{n-1}\right]\Big/2^n, \text{if } -0.5\le u\le 0.5 \\ 0, \quad \text{otherwise}\end{cases}$$

the 95% confidence interval of θ can be got as

$$\left[\frac{1}{2}\left(\min\{X_i\}+\max\{X_i\}\right)\pm\delta\right] \quad (*)$$

where δ is the solution of

$$n\left[(1+2\delta)^n-(1-2\delta)^n\right]\Big/2^n=0.95$$

The length of interval (*) is independent of the sample value! Assam that

$$\min\{X_i\}=0.58, \max\{X_i\}=1.39$$

Is got in a certain sample (Note that

$$\boldsymbol{P}\left\{0.58<X_j<1.39, j=1,2,\cdots,n\right\}=0.85^n\approx 0.08,$$

the above data can illustrate the common problems). The probability that

$$\max\{X_i\}-0.5\le\theta\le\min\{X_i\}+0.5$$

i.e. $\theta\in[0.89,1.08]$, is 1, the length of it is 0.19, but the length of (*) is $2\delta\approx 0.619$!

Fisher's fiducial inference offered a selection in solving the problems similar with above.

2. Fiducial Distribution

Let that $X_1, X_2, \cdots, X_n$ is i.i.d., $X_j \sim \mathcal{U}_{[\alpha,\beta]}$. As well known, their sufficient statistics of least dimension is $\left(\min\{X_i\}, \max\{X_i\}\right)$.Set

$$Y = \frac{\min\{X_i\} - \alpha}{\beta - \alpha}, Z = \frac{\max\{X_i\} - \alpha}{\beta - \alpha} \quad (2.1)$$

It is not difficult to show that *Y* and *Z* are the minimum and maximum order statistics of the sample from $\mathcal{U}_{[0,1]}$ respectively, and by [3] p. 16 Corollary 3.2, the density function of (Y, Z) is

$$f(y,z) = \begin{cases} n(n-1)(z-y)^{n-2}, \text{if } 0 \le y \le z \le 1 \\ 0, \quad \text{otherwise} \end{cases} \quad (2.2)$$

See parameters α and β as r.v.'s, see $\min\{X_i\}$ and $\max\{X_i\}$ as constants now. It can be got from Equation (2.1) that

$$\begin{aligned} \alpha &= \frac{Y \max\{X_i\} - Z \min\{X_i\}}{Y - Z} \\ \beta &= \frac{(1-Y)\max\{X_i\} - (1-Z)\min\{X_i\}}{Z - Y} \end{aligned} \quad (2.3)$$

Applying the relative results about the transformation of r.v.'s, it can be show that:

Theorem 1. The fiducial density function of vector (α, β) is

$$f_{\alpha,\beta}(a,b) = \begin{cases} \dfrac{n(n-1)\left[\max\{X_i\} - \min\{X_i\}\right]^{n-1}}{\left(b - a^{n+1}\right)}, \\ \quad \text{if } a < \min\{X_i\} \text{ and } b > \max\{X_i\} \\ 0, \quad \text{otherwise} \end{cases} \quad (2.4)$$

If only one parameter need to be considered, the another parameter is then so-called *nuisance parameter*. We insist that the *marginal distribution* should be used in this situation. Hence find the two marginal density functions of $f_{\alpha,\beta}(a,b)$

$$f_\alpha(a) = \begin{cases} \dfrac{(n-1)\left[\max\{X_i\} - \min\{X_i\}\right]^{n-1}}{\left(\max\{X_i\} - a\right)^n}, \text{if } a < \min\{X_i\} \\ 0, \quad \text{if } a \ge \min\{X_i\} \end{cases} \quad (2.5)$$

$$f_\beta(b) = \begin{cases} \dfrac{(n-1)\left[\max\{X_i\} - \min\{X_i\}\right]^{n-1}}{\left(b - \min\{X_i\}\right)^n}, \text{if } b > \max\{X_i\} \\ 0, \quad \text{if } b \le \max\{X_i\} \end{cases} \quad (2.6)$$

Corollary 1. The fiducial density functions of only one parameters α or β as r.v.'s are given by (2.5) and (2.6).

3. Estimation

It is easy to see that fiducial density $f_{\alpha,\beta}(a,b)$ has achieved its maxima at

$$\hat{\alpha}_m = \min\{X_i\} \quad (3.1)$$

$$\hat{\beta}_m = \max\{X_i\} \quad (3.2)$$

Theorem 2. The maximum fiducial estimators of α and β are given by (3.1) and (3.2).

It can also be got that $f_\alpha(a)$ has achieved its maxima at $\hat{\alpha}_m$, and $f_\beta(b)$ has achieved its maxima at $\hat{\beta}_m$ as well. The estimators $\hat{\alpha}_m$ and $\hat{\beta}_m$ are coincided with the maximum likelihood estimators of α and β.

To find the median of $f_\alpha(a)$, solve

$$\int_{-\infty}^{Me_\alpha} f_\alpha(a) = \frac{1}{2} \quad (3.3)$$

And get

$$Me_\alpha = \max\{X_i\} - 2^{\frac{1}{n-1}}\left[\max\{X_i\} - \min\{X_i\}\right] \quad (3.4)$$

Found the median of $f_\beta(b)$ by using the same method, and have

$$Me_\beta = \min\{X_i\} + 2^{\frac{1}{n-1}}\left[\max\{X_i\} - \min\{X_i\}\right] \quad (3.5)$$

Theorem 3. The fiducial median estimators of α and β are given by (3.4) and (3.5).

The maximum fiducial estimators $\hat{\alpha}_m$ and $\hat{\beta}_m$ are extreme a little, Equation (3.1) can be written as

$$\hat{\alpha}_m = \max\{X_i\} - \left[\max\{X_i\} - \min\{X_i\}\right]$$

Since

$$2^{\frac{1}{n-1}} > 1$$

Me_α is a modify to $\hat{\alpha}_m$, and Me_β is a modify to $\hat{\beta}_m$ too.

It can be shown that:

Theorem 4. The fiducial expect estimators of α and β are given by

$$\begin{aligned} \hat{\alpha}_e &= \min\{X_i\} - \frac{1}{n-2}\left[\max\{X_i\} - \min\{X_i\}\right] \\ \hat{\beta}_e &= \max\{X_i\} + \frac{1}{n-2}\left[\max\{X_i\} - \min\{X_i\}\right] \end{aligned} \quad (3.6)$$

Proof.

$$E(\alpha)=\int_{-\infty}^{\min\{X_i\}} a\frac{(n-1)\left[\max\{X_i\}-\min\{X_i\}\right]^{n-1}}{(\max\{X_i\}-a)^n}\mathrm{d}a$$
$$=\frac{n-1}{n-2}\min\{X_i\}-\frac{1}{n-2}\max\{X_i\}$$

$E(\beta)$ can be calculated by using the same method. □

$\hat{\alpha}_e$ is a better modify to $\hat{\alpha}_m$, and $\hat{\beta}_e$ is a better modify to $\hat{\beta}_m$ as well. We suggest using $\hat{\alpha}_e$ and $\hat{\beta}_e$.

The fiducial probability that α belongs to a certain interval estimator $[\hat{\alpha}_1,\hat{\alpha}_2]$ can be calculated using $f_\alpha(a)$ as follows

$$P_f\{\hat{\alpha}_1\le\alpha\le\hat{\alpha}_2\}=\int_{\hat{\alpha}_1}^{\hat{\alpha}_2} f_\alpha(a)\mathrm{d}a$$
$$=\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\hat{\alpha}_2\right)^{n-1}}-\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\hat{\alpha}_1\right)^{n-1}},\quad \text{if } \hat{\alpha}_2\le\mathrm{Min}\{X_i\} \tag{3.7}$$

In the same way

$$P_f\{\hat{\beta}_1\le\beta\le\hat{\beta}_2\}=\int_{\hat{\beta}_1}^{\hat{\beta}_2} f_\beta(b)\mathrm{d}b$$
$$=\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\hat{\beta}_1-\mathrm{Min}\{X_i\}\right)^{n-1}}-\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\hat{\beta}_2-\mathrm{Min}\{X_i\}\right)^{n-1}},\quad \text{if } \hat{\beta}_1\ge\mathrm{Max}\{X_i\} \tag{3.8}$$

Give a fiducial probability $1-\delta(0<\delta<1)$ let us consider the $1-\delta$ fiducial interval problem. In order to set the length of the interval as shorter as possible, we choice $\mathrm{Min}\{X_i\}$ as the right end point of the fiducial interval of α, because $f_\alpha(a)$ increases; and choice $\mathrm{Max}\{X_i\}$ as the left end point of the fiducial interval of β, because $f_\beta(b)$ decreases.

Theorem 5. The $1-\delta$ fiducial interval of α is

$$\left[\mathrm{Max}\{X_i\}-\delta^{-\frac{1}{n-1}}\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right),\mathrm{Min}\{X_i\}\right] \tag{3.9}$$

The $1-\delta$ fiducial interval of β is

$$\left[\mathrm{Max}\{X_i\},\mathrm{Min}\{X_i\}+\delta^{-\frac{1}{n-1}}\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)\right] \tag{3.10}$$

Proof. Denote that

$$\breve{\alpha}=\mathrm{Max}\{X_i\}-\delta^{-\frac{1}{n-1}}\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)$$
$$\breve{\beta}=\mathrm{Min}\{X_i\}+\delta^{-\frac{1}{n-1}}\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)$$

Using (3.7) it can be derived that

$$P_f\{\breve{\alpha}\le\alpha\le\mathrm{Min}\{X_i\}\}$$
$$=\int_{\mathrm{Min}\{X_i\}}^{\mathrm{Max}\{X_i\}-\delta^{-\frac{1}{n-1}}\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)} f_\alpha(a)\mathrm{d}a$$
$$=\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}-\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\breve{\alpha}\right)^{n-1}}$$
$$=1-\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left[\delta^{-\frac{1}{n-1}}\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)\right]^{n-1}}$$
$$=1-\delta$$

And the bellow equation can be got by using (3.8)

$$P_f\{\mathrm{Max}\{X_i\}\le\beta\le\breve{\beta}\}$$
$$=\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}-\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\breve{\beta}-\mathrm{Min}\{X_i\}\right)^{n-1}},$$
$$=1-\delta$$

□

Let us consider the $1-\delta$ fiducial region of (α,β). In order to set the area of the region as smaller as possible, we choice the region as the following rectangular triangle:

$$\begin{aligned}&\{\mathrm{Min}\{X_i\}-d\le a\le\mathrm{Min}\{X_i\},\\&\mathrm{Max}\{X_i\}\le b\le\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}+a+d\}\end{aligned} \tag{3.11}$$

for a certain $d > 0$, because $f_{\alpha,\beta}(a,b)$ choice the same value when $b-a$ equals to a constant, and $f_{\alpha,\beta}(a,b)$ increases in a when b is invariant, decreases in b when a is invariant.

Theorem 6. The $1-\delta$ fiducial region of (α,β) is given by (3.11) if positive d satisfies

$$\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}\left[\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}+nd\right]}{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}+d\right)^n}=\delta \tag{3.12}$$

Proof. At first Equation (3.12) has a positive solution

d because its left side equal to 1 when $d=0$ and tends to 0 when d tends to $+\infty$. Hence the fiducial probability that (α,β) belongs to the region given by (3.11) is

$$\int_{\mathrm{Min}\{X_i\}-d}^{\mathrm{Min}\{X_i\}}\left[\int_{\mathrm{Max}\{X_i\}}^{\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}+d+a} f_{\alpha,\beta}(a,b)\mathrm{d}b\right]\mathrm{d}a$$

$$=1-\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}+d\right)^{n-1}}$$

$$-\frac{(n-1)\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}d}{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}+d\right)^{n}}$$

$$=1-\delta$$

Equation (3.12) is used here. □

4. The Case That One Parameter Is in Variation

Let us consider the case that only one parameter is in variation.

$\mathcal{U}_{[\alpha,\beta]}$ is a distribution with single-parameter when one end point of (α,β) is constant. For constant b_0 $\mathrm{Min}\{X_i\}$ is sufficient for $\mathcal{U}_{[\alpha,b_0]}$. It can be got that the fiducial density of parameter α in $\mathcal{U}_{[\alpha,b_0]}$ is

$$f_\alpha^*(a)=\frac{n\left(b_0-\mathrm{Min}\{X_i\}\right)^n}{(b_0-a)^{n+1}} \tag{4.1}$$

It should noted that using (2.4) and (2.6) the conditional density of α under $\beta=b$ can be got as

$$f_{\alpha|\beta}(a|b)=\frac{n\left(b-\mathrm{Min}\{X_i\}\right)^n}{(b-a)^{n+1}} \tag{4.2}$$

Comparing (4.1) and (4.2) is to say that (4.1) is coincided with the conditional density of α under $\beta=b_0$!

The maximum fiducial estimators, the fiducial median estimators and the fiducial expect estimators of α can be got easily by using (4.1).

The fiducial probability of one interval estimator $[\hat{\alpha}_1,\hat{\alpha}_2]$ for α can be calculated as

$$P_f\{\hat{\alpha}_1\le\alpha\le\hat{\alpha}_2\}=\int_{\hat{\alpha}_1}^{\hat{\alpha}_2} f_\alpha^*(a)\mathrm{d}a$$

$$=\frac{\left(b_0-\mathrm{Min}\{X_i\}\right)^n}{(b_0-\hat{\alpha}_2)^n}-\frac{\left(b_0-\mathrm{Min}\{X_i\}\right)^n}{(b_0-\hat{\alpha}_1)^n}, \tag{4.3}$$

if $\hat{\alpha}_2\le\mathrm{Min}\{X_i\}$

The $1-\delta$ fiducial interval of α can be got as follows by using (4.3).

$$\left[b_0-\delta^{-\frac{1}{n}}\left(b_0-\mathrm{Min}\{X_i\}\right),\mathrm{Min}\{X_i\}\right] \tag{4.4}$$

The similar results for β can be got easily as well.

If there is a relation between the parameters, such as the example in Section 1, this situation may be thought as missing parameter(s). We insist that the *conditional distribution* should be used in this situation. Under the condition that $\beta-\alpha=C$ for a constant C, the conditional density $f_{\alpha|\beta-\alpha=c}(a)$, or $f_{\alpha|\beta-\alpha}(a|C)$, is a constant in the interval on which its value isn't zero, because $f_{\alpha,\beta}(a,b)$ is a constant when $b-a=C$. Since

$$\alpha\le\mathrm{Min}\{X_i\},$$

$$\beta=\alpha+C\ge\mathrm{Max}\{X_i\},$$

the conditional density $f_{\alpha|\beta-\alpha=c}(a)$ is the density of $\mathcal{U}_{[\mathrm{Max}\{X_i\}-C,\mathrm{Min}\{X_i\}]}$. This is the fiducial density of the parameter α of a totality $\mathcal{U}_{[\alpha,\alpha+C]}$.

It can be seen that for distribution $\mathcal{U}_{[\alpha,\alpha+C]}$,

$$\left[\mathrm{Max}\{X_i\}-C,\mathrm{Min}\{X_i\}\right]$$

is a 100% fiducial interval of α. Any subinterval of

$$\left[\mathrm{Max}\{X_i\}-C,\mathrm{Min}\{X_i\}\right]$$

is a $1-\delta$ fiducial interval of α in this problem only if it has the length

$$(1-\delta)\left[\mathrm{Min}\{X_i\}+C-\mathrm{Max}\{X_i\}\right].$$

Using the above results to the example in Section 1 it can be got that any subinterval of [0.89, 1.08] with the length 0.95 × 0.19 is the 95% fiducial interval of θ. Its length 0.1805 is much smaller than $2\delta(\approx 0.619)$, the length of interval (*).

5. Hypothesis Testing

Let us consider the hypothesis testing problem. Equation (3.7) and (3.8) can be used to calculate the fiducial probability when the parameter would belong to the range that a certain hypothesis is true.

Theorem 7. For hypothesis

$$H_0:\alpha\ge\alpha_0 \rightleftarrows H_1:\alpha<\alpha_0$$

$$P_f\{\alpha<\alpha_0\}=1-\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\alpha_0\right)^{n-1}},$$

$$\text{if } \alpha_0\le\mathrm{Min}\{X_i\} \tag{5.1}$$

$$P_f\{\alpha\ge\alpha_0\}=\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\alpha_0\right)^{n-1}},$$

if $\alpha_0\le\mathrm{Min}\{X_i\}$

And should rejected H_1 w.p.1 if $\mathrm{Min}\{X_i\}<\alpha_0$.

Proof. Choice $\hat{\alpha}_2=\mathrm{Min}\{X_i\}$ and $\hat{\alpha}_1=\alpha_0$ in (3.7). □

If for a certain $\lambda>0$, the decision is made by comparing $P_f\{\alpha<\alpha_0\}$ with $\lambda P_f\{\alpha\ge\alpha_0\}$, then the criterion is that reject H_0 when

$$\max\{X_i\}-(1+\lambda)^{\frac{1}{n-1}}\left[\max\{X_i\}-\min\{X_i\}\right]<\alpha_0 \quad (5.2)$$

Note that the left hand of (5.2) is the quantile of order $\frac{\lambda}{\lambda+1}$ of the fiducial distribution of α. Especially for $\lambda=1$ the criterion is reject H_0 when

$$Me_\alpha<\alpha_0 \quad (5.3)$$

Theorem 8. For hypothesis

$$H_0:\alpha\in(\alpha_0-\eta,\alpha_0+\eta)$$
$$\rightleftarrows H_1:\alpha\notin(\alpha_0-\eta,\alpha_0+\eta)$$

The fiducial probability

$$P\{\alpha\in(\alpha_0-\eta,\alpha_0-\eta)\}$$
$$=\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\alpha_0-\eta\right)^{n-1}}$$
$$-\frac{\left(\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}\right)^{n-1}}{\left(\mathrm{Max}\{X_i\}-\alpha_0+\eta\right)^{n-1}}$$
$$\text{if } \alpha_0+\eta\le\mathrm{Min}\{X_i\}$$

Proof. The result can be got just like theorem 7. □

The parallel results for β can be got by using the same method as well.

Theorem 9. Hypothesis

$$H_0:(\alpha,\beta)$$
$$\in\{\alpha\in(\alpha_0-\eta,\alpha_0+\eta),\beta\in(\beta_0-\eta,\beta_0+\eta)\}$$
$$\rightleftarrows H_1:(\alpha,\beta)$$
$$\notin\{\alpha\in(\alpha_0-\eta,\alpha_0+\eta),\beta\in(\beta_0-\eta,\beta_0+\eta)\}$$

The fiducial probability

$$P_f\left[(\alpha,\beta)\in\begin{Bmatrix}\alpha\in(\alpha_0-\eta,\alpha_0+\eta)\\ \beta\in(\beta_0-\eta,\beta_0+\eta)\end{Bmatrix}\right]$$
$$=\left(\frac{\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}}{\beta_0-\alpha_0-2\eta}\right)^{n-1}$$
$$-2\left(\frac{\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}}{\beta_0-\alpha_0}\right)^{n-1}$$
$$+\left(\frac{\mathrm{Max}\{X_i\}-\mathrm{Min}\{X_i\}}{\beta_0-\alpha_0+2\eta}\right)^{n-1}$$
$$\text{if } \alpha_0+\eta\le\mathrm{Min}\{X_i\} \text{ and } -\beta_0-\eta\ge\mathrm{Max}\{X_i\}$$

Proof.

$$P_f\left[(\alpha,\beta)\in\begin{Bmatrix}\alpha\in(\alpha_0-\eta,\alpha_0+\eta)\\ \beta\in(\beta_0-\eta,\beta_0+\eta)\end{Bmatrix}\right]$$
$$=\int_{\alpha_0-\eta}^{\alpha_0+\eta}\left[\int_{\beta_0-\eta}^{\beta_0+\eta}f_{\alpha,\beta}(a,b)\mathrm{d}b\right]\mathrm{d}a$$

This theorem can be got by calculating the above integral. □

The fiducial probability in the situation that the parameters would belong to the range that a certain hypothesis in Theorem 7 or 8 is true can be easily got by using (4.3) in the case that one parameter is in variation.

Example. For the example in Section 1, consider the hypothesis

$$H_0:\theta\ge\theta_0\rightleftarrows H_1:\theta<\theta_0$$

It can be shown that

$$P_f\{\theta<\theta_0\}=\frac{\theta_0+0.5-\mathrm{Max}\{X_i\}}{\mathrm{Min}\{X_i\}+1-\mathrm{Max}\{X_i\}},$$
$$\text{if } \mathrm{Max}\{X_i\}-0.5<\theta_0\le\mathrm{Min}\{X_i\}+0.5$$

If for a certain $\lambda>0$, the decision is made by comparing $P_f\{\theta<\theta_0\}$ with $\lambda P_f\{\theta\ge\theta_0\}$, when the one in front is greater,

$$\frac{\theta_0+0.5-\mathrm{Max}\{X_i\}}{\mathrm{Min}\{X_i\}+1-\mathrm{Max}\{X_i\}}$$
$$>\lambda\left[1-\frac{\theta_0+0.5-\mathrm{Max}\{X_i\}}{\mathrm{Min}\{X_i\}+1-\mathrm{Max}\{X_i\}}\right]$$
$$\times(\lambda+1)\frac{\theta_0+0.5-\mathrm{Max}\{X_i\}}{\mathrm{Min}\{X_i\}+1-\mathrm{Max}\{X_i\}}$$
$$>\lambda$$

So the criterion is that reject H_0 when

$$\frac{\max\{X_i\}+\lambda\min\{X_i\}+0.5\lambda-0.5}{\lambda+1}<\theta_0 \quad (5.4)$$

Please note that the left hand of (5.4) is the quantile of order $\frac{\lambda}{\lambda+1}$ of the fiducial distribution of θ. Especially for $\lambda=1$ the criterion is that reject H_0 when

$$\frac{\max\{X_i\}+\min\{X_i\}}{2}<\theta_0 \quad (5.5)$$

That is

$$Me_\theta<\theta_0 \quad (5.6)$$

6. Discussion

Up to now, the discussion on Fisher's fiducial inference has still remained intuitive and imprecise. There are two problems: 1) Just what a fiducial probability means? 2) How can one derive the only fiducial distribution of the parameter(s)? Paper [4] considered the 1st problem. For the 2nd problem we guess that two sufficient statistics of least dimension, whose dimension is coincides with the parameter(s), must derive the same fiducial distribution of the parameter(s). And we insist that the *marginal distribution* should be used in the situation when there is (are) *nuisance parameter*(*s*); *and* that the *conditional distribution* should be used in the situation when there is (are) (a) relation(s) between the parameters.

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