Applied Mathematics, **2013, 4, 348-351**
http://dx.doi.org/10.4236/am.2013.42053 Published Online February 2013 (http://www.scirp.org/journal/am)

Super Cyclically Edge Connected Half Vertex Transitive Graphs*

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Received October 21, 2012; revised December 26, 2012; accepted January 3, 2013

ABSTRACT

Tian and Meng in [Y. Tian and J. Meng, λ_c-Optimally half vertex transitive graphs with regularity k, Information Processing Letters 109 (2009) 683-686] shown that a connected half vertex transitive graph with regularity k and girth $g(G) \geq 6$ is cyclically optimal. In this paper, we show that a connected half vertex transitive graph G is super cyclically edge-connected if minimum degree $\delta(G) \geq 4$ and girth $g(G) \geq 6$.

Keywords: Cyclic Edge-Connectivity; Cyclically Optimal; Super Cyclically Edge-Connected; Half Vertex Transitive Graph

1. Introduction

The traditional connectivity and edge-connectivity, are important measures for networks, which can correctly reflect the fault tolerance of systems with few processors, but it always underestimates the resilience of large networks. The discrepancy incurred is because events whose occurrence would disrupt a large network after a few processors, therefore, the disruption envisaged occurs in a worst case scenario. To overcome such a shortcoming, Latifi *et al*. [1] proposed a kind of conditional edge-connectivity, denoted by $\lambda^k(G)$, which is the minimum size of edge-cut S such that each vertex has degree at least k in $G-S$.

Throughout the paper graphs are undirected finite connected without loops or multiple edges.

Let $G=(V,E)$ be a graph, an edge set F is a *cyclic edge-cut* if $G-F$ is disconnected and at least two of its components contain cycles. Clearly, a graph has a cyclic edge-cut if and only if it has two vertex-disjoint cycles. A graph G is said to be *cyclically separable* if G has a cyclic edge-cut. Note that Lovász [2] characterized all multigraphs without two vertex-disjoint cycles. The characterization can also be found in Bollobás [3]. So, it is natural to further study the cyclically separable graphs. For a cyclically separable graph G, The *cyclic edge-connectivity* of G, denoted by $\lambda_c(G)$, is defined as the minimum cardinality over all cyclic edge-cuts of G by following Plummer [4]. The concept of cyclic edge-connectivity as applied to planar graphs dates to the famous incorrect conjecture of Tait [5].

The cyclic edge-connectivity plays an important role in some classic fields of graph theory such as Hamiltonian graphs (Máčajová and Šoviera [6]), fullerence graphs (Kardoš and Škrekovski [7]), integer flow conjectures (Zhang [8]), n-extendable graphs (Holton *et al*. [9]; Lou and Holton [10]), etc.

For two vertex sets $X,Y \subseteq V, [X,Y]_G$ is the set of edges with one end in X and the other end in Y. For any vertex set X, $G[X]$ is the subgraph of G induced by X, $\overline{X}$ is the complement of X. Clearly, if $[X,\overline{X}]$ is a minimum cyclic edge-cut, then both $G[X]$ and $G[\overline{X}]$ are connected. We set

$$\zeta(G) = \min\{\omega(X) \mid X \text{ induces a shortest cycle in } G\},$$

where $\omega(X)$ is the number of edges with one end in X and the other end in $V(G)\setminus X$. It has been proved in Wang and Zhang [11] that $\lambda_c(G) \leq \zeta(G)$ for any cyclically separable graph. Hence, a cyclically separable graph G is called cyclically optimal, in short, λ_c-optimal, if $\lambda_c(G)=\zeta(G)$, and *super cyclically edge-connected*, in short, super-λ_c, if the removal of any minimum cyclic edge-cut of graph G results in a component which is a shortest cycle.

Cyclic edge-fragment and cyclic edge-atom play a fundamental role. A vertex set X is a *cyclic edge-fragment*, in short, *fragment*, if $[X,\overline{X}]$ is a minimum cyclic edge-cut. A cyclic edge-fragment with the mini-

*This research is supported by NSFC (10671165) and NSFCXJ (2010211A06).
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mum cardinality is called a *cyclic edge-atom*, in short, *atom*. A cyclic edge-fragment of G is said to be *super*, if neither X nor $\bar{X}$ induces a shortest cycle, in short, *super fragment*. A super cyclic edge-fragment with the minimum cardinality is called a *super cyclic edge-atom*, in short, *super atom*. A cyclic edge-fragment is said to be trivial, if it induces a cycle, otherwise it is nontrivial.

A graph G is said to be *vertex transitive* if $Aut(G)$ acts transitively on $V(G)$, and is *edge transitive* if $Aut(G)$ acts transitively on $E(G)$. A bipartite graph is *biregular*, if all the vertices from the same partite set have the same degree. We abbreviate the bipartite graph as a (p,q)-biregular graph, if the two distinct degrees are p and q respectively $(p \ge q)$. A bipartite graph G with bipartition $X_1 \cup X_2$ is called *half vertex transitive* [12], if $Aut(G)$ acts transitively both on X_1 and X_2. Clearly, the half vertex transitive graph is biregular graph. Let $x \in V(G)$, we call the set $\{x^g : g \in Aut(G)\}$ an orbit of $Aut(G)$. Clearly, $Aut(G)$ acts transitively on each orbit of $Aut(G)$. Transitive graphs have been playing an important role in designing network topologies, since they possess many desirable properties such as high fault tolerance, small transitive delay, etc. [13,14].

In Nedela and Škoviera [15], it was proved that a cubic-transitive or edge-transitive graph (expect for K_4 and $K_{3,3}$) is λ_c-optimal. From Wang and Zhang [11], Xu and Liu [16], we have known that a $k(\ge 4)$-regular vertex-transitive graph G is λ_c-optimal if it has girth $g(G) \ge 5$. It was also shown that an edge-transitive graph G with minimum degree $\delta(G) \ge 4$ and order $n \ge 6$ is λ_c-optimal in Wang and Zhang [11]. Recently, Zhang and Wang [17] showed that a connected vertex-transitive or edge-transitive graph is super-λ_c if either G is cubic with girth $g(G) \ge 7$ or G has minimum degree $\delta(G) \ge 4$ and girth $g(G) \ge 6$. Zhou and Feng [18] characterized all possible λ_c-superatoms for λ_c-optimal nonsuper-λ_c graphs, and classified all λ_c-optimal nonsuper-λ_c edge-transitive graphs.

Theorem 1.1 ([19]) *Let G be a* $k(\ge 4)$*-regular connect half vertex transitive graph with bipartition* $X_1 \cup X_2$*, and girth* $g \ge 6$*, then G is* λ_c*-optimal.*

Motivated by the work in Tian and Meng [19], in this article we aim to study a connected half vertex transitive graph, and we show that a connected half vertex transitive graph G is super cyclically edge-connected if minimum degree $\delta(G) \ge 4$ and girth $g(G) \ge 6$.

2. Preliminaries

Lemma 2.1 ([11]) *Let G be a simple connected graph with* $\delta(G) \ge 3$ *and* $g(G) \ge 5$ *or* $\delta(G) \ge 4$ *and order* $n \ge 6$*. Then G is cyclically separable.*

Lemma 2.2 ([11]) *Let G be a cyclically separable (p, q)-biregular graph with* $p \ge q \ge 4$*. Suppose G is not cyclically optimal and* $g(G) \ge 6$*. Then for any distinct atoms X and Y,* $X \cap Y = \varnothing$*.*

An *imprimitive block* of G is a proper nonempty subset A of $V(G)$ such that for any automorphism $\phi \in Aut(G)$, either $\phi(A) = A$ or $\phi(A) = \varnothing$.

Lemma 2.3 ([20]) *Let* $G = (V, E)$ *be a graph and let Y be the subgraph of G induced by an imprimitive block A of G. If G is vertex-transitive, then so is Y. If G is edge-transitive, then A is an independent set of G.*

If X is a super atom, and X' is a proper subset of X such that $[X', \bar{X}']$ is a cyclic edge-cut and $G[X']$ is not a shortest cyclic, then

$$\omega(X') > \omega(X).$$

The observation is used frequently in the proofs.

Lemma 2.4 ([11]) *Let G be a connected graph with* $\delta(G) \ge 3$ *and* X *be a fragment. Then*

(1) $\delta(G[X]) \ge 2$;

(2) If $\delta(G[X]) \ge 3$, then $d_X(v) \ge d_{\bar{X}}(v)$ holds for any $v \in X$;

(3) If $G[X]$ is not a cycle and v is a vertex in X with $d_X(v) = 2$, then $d_X(v) \ge d_{\bar{X}}(v)$ holds for any $v \in X$;

(4) If $\delta(G) \ge 4$, and X is a non-trivial atom of G, then $\delta(G[X]) \ge 3$. Furthermore, $d_X(v) > d_{\bar{X}}(v)$ holds for any $v \in X$. and $d_X(v) > d_{\bar{X}}(v)$ holds for any $v \in X$.

Lemma 2.5 ([17]) *Let G be a connected graph with* $\delta(G) \ge 3$ *and* $g(G) \ge 6$*. Then G has two vertex-disjoint cycles and* $|V(G)| > (\delta(G) - 1) g \ge 2g$.

Lemma 2.6 ([17]) *Let G be a (p,q)-biregular graph with* $\delta(G) \ge 4$ *and girth* $g(G) \ge 6$*. Suppose G is cyclically optimal but not super cyclically edge-connected. Then any two distinct super atoms X and Y of G satisfies* $X \cap Y = \Phi$.

Lemma 2.7 *Let G be a connected (p,q)-half vertex transitive graph with bipartition* $X_1 \cup X_2$*,* $\delta(G) \ge 4$ *and girth* $g(G)$*. Suppose A is a atom of G and* $Y = G[A]$*. If G is not* λ_c*-optimal, then*

(1) $V(G)$ is a disjoint union of distinct atoms;

(2) Y is a (p', q')-half vertex transitive graph, where $3 \le p' \le p-1, 3 \le q' \le q-1$.

Proof. Let

$$A_1 = A \cap X_1 \text{ and } A_2 = A \cap X_2,$$

then

$$A = A_1 \cup A_2.$$

Since A is a λ_c-atom, we have

$$|A_i| \ge 2 (1 \le i \le 2).$$

(1) Since $A_i \neq \varnothing (i=1,2)$ and *Aut*(*X*) acts transitively both on X_1 and X_2, each vertex of *G* lies in a λ_c-atom. by Lemma 2.3, we have that $V(G)$ is a disjoint union of distinct λ_c-atoms.

(2) Let $u_1, v_1 \in A_1$, then there exits an automorphism φ of G with $\varphi(u_1)=v_1$ and so $\varphi(A)\bigcap A \neq \varnothing$. By Lemma 2.3, $\varphi(A)=A$. Thus the restriction of φ on *A* induces an automorphism of *Y*, and then *Aut*(*Y*) acts transitively on A_1. Similarly, *Aut*(*Y*) acts transitively on A_2. X_1 and X_2 are two orbits of *Aut*(*G*). By (1), there exists $\sigma_i \in Aut(G)(i=1,\cdots,m)$, such that

$$V(G)=\bigcup_{i=1}^{m}\sigma_i(A).$$

Since *Aut*(*G*) has two orbits X_1 and X_2, for any $1 \leq i,j \leq m$ and $i \neq j$, $\sigma_i(A_1)\bigcap \sigma_j(A_1)=\varnothing$ and $\sigma_i(A_1),\sigma_j(A_1)\subseteq X_1$. Thus, we have $X_1=\bigcup_{i=1}^{m}\sigma_i(A_1)$, $X_2=\bigcup_{i=1}^{m}\sigma_i(A_2)$, and $|X_i|=m|A_i|(i=1,2)$. Thus *Y* is a (p',q')-half vertex transitive graph, where $3 \leq p' \leq p-1, 3 \leq q' \leq q-1$ (by Lemma 2.4).

Lemma 2.8 ([17]) *A cyclically optimal graph is not super cyclically edge-connected if and only if it has a super atom.*

Lemma 2.9 *Let G be a connected* (*p*,*q*)*-half vertex transitive graph with bipartition* $X_1 \bigcup X_2, \delta(G)\geq 4$ *and girth* $g(G)$. *Suppose A is a super atom of G and* $Y=G[A]$. *If G is* λ_c*-optimal but not super-*λ_c, *then*

(1) $V(G)$ is a disjoint union of distinct super atoms;

(2) Y is a (p',q')-half vertex transitive graph, where $3 \leq p' \leq p-1, 3 \leq q' \leq q-1$

With a similar argument as the proof of Lemma 2.7, we can prove it.

3. Super-λ_c Half Vertex Transitive Graphs

Theorem 3.1 *Let G be a connected* (*p*,*q*)*-half vertex transitive graph with bipartition* $X_1 \bigcup X_2$, $\delta(G)\geq 4$ *and girth* $g \geq 6$, *then G is* λ_c*-optimal.*

Proof. By Lemma 2.1, *G* is cyclically separable. Suppose *G* is not λ_c-optimal. By Lemma 2.2, every atom is impimitive block. Let A be a atom of G, by Lemma 2.3, $G[A]$ is half-vertex transitive. Let $A_1=A\bigcap X_1$ and $A_2=A\bigcap X_2$, then $A=A_1\bigcup A_2$. Suppose $G[A]$ is (p',q') by Lemma 2.4 (2), $p',q'\geq 3$. Let *C* be a shortest cycle of $G[A]$. Then by Lemma 2.4 (2) and Lemma 2.5, $G[A]$ contains two disjoint cycles, and $[V(C),A-V(C)]$ is s cyclic edge-cut. Clearly, $|A-V(C)|\geq \omega_A(C)$ since no two vertices of *C* have common neighbor in $A-V(C)$. Then,

$$\begin{aligned}
\omega(A) &= (p-p')|A_1|+(q-q')|A_2| \\
&= (p-p')|A_1-V_1(C)|+(q-q')|A_2-V_2(C)| \\
&\quad +(p-p')|V_1(C)|+(q-q')|V_2(C)| \\
&\geq |A_1-V_1(C)|+|A_2-V_2(C)| \\
&\quad +(p-p')|V_1(C)|+(q-q')|V_2(C)| \\
&= |A-V(C)|+(p-p')|V_1(C)|+(q-q')|V_2(C)| \\
&\geq \omega_{G[A]}(C)+(p-p')|V_1(C)|+(q-q')|V_2(C)| \\
&= \omega_G(C).
\end{aligned}$$

a contradiction.

Theorem 3.2 *Let G be a connected* (p,q)*-half vertex transitive graph with bipartition* $X_1 \bigcup X_2$, $\delta(G)\geq 4$ *and girth* $g \geq 6$, *then G is super-*λ_c.

Proof. By Theorem 3.1, G is λ_c-optimal. Suppose *G* is not super-λ_c. By Lemma 2.8, *G* has a super atom. By Lemma 2.9, every super atom is impimitive block. Let A be a super atom of *G*, by Lemma 2.3, $G[A]$ is half-vertex transitive. Let $A_1=A\bigcap X_1$ and $A_2=A\bigcap X_2$, then $A=A_1\bigcup A_2$. Suppose $G[A]$ is (p',q') by Lemma 2.4 (2), $p',q'\geq 3$. Let *C* be a shortest cycle of $G[A]$. With a similar proof as Theorem 3.1, we can get $\omega(A)\geq \omega_G(C)$, a contradiction.

4. Acknowledgements

We would like to appreciate the anonymous referees for the valuable suggestions which help us a lot in refining the presentation of this paper.

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