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Wavelet Density Estimation and Statistical Evidences Role for a GARCH Model in the Weighted Distribution

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ABSTRACT

We consider n observations from the GARCH-type model: $Z = UY$, where U and Y are independent random variables. We aim to estimate density function Y where Y have a weighted distribution. We determine a sharp upper bound of the associated mean integrated square error. We also make use of the measure of expected true evidence, so as to determine when model leads to a crisis and causes data to be lost.

Keywords: Density Estimation; GARCH Model; Weighted Distribution; Wavelets; Statistical Evidences; Strongly Mixing

1. Introduction

We suppose that $Z_1,\cdots,Z_n$ is a sample of a strictly stationary and exponentially strongly mixing process $(Z_i)_{i\in\mathbb{Z}}$ where, for any $i\in\mathbb{Z}$,

$$Z_i = U_i Y_i, \tag{1}$$

$(U_i)_{i\in\mathbb{Z}}$ is a sequence of identically distributed random variables with common known density f_U and $(Y_i)_{i\in\mathbb{Z}}$ is a sequence of identically distributed random variables with common unknown density f_Y. For any $i\in\mathbb{Z}$, U_i and Y_i are independent. We suppose that f_Y is a weighted density of the form

$$f_Y(x) = \frac{w(x) f_X(x)}{\theta}, x\in[0,1], \tag{2}$$

where w is a known positive function, f_X an unknown density of a random variable X and θ is the unknown normalization parameter:

$$\theta = \mathbb{E}(w(X)) = \int_0^1 w(x) f_X(x)\,\mathrm{d}x.$$

Our goal is to estimate f_X when only $Z_1,\cdots,Z_n$ are observed. The Equation (1) is a GARCH-type time series model classically encountered in financial models see [1] and practical examples of Equation (2) can be found in e.g. [2-4].

In this article, we construct a linear wavelet estimator and measure its performance by determining upper bounds of the mean integrated squared error (MISE) over Besov space.

In what follows, we have also surveyed the role of data and evidential inference in the model. The data play a very important essential role in statistical analysis, to the extent that many statistical researchers believe in the famous saying: "Ask the data." We consider the Test

$$\begin{cases} H_1 : \theta = \theta_1 \\ H_2 : \theta = \theta_2 = \alpha\theta_1 (\alpha > 1), \end{cases} \tag{3}$$

for the model and we evaluate the sensitivity of the value in the test hypotheses. In this test, the evaluation criterion is the area between the curves of the cumulative distribution functions under H_1 and H_2 hypotheses. Details on evidential inference can be found in [5,6]. Also [7] have studied about Comparing of record data and random observation based on statistical evidence.

Through the rest of the paper, at first assumptions and then an introduction about wavelets are presented in Section 2. The estimators and results are given in Section 3. In Section 4, general explanations regarding evidential inference and its application in a test. The proofs are gathered in Section 5.

2. Assumptions and Wavelets

2.1. Assumptions

We formulate the following assumptions:

- Without loss of generality, we assume that f_X and f_Y have the support $[0,1]$ and $f_X \in \mathbb{L}^2([0,1])$ where

$$\mathbb{L}^2([0,1]) = \left\{ g:[0,1]\to\mathbb{R}; \left(\int_0^1 g^2(x)\,\mathrm{d}x\right)^{1/2} < \infty \right\}.$$

- We suppose that for any $m \in \mathbb{Z}$, the m-th strongly mixing coefficient of $(Z_i)_{i\in\mathbb{Z}}$ by

$$a_m = \sup_{(A,B)\in F_{-\infty,0}^Z \times F_{m,\infty}^Z} \left|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)\right|,$$

where, for any $u \in \mathbb{Z}$, let $F_{-\infty,u}^Z$ be the σ-algebra generated by $\cdots, Z_{u-1}, Z_u$ and $F_{u,\infty}^Z$ is the σ-algebra generated by $Z_u, Z_{u+}, \cdots$.

We suppose that there exist three (known) constants, $\gamma > 0, c > 0$ and $\theta > 0$ such that

$$a_m \le \gamma \exp\left(-c|m|^\theta\right).$$

This assumption is satisfied by a large class of GARCH processes. See e.g. [8-10].

- For any $x \in [0,1]$, it follows from the independence of U_1 and Y_1 that the density of Z_1 is

$$f_Z(x) = \int_x^1 f_U\left(\frac{x}{y}\right) f_Y(y) \frac{1}{y} \mathrm{d}y.$$

- We suppose that there exists two constants, $C > 0$ and $c > 0$, such that

$$\sup_{x\in[0,1]} f_Z(x) \le C. \tag{4}$$

and

$$\sup_{x\in[0,1]} w(x) \le C, \sup_{x\in[0,1]} w'(x) \le C, \inf_{x\in[0,1]} w(x) \ge c. \tag{5}$$

2.2. Wavelets and Besov Balls

Let N be a positive integer, and ϕ and ψ be the Daubechies wavelets dbN which satisfy

$$supp(\phi) = supp(\psi) = 1-N, N.$$

Set

$$\phi_{j,k}(x) = 2^{j/2}\phi\left(2^j x - k\right),$$
$$\psi_{j,k}(x) = 2^{j/2}\psi\left(2^j x - k\right).$$

Then, there exists an integer τ such that, for any integer $\ell \ge \tau$, the collection

$$\mathcal{B} = \Big\{\phi_{\ell,k}(\cdot), k \in \{0,\cdots,2^\ell - 1\}; \psi_{j,k}(\cdot);$$
$$j \in \mathbb{N} - \{0,\cdots,\ell-1\}, k \in \{0,\cdots,2^j - 1\}\Big\}$$

is an orthonormal basis of $\mathbb{L}^2[0,1]$ (the space of square-integrable functions on 0,1). We refer to [11].

For any integer $\ell \ge \tau$, any $h \in \mathbb{L}^2[0,1]$ can be expanded on $\mathcal{B}$ as

$$h(x) = \sum_{k=0}^{2^\ell - 1} \alpha_{\ell,k}\phi_{\ell,k}(x) + \sum_{j=\ell}^{\infty}\sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x), x \in [0,1],$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ are the wavelet coefficients of h defined by

$$\alpha_{j,k} = \int_0^1 h(x)\phi_{j,k}(x)\mathrm{d}x, \beta_{j,k} = \int_0^1 h(x)\psi_{j,k}(x)\mathrm{d}x. \tag{6}$$

Let $M > 0, s > 0, p \ge 1$ and $r \ge 1$. A function h belongs to $B_{p,r}^s(M)$ if and only if there exists a constant $M^* > 0$ (depending on M) such that the associated wavelet coefficients Equation (6) satisfy

$$\left(\sum_{j=\tau-1}^{\infty}\left(2^{j(s+1/2-1/p)}\left(\sum_{k=0}^{2^j-1}\left|\beta_{j,k}\right|^p\right)^{1/p}\right)^r\right)^{1/r} \le M^*.$$

We set $\beta_{\tau-1,k} = \alpha_{\tau,k}$. Details on Besov balls can be found in [12].

3. Estimators and Results

Firstly, we consider the following estimator for θ

$$\hat{\theta} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{w(Z_i) - Z_i w'(Z_i)}{w^2(Z_i)}\right)^{-1}. \tag{7}$$

Then, for any integer $j \ge \tau$ and any $k \in \{0,\cdots,2^j - 1\}$, we estimate

$$\alpha_{j,k} = \int_0^1 f_X(x)\phi_{j,k}(x)\mathrm{d}x, \hat{\alpha}_{j,k} = \frac{\hat{\theta}}{n}\sum_{i=1}^{n} T(\phi_{j,k})(Z_i), \tag{8}$$

where, for any $h : \mathcal{C}[0,1] \to \mathbb{R}$, T is the operator

$$T(h)(x) = \frac{h(x)w(x) + xh'(x)w(x) - xh(x)w'(x)}{w^2(x)}. \tag{9}$$

$\hat{\alpha}_{j,k}$ and $\hat{\theta}$ are similar with multiplicative censoring model (see [13]).

We are now in the position to define the considered estimators for f_X. Suppose that $f_X \in B_{p,r}^s(H)$. We define the linear estimator $\hat{f}$ by

$$\hat{f}(x) = \sum_{k=0}^{2^{j_0}-1}\hat{\alpha}_{j_0,k}\phi_{j_0,k}(x), x \in [0,1], \tag{10}$$

where $\hat{\alpha}_{j,k}$ is defined by Equation (8) and j_0 is the integer satisfying

$$(1/2)n^{1/(2s+q+3)} < 2^{j_0} \le n^{1/(2s+q+3)}. \tag{11}$$

Lemma 3.1

- Let $\hat{\theta}$ be Equation (7) and $\theta = \int_0^1 w(x) f_X(x)\mathrm{d}x$.

  Then we have

$$\mathbb{E}\left(\frac{1}{\hat{\theta}}\right) = \frac{1}{\theta}.$$

- Let $Z_1, \cdots, Z_n$ be Equation (1), T be Equation (9) and for any integer $j \ge \tau$ and any $k \in \{0,\cdots,2^j-1\}$, $\alpha_{j,k} = \int_0^1 f_X(x)\phi_{j,k}(x)\mathrm{d}x$. Then we have

$$\mathbb{E}\left(\frac{\theta}{n}\sum_{i=1}^{n} T(\phi_{j,k})(Z_i)\right) = \alpha_{j,k}.$$

- For any integer $j \ge \tau$ and any $k \in \{0,\cdots,2^j-1\}$, let $\hat{\alpha}_{j,k}$ be Equation (8) and $\alpha_{j,k} = \int_0^1 f_X(x)\phi_{j,k}(x)\mathrm{d}x$. Then, under the assumptions of Subsection 2.1, there exists a constant $C>0$ such that

$$\left|\hat{\alpha}_{j,k} - \alpha_{j,k}\right| \le C\left(\left|\frac{\theta}{n}\sum_{i=1}^{n} T(\phi_{j,k})(Z_i) - \alpha_{j,k}\right| + \left|\frac{1}{\hat{\theta}} - \frac{1}{\theta}\right|\right).$$

Proposition 3.1 *For any integer* $j \ge \tau$ *and any* $k \in \{0,\cdots,2^j-1\}$ *Then, under the assumptions of Subsection* 2.1,

- $\sup_{x\in[0,1]} \left|T(\phi_{j_0,k})(x)\right| \le C2^{\frac{3}{2}j}$.
- $\mathbb{E}\left(T(\phi_{j_0,k})(Z_0)\right)^2 \le C2^{2j}$.

Proposition 3.2 *Let* $q \in (0,1)$, *for any integer* $j \ge \tau$ *and any* $k \in \{0,\cdots,2^j-1\}$, *let* $\hat{\alpha}_{j,k}$ *be Equation* (8) *and* $\alpha_{j,k} = \int_0^1 f_X(x)\phi_{j,k}(x)\mathrm{d}x$. *Then,*

- there exists a constant $C>0$ such that

$$\mathbb{V}\left(\frac{\theta}{n}\sum_{i=1}^{n} T(\phi_{j,k})(Z_i)\right) \le \frac{C}{n}2^{j(2+q)}.$$

- there exists a constant $C>0$ such that

$$\mathbb{V}\left(\frac{1}{\hat{\theta}}\right) \le \frac{C}{n}.$$

- there exists a constant $C>0$ such that

$$\mathbb{E}\left(\left(\hat{\alpha}_{j,k} - \alpha_{j,k}\right)^2\right) \le \frac{C}{n}2^{j(2+q)}.$$

Theorem 3.1 (Upper bound for $\hat{f}$) *Consider Equation* (1) *under the assumptions of Subsection* 2.1. *Suppose that* $f_X \in B^s_{p,r}(H)$ *with* $s>0, p\ge 2, r\ge 1$. *For any* $q \in (0,1)$ *and* $\hat{f}$ *be Equation* (10), *then there exists a constant* $C>0$ *such that*

$$\mathbb{E}\left(\int_0^1 \left(\hat{f}(x) - f_X(x)\right)^2 \mathrm{d}x\right) \le Cn^{-2s/(2s+q+3)}.$$

Remark that $n^{-2s/(2s+q+3)}$ is the slower than the optimal one in the standard density estimation problem *i.e.* $n^{-2s/(2s+1)}$ (see e.g. [14, Chapter 10]). This deterioration is due to the presence of GARCH model and weighted distribution.

4. Statistical Evidence

4.1. Statistical Inference

The evidential approach to statistical inference concerns a novel approach in statistical analysis. Evidential inference is solely based on data as evidence and calculation of the evidence strength. It is not influenced by mental and personal components and factors such as former beliefs and loss functions. Using evidential inference in the model Equation (1), we will survey when censoring of data will lead to considerable data loss, and we will determine the time when data is lost by determining an appropriate criterion. In the model $Z_i = U_i Y_i$ for $i \in \{1,\cdots,n\}$, the data observed from the variable Z_i are denoted by the subscript (cen), and the data observed from the variable Y_i are denoted by the subscript (ncen). Considering the Test Equation (3) in the above model, due to the symmetry of the test hypotheses in evidential methods and without losing the generality of the problem, the value of is assumed to be $\alpha > 1$. In order to support H_1 and H_2 hypotheses, we now use the following criterion:

$$\gamma = \frac{\boldsymbol{abc}_{cen}(\eta)}{\boldsymbol{abc}_{ncen}(\eta)}, \tag{12}$$

where $\boldsymbol{abc}_{cen}(\eta)$ and $\boldsymbol{abc}_{ncen}(\eta)$ are the measure of expected true evidence in the censored and uncensored data respectively, and η is the criterion of the support of data from H_1 hypothesis against H_2 hypothesis. This support criterion is optimal when the area between the two curves of η cumulative functions under H_1 and H_2 is maximum, please see [15]. This area which is denoted by $\boldsymbol{abc}(\eta)$ in the form of

$$\begin{aligned}\boldsymbol{abc}(\eta) &= \int_0^1 \kappa_2(\eta)\mathrm{d}\eta - \int_0^1 \kappa_1(\eta)\mathrm{d}\eta \\ &= \int_0^1 \left(1-\kappa_1(\eta)\right)\mathrm{d}\eta - \int_0^1 \left(1-\kappa_2(\eta)\right)\mathrm{d}\eta \\ &= \boldsymbol{E}_1(\eta) - \boldsymbol{E}_2(\eta),\end{aligned}$$

where $\kappa_i(\eta)$ is the cumulative distribution function of η and $\boldsymbol{E}_i(\eta)$ is the mean value of η under $H_i:(i=1,2)$ hypotheses. In view of [6], the support criterion η is defined as follows:

$$\eta = \frac{\lambda}{1+\lambda}, \tag{13}$$

where λ is the likelihood ratio and for the two censored and uncensored cases we have

$$\lambda_{cen} = \frac{f_Z^{\theta 1}(\underline{z})}{f_Z^{\theta 2}(\underline{z})} \text{ and } \lambda_{ncen} = \frac{f_Y^{\theta 1}(\underline{y})}{f_Y^{\theta 2}(\underline{y})}, \tag{14}$$

where $f_Z^{\theta i}(\underline{z})$ and $f_Y^{\theta i}(\underline{y})$ are likelihood functions for Z_i and Y_i variables respectively, in the Equation (1) under $H_i:(i=1,2)$ hypotheses.

4.2. Measuring Statistical Evidence

We consider i.i.d case for variables in the Equations (1) and (2), also set $w(x)=1$ and then, we investigate the behavior of α by means of simulation. In addition, we analyze H_1 and H_2 hypotheses in Test Equation (3) by determining support criterion η of the measure of expected true evidence $\boldsymbol{abc}(\eta)$. The programming codes of this part are written in the MAPLE (15) environment.

Example 1. In this example, we generate data from gamma and uniform distribution as follows, considering multiplicative censoring model:

$$Y_i \sim \text{Gamma}(2,\theta_i) \text{ and } U_i \sim \text{Uniform}(0,1)$$
$$\text{and } H_i:\theta=\theta_i \text{ for } (i=1,2)$$

Then, according to Equations (13) and (14), we calculate the support criterion and the likelihood ratio via below relations:

$$\eta_{cen}=\frac{\lambda_{cen}}{1+\lambda_{cen}},$$

such that

$$\lambda_{cen}=\frac{f_Z^{\theta_1}(\underline{z})}{f_Z^{\theta_2}(\underline{z})}=\left(\frac{\theta_2}{\theta_1}\right)^n \mathrm{e}^{-\sum_{i=1}^{n} z_i\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right)}.$$

and

$$\eta_{ncen}=\frac{\lambda_{ncen}}{1+\lambda_{ncen}},$$

such that

$$\lambda_{ncen}=\frac{f_Y^{\theta_1}(\underline{y})}{f_Y^{\theta_2}(\underline{y})}=\left(\frac{\theta_2}{\theta_1}\right)^{2n} \mathrm{e}^{-\sum_{i=1}^{n} y_i\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right)}.$$

For different values of α and n, we calculate the value of γ according to Equation (12). The results can be observed in **Table 1**. By carefully considering this table, it is observed that as the value of α increases (which implies the distance growth between θ_1 and θ_2), the value of γ gets closer to one. In other words, $\boldsymbol{abc}_{cen}$ approaches $\boldsymbol{abc}_{ncen}$ more and more. This fact can be interpreted in this way that if the distance between θ_1 and θ_2 is large, the data lost in censored data Y_i is negligible. That is to say, evidential inference draws our attention to the time when θ_2 is close to θ_1. The above analysis can also be observed in **Figure 1**.

In what follows, the variations of sample volume ratio increase against α are investigated, and the value of α in the $\text{ArgMax}\,\frac{\gamma(n_2,\alpha)}{\gamma(n_1,\alpha)}$ equation is determined for different values of n_1 and n_2.

If $\frac{n_2}{n_1}=\frac{25}{20}=1.25$ then $\alpha=1.6$. The result can be viewed in **Figure 2**, If $\frac{n_2}{n_1}=\frac{15}{10}=1.5$ then $\alpha=2.1$. The result can be viewed in **Figure 3**, If $\frac{n_2}{n_1}=\frac{20}{10}=2$ then $\alpha=2.5$. The result can be viewed in **Figure 4**.

The above results can be interpreted in this way that as the sample volume increases from a certain stage on, the value of α remains constant. In other words, it can be intuitively said that the $\text{ArgMax}\,\frac{\gamma(n_2,\alpha)}{\gamma(n_1,\alpha)}$ ratio tends to the constant α value, which this also leads to an increase in evidential strength.

5. Proofs

In this section, C denotes any constant that does not depend on j,k and n. Its value may change from one term to another and may depends on ϕ or ψ.

Table 1. Computed values for *γ*.

n value	α value						
	$\alpha=1.1$	$\alpha=1.6$	$\alpha=2.1$	$\alpha=2.6$	$\alpha=3.1$	$\alpha=3.6$	$\alpha=4.1$
10	0.5122	0.6427	0.7580	0.8398	0.8994	0.9317	0.9596
15	0.5165	0.6879	0.8273	0.9141	0.9572	0.9799	0.9894
20	0.5291	0.7251	0.8773	0.9516	0.9824	0.9931	0.9970
25	0.5219	0.7614	0.9145	0.9738	0.9923	0.9981	0.9991
30	0.5329	0.7881	0.9409	0.9862	0.9968	0.9992	0.9995

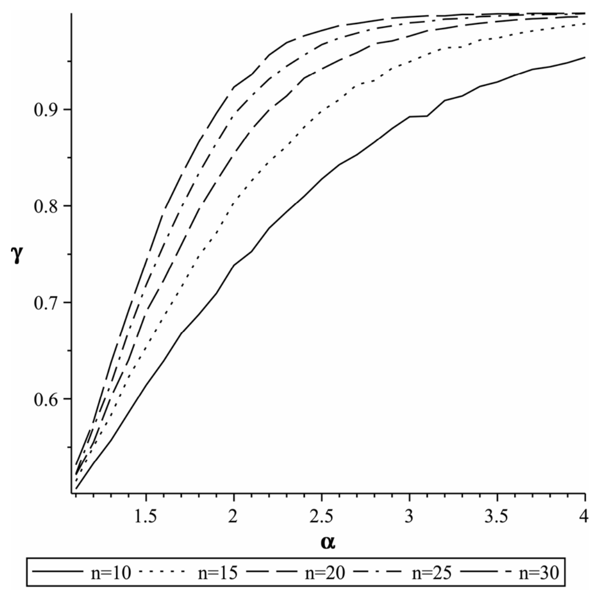

Figure 1. *γ* computed from gamma distribution for different values of *n*.

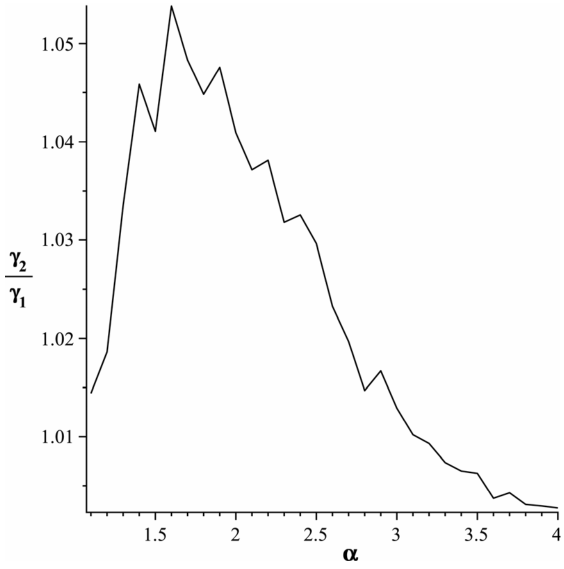

Figure 2. The most relative changes when sample size n_1 = 20 increases to n_2 = 25 happens in α = 1.6.

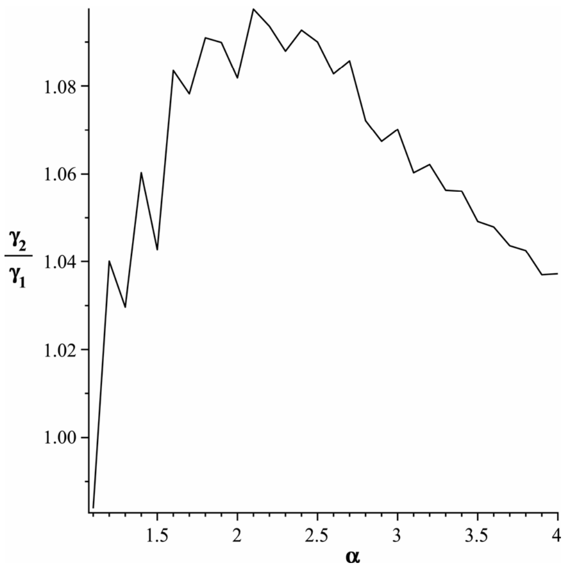

Figure 3. The most relative changes when sample size n_1 = 10 increases to n_2 = 15 happens in α = 2.1.

Proof of Lemma 3.1.

Proof can be found in [13].

Proof of proposition 3.1.

1. By Equation (9) and Equation (5), we have

$$\begin{aligned}&\left|T\left(\phi_{j,k}\right)(x)\right| \\ &\le \frac{1}{w^2(x)}\left(\begin{aligned}&\left|\phi_{j,k}(x)w(x)\right|+\left|x\left(\phi_{j,k}\right)'(x)w(x)\right| \\ &+\left|x\phi_{j,k}(x)w'(x)\right|\end{aligned}\right) \\ &\le C\left(\left|\phi_{j,k}(x)\right|+\left|\left(\phi_{j,k}\right)'(x)\right|\right).\end{aligned} \tag{15}$$

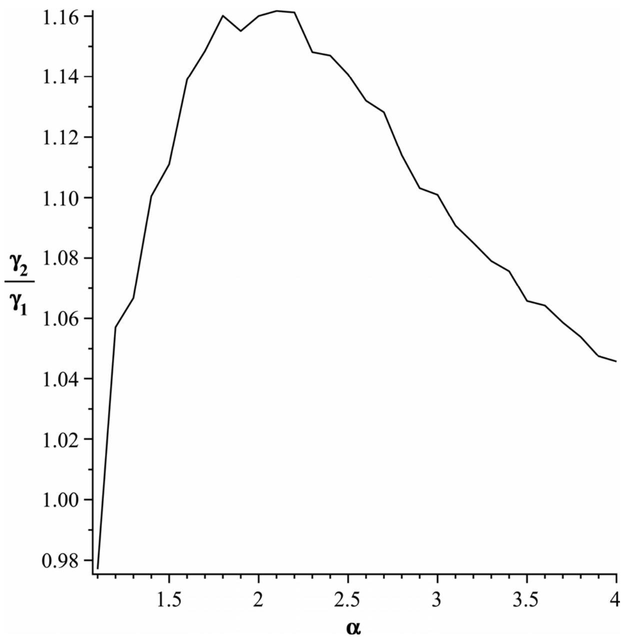

Figure 4. The most relative changes when sample size n_1 = 10 increases to n_2 = 20 happens in α = 2.2.

Therefore

$$\begin{aligned}&\sup_{x\in[0,1]}\left|T\left(\phi_{j_0,k}\right)(x)\right| \\ &\le C\left(\sup_{x\in[0,1]}\left(\phi_{j_0,k}\right)(x)+\sup_{x\in[0,1]}\left(\phi_{j_0,k}\right)'(x)\right) \\ &\le C\left(2^{\frac{j}{2}}+2^{\frac{3}{2}j}\right)\le C2^{\frac{3}{2}j}.\end{aligned} \tag{16}$$

2. By Equation (9) and Equation (5), we have

$$\begin{aligned}&\mathbb{E}\left(\left|T\left(\phi_{j,k}\right)(Z_0)\right|^2\right) \\ &\le C\left(\mathbb{E}\left(\left|\phi_{j,k}(Z_0)\right|^2\right)+\mathbb{E}\left(\left|\left(\phi_{j,k}\right)'(Z_0)\right|^2\right)\right).\end{aligned} \tag{17}$$

By Equation (4) and under the assumptions of Subsection 2.1 and also doing the change of variables $y=2^jx-k$, we have

$$\begin{aligned}&\mathbb{E}\left(\left|\phi_{j,k}(Z_0)\right|^2\right) \\ &=\int_0^1\left|\phi_{j,k}(x)\right|^2 f_Z(x)\,\mathrm{d}x\le C\int_0^1\left|\phi_{j,k}(x)\right|^2\mathrm{d}x \\ &=C\int_{-k}^{2^j-k}\left|\phi(y)\right|^2\mathrm{d}y\le C.\end{aligned} \tag{18}$$

Using again Equation (4) and the assumptions of Subsection 2.1, the equality $\left(\phi_{j,k}\right)'(x)=2^{3j/2}\phi'\left(2^jx-k\right)$ and doing the change of variables $y=2^jx-k$, we obtain

$$\mathbb{E}\left(\left|\left(\phi_{j,k}\right)'\left(Z_0\right)\right|^2\right) = \int_0^1 \left|\left(\phi_{j,k}\right)'(x)\right|^2 f_Z(x)\,\mathrm{d}x \le C\int_0^1 \left|\left(\phi_{j,k}\right)(x)\right|^2 \mathrm{d}x \le C2^{2j}\int_{-k}^{2^j-k}\left|\phi'(y)\right|^2 \mathrm{d}y \le C2^{2j}. \quad (19)$$

Combining Equations (17)-(19), we obtain

$$\mathbb{E}\left(\left|T\left(\phi_{j,k}\right)\left(Z_0\right)\right|^2\right) \le C2^{2j}. \quad (20)$$

The proof of Proposition 3.1 is complete.

Proof of Proposition 3.2.

1. We have

$$\begin{aligned}\mathbb{V}\left(\frac{\theta}{n}\sum_{i=1}^n T\left(\phi_{j,k}\right)\left(Z_i\right)\right) &= \frac{\theta^2}{n^2}\sum_{v=1}^n\sum_{\ell=1}^n \mathbb{C}\left(T\left(\phi_{j_0,k}\right)\left(Z_v\right), T\left(\phi_{j_0,k}\right)\left(Z_\ell\right)\right)\\ &= \frac{\theta^2}{n^2}\mathbb{V}\left(T\left(\phi_{j,k}\right)\left(Z_1\right)\right) + 2\frac{\theta^2}{n^2}\sum_{v=2}^n\sum_{\ell=1}^{v-1}\mathbb{C}\left(T\left(\phi_{j_0,k}\right)\left(Z_v\right), T\left(\phi_{j_0,k}\right)\left(Z_\ell\right)\right)\\ &\le \frac{\theta^2}{n^2}\mathbb{V}\left(T\left(\phi_{j,k}\right)\left(Z_1\right)\right) + 2\frac{\theta^2}{n^2}\left|\sum_{v=2}^n\sum_{\ell=1}^{v-1}\mathbb{C}\left(T\left(\phi_{j_0,k}\right)\left(Z_v\right), T\left(\phi_{j_0,k}\right)\left(Z_\ell\right)\right)\right|.\end{aligned} \quad (21)$$

Using Equation (20) we obtain

$$\mathbb{V}\left(T\left(\phi_{j,k}\right)\left(Z_1\right)\right) \le \mathbb{E}\left(\left(T\left(\phi_{j,k}\right)\left(Z_1\right)\right)^2\right) \le C2^{2j}. \quad (22)$$

also

$$\begin{aligned}&\left|\sum_{v=2}^n\sum_{\ell=1}^{v-1}\mathbb{C}\left(T\left(\phi_{j_0,k}\right)\left(Z_v\right), T\left(\phi_{j_0,k}\right)\left(Z_\ell\right)\right)\right|\\ &= \left|\sum_{m=1}^n (n-m)\mathbb{C}\left(T\left(\phi_{j_0,k}\right)\left(Z_0\right), T\left(\phi_{j_0,k}\right)\left(Z_m\right)\right)\right|\\ &\le n\sum_{m=1}^n\left|\mathbb{C}\left(T\left(\phi_{j_0,k}\right)\left(Z_0\right), T\left(\phi_{j_0,k}\right)\left(Z_m\right)\right)\right|.\end{aligned} \quad (23)$$

By the Davydov inequality (see [16]) and for any $q \in (0,1)$ we obtain

$$\mathbb{C}\left(T\left(\phi_{j_0,k}\right)\left(Z_0\right), T\left(\phi_{j_0,k}\right)\left(Z_m\right)\right) \le 10a_m^q\left(\sup_{x\in[0,1]}\left|T\left(\phi_{j_0,k}\right)(x)\right|\right)^{2q}\left(\mathbb{E}\left(T\left(\phi_{j_0,k}\right)\left(Z_0\right)\right)^2\right)^{1-q}. \quad (24)$$

Putting Equation (24), Equation (16) and Equation (20) together, we have

$$\mathbb{C}\left(T\left(\phi_{j_0,k}\right)\left(Z_0\right), T\left(\phi_{j_0,k}\right)\left(Z_m\right)\right) \le 10a_m^q C2^{3jq}2^{2j(1-q)} = Ca_m^q 2^{j(2+q)}. \quad (25)$$

Since $\sum_{m=1}^n a_m^q \le \sum_{m=1}^\infty a_m^q = C$, we obtain

$$\left|\sum_{v=2}^n\sum_{\ell=1}^{v-1}\mathbb{C}\left(T\left(\phi_{j_0,k}\right)\left(Z_v\right), T\left(\phi_{j_0,k}\right)\left(Z_\ell\right)\right)\right| \le nC2^{j(2+q)}\sum_{m=1}^n a_m^q \le nC2^{j(2+q)}. \quad (26)$$

Putting Equations (21) and (22) and Equation (26) together, we have

$$\mathbb{V}\left(\frac{\theta}{n}\sum_{i=1}^n T\left(\phi_{j,k}\right)\left(Z_i\right)\right) \le \frac{C}{n}2^{j(2+q)}. \quad (27)$$

2. We have

$$\begin{aligned}\mathbb{V}\left(\frac{1}{\hat\theta}\right) &= \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^n \frac{w\left(Z_i\right) - Z_i w'\left(Z_i\right)}{w^2\left(Z_i\right)}\right)\\ &= \frac{1}{n}\mathbb{V}\left(\frac{w\left(Z_i\right) - Z_i w'\left(Z_i\right)}{w^2\left(Z_i\right)}\right) + \frac{2}{n^2}\sum_{v=2}^n\sum_{\ell=1}^{v-1}\mathbb{C}\left(\frac{w\left(Z_v\right) - Z_v w'\left(Z_v\right)}{w^2\left(Z_v\right)}, \frac{w\left(Z_\ell\right) - Z_\ell w'\left(Z_\ell\right)}{w^2\left(Z_\ell\right)}\right).\end{aligned} \quad (28)$$

By Equation (5), we have

$$\mathbb{V}\left(\left|\frac{w\left(Z_i\right) - Z_i w'\left(Z_i\right)}{w^2\left(Z_i\right)}\right|\right) \le C. \quad (29)$$

By the Davydov inequality (see [16]) we obtain

$$\mathbb{C}\left(\frac{w\left(Z_v\right) - Z_v w'\left(Z_v\right)}{w^2\left(Z_v\right)}, \frac{w\left(Z_\ell\right) - Z_\ell w'\left(Z_\ell\right)}{w^2\left(Z_\ell\right)}\right) \le Ca_m^q. \quad (30)$$

Combining Equations (28)-(30), we obtain

$$\mathbb{V}\left(\frac{1}{\hat\theta}\right) \le \frac{C}{n}\left(1 + \sum_{m=1}^n a_m^q\right) \le \frac{C}{n}. \quad (31)$$

3. Using Lemma (3.1), Equation (27) and Equation (31), then

$$\begin{aligned}\mathbb{E}\left(\left(\hat\alpha_{j,k} - \alpha_{j,k}\right)^2\right) &\le C\left(\mathbb{E}\left(\frac{\theta}{n}\sum_{i=1}^n T\left(\phi_{j,k}\right)\left(Z_i\right) - \alpha_{j,k}\right)^2 + \mathbb{E}\left(\frac{1}{\hat\theta} - \frac{1}{\theta}\right)^2\right)\\ &= C\left(\mathbb{V}\left(\frac{\theta}{n}\sum_{i=1}^n T\left(\phi_{j,k}\right)\left(Z_i\right)\right) + \mathbb{V}\left(\frac{1}{\hat\theta}\right)\right) \le \frac{C}{n}2^{j(2+q)}.\end{aligned} \quad (32)$$

The proof of Proposition 3.2 is complete.

Proof of Theorem 3.1. For any integer $\ell \ge \tau$, any $f_X \in \mathbb{L}^2(0,1)$ can be expanded on $\mathcal{B}$ as

$$f_X(x)=\sum_{k=0}^{2^{j_0}-1}\alpha_{j_0,k}\phi_{j_0,k}(x)+\sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}\psi_{j,k}(x),$$

where

$$\alpha_{j_0,k}=\int_0^1 f_X(x)\phi_{j_0,k}(x)\mathrm{d}x,\beta_{j,k}=\int_0^1 f_X(x)\psi_{j,k}(x)\mathrm{d}x.$$

We obtain

$$\mathbb{E}\left(\int_0^1\left(\hat{f}(x)-f_X(x)\right)^2\mathrm{d}x\right)=A+B,$$

where

$$A=\sum_{k=0}^{2^{j_0}-1}\left(\left(\hat{\alpha}_{j_0,k}-\alpha_{j_0,k}\right)^2\right),B=\sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}^2.$$

Using Proposition 3.2 and inequalitity Equation (11)

$$A\le\frac{C}{n}2^{j_0}2^{j_0(2+q)}\le Cn^{-2s/(2s+q+3)}.$$

and since $p\ge 2$, we have $B_{p,r}^s(H)\subseteq B_{2,\infty}^s(H)$. Hence

$$B\le C2^{-2j_0s}\le Cn^{-2s/(2s+q+3)}.$$

Hence we have

$$\mathbb{E}\left(\int_0^1\left(\hat{f}(x)-f_X(x)\right)^2\mathrm{d}x\right)\le Cn^{-2s/(2s+q+3)}.$$

This ends the proof of Theorem 3.1.

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