Applied Mathematics, 2013, 4, 271-278
http://dx.doi.org/10.4236/am.2013.42041 Published Online February 2013 (http://www.scirp.org/journal/am)

Bifurcation Analysis of Homoclinic Flips at Principal Eigenvalues Resonance*

Tiansi Zhang[1], **Deming Zhu**[2]
[1]College of Science, University of Shanghai for Science and Technology, Shanghai, China
[2]Department of Mathematics, East China Normal University, Shanghai, China
Email: zhangts1209@163.com

Received December 15, 2012; revised January 15, 2013; accepted January 22, 2013

ABSTRACT

One orbit flip and two inclination flips bifurcation is considered with resonant principal eigenvalues. We introduce a local active coordinate system to establish bifurcation equation and obtain the conditions when the original homoclinic orbit is kept or broken. We also prove the existence and the existence regions of double 1-periodic orbit bifurcation. Moreover, the complicated homoclinic-doubling bifurcations are found and expressed approximately, and are well located.

Keywords: Orbit Flip; Inclination Flips; Homoclinic Orbit; Resonance

1. Introduction

Homoclinic bifurcations have been comprehensively investigated from the initial work of Silnikov in [1] who gave a detailed study of a system which permits an orbit homoclinic to a saddle-focus. After that many flips cases attract researcher's interests, including resonant eigenvalues case in [2], orbit flips in [3,4], inclination flips in [5-7], and also resonant homoclinic flips in [8-11]. In these cases homoclinic-doubling bifurcation has been expensively studied, which is a codimension-two transition from an n-homoclinic to a 2n-homoclinic orbit. Some applications of these cases may be referred to a model for electro-chemical oscillators, the FitzHugh-Nagumo nerve-axon equations [12], a Shimitzu-Morioka equation for convection instabilities [13], and a Hodgkin-Huxley model of thermally sensitive neurons [14], etc.

More recently, the flip of heterodimensional cycles or accompanied by transcritical bifurcation is got attention, see [15-17], the double and triple periodic orbit bifurcation are proved to exist, and also some coexistence conditions for the homoclinic orbit and the periodic orbit. But the research is not concerned with multiple flips. While multiple cases may have more complicated bifurcation behaviors and even chaos, it is necessary to give a deep study. This paper produces mainly a theoretical study of homoclinic bifurcation with one orbit flip and two inclination flips, which can take place at least in a four-dimensional system. Compared with the above work mentioned, our problem has higher codimension with resonant, and we get not only the existence of 1-periodic orbit, 1-homoclinic orbit, and double periodic orbit, but also the 2^n-homoclinic orbit and their corresponding bifurcation surfaces.

In the present context, we consider the following C^r system

$$\dot{z} = f(z) + g(z,\mu), \tag{1.1}$$

and its unperturbed system

$$\dot{z} = f(z), \tag{1.2}$$

where $r \geq 6$, $z \in \mathbb{R}^4$, $\mu \in \mathbb{R}^l, l \geq 4, 0 < |\mu| \ll 1, f(0) = 0,$ $g(0,\mu) = g(z,0) = 0$.

Hypothesis

We assume that system (1.2) has a homoclinic orbit $\Gamma = \{z = \gamma(t) : t \in \mathbb{R}, \gamma(\pm\infty) = 0\}$ to an equilibrium $z = 0$, which is hyperbolic and has two negative and two positive eigenvalues, denoted by $\lambda_1, \lambda_2, -\rho_1, -\rho_2$, and additionally $\lambda_2 > \lambda_1 > 0 > -\rho_1 > -\rho_2$. Set W^s (resp. W^{ss}) and W^u (resp. W^{uu}) the stable (resp. strong stable) manifold and unstable (resp. strong unstable) manifold of the equilibrium $z = 0$, respectively. Now we further make three assumptions:

(H1) (Resonance) $\lambda_1(\mu) \equiv \rho_1(\mu)$ for $|\mu| \ll 1$, where $\lambda_1(0) = \lambda_1$ and $\rho_1(0) = \rho_1$.

(H2) (Orbit flip) Define $e^+ = \lim_{t\to-\infty} \dot{\gamma}(t)/|\dot{\gamma}(t)|$, $e_s^- = \lim_{t\to+\infty} \dot{\gamma}(t)/|\dot{\gamma}(t)|$, then $e^+ \in T_0W^u$ and $e_s^- \in T_0W^{ss}$

*Project supported by National Natural Science Foundation of China (Grant: 11126097) and by SRF for ROCS, SEM.

are unit eigenvectors corresponding to λ_1 and $-\rho_2$ respectively, where T_0W^u is the tangent space of the corresponding manifold W^u at the saddle $z=0$, and the similar meaning for T_0W^{ss}.

(H3) (Inclination flips) Denote by e_u^+ and e^- the unit eigenvectors corresponding to λ_2 and $-\rho_1$ respectively, let

$$T_{\gamma(t)}W^u \to span\{e_s^-, e^+\} \text{ as } t\to+\infty, T_{\gamma(t)}W^s \to span\{e^-, e^+\} \text{ as } t\to-\infty.$$

The paper is organized as follows. In Section 2 we will construct the Poincaré map by the method used in [18] to get the associated successor function. In Section 3, we first establish bifurcation equation. Then a delicate study shows our main results about the existence of double 1-periodic orbit, 1-homoclinic orbit and also 2^n-homoclinic orbit. The last section gives a conclusion of the work.

2. Two Normal Forms and Successor Function

From the above hypotheses, the normal form theory provides a C^{r-4} system as follows after four successive C^r to C^{r-3} transformations in U (see [10,11,18])

$$\begin{aligned}\dot{x} &= \left[\lambda_1(\mu)+a(\mu)xy+o(|xy|)\right]x + O(u)\left[O(x^2y)+O(v)\right],\\ \dot{y} &= \left[-\rho_1(\mu)+b(\mu)xy+o(|xy|)\right]y + O(v)\left[O(xy^2)+O(\mu)\right],\\ \dot{u} &= \left[\lambda_2(\mu)+c(\mu)xy+o(|xy|)\right]u+x^2H_1(x,y,v),\\ \dot{v} &= \left[-\rho_2(\mu)+d(\mu)xy+o(|xy|)\right]v+y^2H_2(x,y,u),\end{aligned} \tag{2.1}$$

with the assumption

(H4) $H_1(x,0,0)=0, H_2(0,y,0)=0$.

Indeed we have

$$\begin{aligned}x^2H_1(x,y,v) &= a_1x^{2+k_1}y^{k_2}+a_2x^{2+k_3}v^{k_4} + a_3x^{2+k_5}y^{k_6}v^{k_7}+h.o.t.,\\ y^2H_2(x,y,u) &= b_1y^{2+l_1}x^{l_2}+b_2y^{2+l_3}u^{l_4} + b_3y^{2+l_5}x^{l_6}u^{l_7}+h.o.t.,\end{aligned}$$

where

$$2+k_i-\frac{\lambda_2}{\lambda_1}>\max\left\{\frac{\rho_2}{\lambda_1},2\right\}, i=1,3,5,$$

$$k_2>\max\left\{\frac{\rho_2}{\lambda_1},2\right\}, k_4>\max\left\{\frac{2\lambda_1}{\rho_2},1\right\}$$

and

$$k_6+k_7\frac{\rho_2}{\rho_1}>\max\left\{\frac{\rho_2}{\rho_1},\frac{2\lambda_1}{\rho_1}\right\},$$

$$2+l_i>\frac{\rho_2}{\lambda_1}, i=1,3,5, l_2,l_4>0$$

and

$$l_6+l_7\frac{\lambda_2}{\lambda_1}>0, \lambda_1(0)=\lambda_1, \lambda_2(0)=\lambda_2, \rho_1(0)=\rho_1$$

and

$$\rho_2(0)=\rho_2, a(\mu), b(\mu), c(\mu)$$

and $d(\mu)$ are parameters depending on μ. Notice that we have straightened the corresponding invariant manifolds. So it is possible to choose some moment T, such that $\gamma(-T)=\{\delta,0,0,0\}$ and $\gamma(T)=\{0,0,0,\delta\}$, where δ is small enough and

$$\{(x,y,u,v): |x|,|y|,|u|,|v|<2\delta\}\subset U.$$

Now we turn to consider the linear variational system and its adjoint system

$$\dot{z}=Df(\gamma(t))z, \tag{2.2}$$

$$\dot{z}=-(Df(\gamma(t)))^* z. \tag{2.3}$$

First we introduce a lemma, see [10,11]

Lemma 2.1 *There exists a fundamental solution matrix* $Z(t)=(z_1(t),z_2(t),z_3(t),z_4(t))$ *of system* (2.2) *satisfying*

$$Z(-T)=\begin{pmatrix} w_{11} & w_{21} & 0 & w_{41}\\ 0 & 0 & 0 & w_{42}\\ w_{13} & 0 & 1 & w_{43}\\ w_{14} & 0 & 0 & w_{44}\end{pmatrix},$$

$$Z(T)=\begin{pmatrix} 0 & 0 & w_{31} & 0\\ w_{12} & 0 & w_{32} & 1\\ 1 & 0 & w_{33} & 0\\ 0 & 1 & w_{34} & 0\end{pmatrix},$$

where

$$\begin{aligned}z_1(t) &\in \left(T_{\gamma(t)}W^u\right)^c \cap \left(T_{\gamma(t)}W^s\right)^c,\\ z_2(t) &= -\dot{\gamma}(t)/|\dot{\gamma}(T)| \in T_{\gamma(t)}W^u \cap T_{\gamma(t)}W^s,\\ z_3(t) &\in T_{\gamma(t)}W^u\end{aligned}$$

and $z_4(t)\in T_{\gamma(t)}W^s$, and $w_{14}w_{21}w_{31}w_{42}\neq 0, w_{21}<0$.

Remark 2.1 The matrix $(Z^{-1}(t))^*$ is a fundamental solution matrix of system (2.3), denote by

$$\Phi(t)=(\phi_1(t),\phi_2(t),\phi_3(t),\phi_4(t))=(Z^{-1}(t))^*,$$

then

$$\phi_1(t) \in \left(T_{\gamma(t)} W^u\right)^c \cap \left(T_{\gamma(t)} W^s\right)^c$$

is bounded and tends to zero exponentially as $|t| \to +\infty$ due to $\langle \phi_1(t), z_1(t) \rangle = 1$ and $z_1(t)$ tends exponentially to infinity.

Let

$$\begin{aligned} s(t) &\triangleq \gamma(t) + Z(t) N^* \\ &= \gamma(t) + z_1(t) n_1 + z_3(t) n_3 + z_4(t) n_4, \end{aligned} \tag{2.4}$$

where $N = (n_1, 0, n_3, n_4)$. We can well regard $(z_1(t), z_2(t), z_3(t), z_4(t))$ as a new local coordinate sys- tem along Γ, and choose

$$S_0 = \left\{ z = s(T) : |x|, |y|, |u|, |v| < 2\delta \right\} \subset U,$$
$$S_1 = \left\{ z = s(-T) : |x|, |y|, |u|, |v| < 2\delta \right\} \subset U$$

as the cross sections of Γ at $t = T$ and $t = -T$ respectively. Under the transformation of (2.4), system (1.1) becomes

$$\dot{n}_i = \phi_i^*(t) g_\mu(\gamma(t), 0) \mu + h.o.t., \; i = 1,3,4.$$

A simple integrating of both sides from $-T$ to T of the above equation, we further achieve

$$n_i(T) = n_i(-T) + M_i \mu + h.o.t., \; i = 1,3,4, \tag{2.5}$$

where

$$M_i = \int_{-T}^{T} \phi_i^*(t) g_\mu(\gamma(t), 0) \mathrm{d}t, i = 1,3,4$$

are the Melnikov vectors (see [18]).

Lemma 2.2

$$M_1 = \int_{-T}^{T} \phi_1^*(t) g_\mu(\gamma(t), 0) \mathrm{d}t = \int_{-\infty}^{+\infty} \phi_1^*(t) g_\mu(\gamma(t), 0) \mathrm{d}t.$$

Actually a regular map is given by (2.5) as (see **Figure 1(a)**)

$$\begin{aligned} &F_1 : S_1 \to S_0; (n_1(-T), 0, n_3(-T), n_4(-T)) \\ &\mapsto (n_1(T), 0, n_3(T), n_4(T)). \end{aligned}$$

But this map is established in the new coordinate system, so we should look for the relationship between two coordinate systems. Set

$$s(T) = q_{2j}\left(x_{2j}, y_{2j}, u_{2j}, v_{2j}\right) \in S_0,$$
$$s(-T) = q_{2j+1}\left(x_{2j+1}, y_{2j+1}, u_{2j+1}, v_{2j+1}\right) \in S_1$$

and

$$N_{2j}(T) = \left(n_{2j,1}, 0, n_{2j,3}, n_{2j,4}\right),$$
$$N_{2j+1}(-T) = \left(n_{2j+1,1}, 0, n_{2j+1,3}, n_{2j+1,4}\right),$$
$$j = 0,1,2\cdots$$

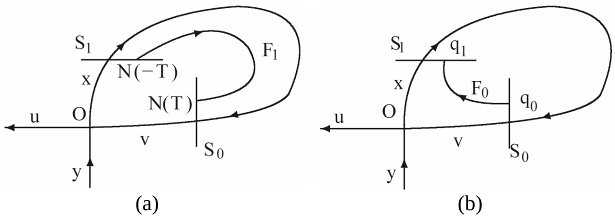

Figure 1. Transition maps. (a) F_1: $S_1 \to S_0$; (b) F_0: $S_0 \to S_1$.

Take $t = -T, T$ respectively in (2.4), we have

$$\begin{aligned} n_{2j,1} &= u_{2j} - w_{33} w_{31}^{-1} x_{2j}, \\ n_{2j,3} &= w_{31}^{-1} x_{2j}, \\ n_{2j,4} &= y_{2j} - w_{12} u_{2j} + \left(w_{12} w_{33} - w_{32}\right) w_{31}^{-1} x_{2j}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} n_{2j+1,1} &= w_{14}^{-1} v_{2j+1} - w_{44} w_{14}^{-1} w_{42}^{-1} y_{2j+1}, \\ n_{2j+1,3} &= u_{2j+1} - w_{13} w_{14}^{-1} v_{2j+1} \\ &\quad + \left(w_{13} w_{44} w_{14}^{-1} - w_{43}\right) w_{42}^{-1} y_{2j+1}, \\ n_{2j+1,4} &= w_{42}^{-1} y_{2j+1}, \end{aligned} \tag{2.7}$$

and

$$x_{2j+1} \approx \delta, v_{2j} \approx \delta. \tag{2.8}$$

Next, we start to set up a singular map

$$F_0 : S_0 \to S_1; q_0(x_0, y_0, u_0, v_0) \mapsto q_1(x_1, y_1, u_1, v_1)$$

(see **Figure 1(b)**) induced by the solutions of system (2.1) in the neighborhood U, for example

$$\begin{aligned} &x(t) \\ &= \mathrm{e}^{\lambda_1(\mu)(t-T-\tau)} \left\{ x_1 + \int_{T+\tau}^{t} a(\mu) \mathrm{e}^{-\lambda_1(\mu)(s-T-\tau)} x^2 y \right\} \mathrm{d}s + h.o.t. \\ &= \mathrm{e}^{\lambda_1(\mu)(t-T-\tau)} x_1 \\ &+ a(\mu) \int_{T+\tau}^{t} \mathrm{e}^{\lambda_1(\mu)(t-s)} \mathrm{e}^{2\lambda_1(\mu)(s-T-\tau)} x_1^2 \mathrm{e}^{-\rho_1(\mu)(s-T)} y_0 \mathrm{d}s + h.o.t., \end{aligned}$$

where τ is the time going from $q_0 \in S_0$ to $q_1 \in S_1$. Denote the Silnikov time $s = \mathrm{e}^{-\lambda_1(\mu)\tau}$, then there is

$$x_0 \triangleq x(T) = s x_1 + O\left(x_1^2 y_0 s^2 \ln s\right).$$

Similarly, there are

$$\begin{aligned} y_1 &= y(T+\tau) = s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}} y_0 + O\left(x_1 y_0^2 s^2 \ln s\right), \\ u_0 &= u(T) = s^{\frac{\lambda_2(\mu)}{\lambda_1(\mu)}} u_1 + O\left(x_1 y_0 u_1 s^{\frac{\lambda_2(\mu)}{\lambda_1(\mu)}+1} \ln s \right), \\ v_1 &= v(T+\tau) = s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}} v_0 + O\left(x_1 y_0 v_0 s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}+1} \ln s \right). \end{aligned} \tag{2.9}$$

With Equations (2.6)-(2.9), Equation (2.5) well defines the Poincaré map $F \triangleq F_1 \circ F_0$,

$$n_{21} = w_{14}^{-1}\delta s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}} - w_{44}w_{14}^{-1}w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0 + M_1\mu + h.o.t.,$$
$$n_{23} = u_1 - w_{13}w_{14}^{-1}\delta s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}} + \left(w_{13}w_{44}w_{14}^{-1} - w_{43}\right)w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0 + M_3\mu + h.o.t.,$$
$$n_{24} = w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0 + M_4\mu + h.o.t..$$

The above fact enables one to achieve the associated successor function $G(s,u_1,y_0) = (G_1,G_3,G_4) = F(q_0) - q_0$ as follows:

$$\begin{aligned} G_1 &= w_{14}^{-1}\delta s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}} - u_1 s^{\frac{\lambda_2(\mu)}{\lambda_1(\mu)}} + w_{33}w_{31}^{-1}\delta s \\ &\quad - w_{44}w_{14}^{-1}w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0 + M_1\mu + h.o.t., \\ G_3 &= u_1 - w_{13}w_{14}^{-1}\delta s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}} - w_{31}^{-1}\delta s \\ &\quad + \left(w_{13}w_{44}w_{14}^{-1} - w_{43}\right)w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0 + M_3\mu + h.o.t., \\ G_4 &= w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0 - y_0 + w_{12}s^{\frac{\lambda_2(\mu)}{\lambda_1(\mu)}}u_1 \\ &\quad + \left(w_{32} - w_{12}w_{33}\right)w_{31}^{-1}\delta s + M_4\mu + h.o.t.. \end{aligned} \tag{2.10}$$

3. Main Results

To begin the bifurcation study, $G_3 = 0$ and $G_4 = 0$ first give

$$u_1 = w_{31}^{-1}\delta s - M_3\mu + h.o.t.,$$
$$y_0 = \left(w_{32} - w_{33}w_{12}\right)w_{31}^{-1}\delta s + M_4\mu + h.o.t..$$

Substitute them into $G_1 = 0$, we get

$$\begin{aligned} F(s,\mu) &\equiv -w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s \\ &\quad - w_{32}w_{44}\left(w_{14}w_{31}w_{42}\right)^{-1}\delta s^2 \\ &\quad + w_{33}w_{31}^{-1}\delta s + w_{14}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} - w_{31}^{-1}\delta s^{\frac{\lambda_2}{\lambda_1}+1} \\ &\quad + M_3\mu s^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + h.o.t. = 0, \end{aligned} \tag{3.1}$$

this is the bifurcation equation. Here we have omitted the parameter μ in $\lambda_i(\mu)$ and $\rho_i(\mu)$, and replaced the exponent ρ_1/λ_1 by one owing to (H1) for concision.

Set $Q = (s,u_1,y_0), \tilde{G} = \partial(G_1,G_3,G_4)/\partial Q$, we find that, when $w_{33} \neq 0$,

$$\det \tilde{G}\Big|_{\substack{Q=0\\ \mu=0}} = \begin{vmatrix} w_{33}w_{31}^{-1}\delta & 0 & 0 \\ -w_{31}^{-1}\delta & 1 & 0 \\ \left(w_{32} - w_{33}w_{12}\right)w_{31}^{-1}\delta & 0 & -1 \end{vmatrix} \neq 0.$$

The implicit function theorem reveals that $G = 0$ has a unique solution

$$s = s(\mu), u_1 = u_1(\mu), y_0 = y_0(\mu),$$

satisfying $s(0) = 0, u_1(0) = 0, y_0(0) = 0.$ So system (1.1) has a unique periodic orbit as $s > 0$ or a unique homoclinic orbit as $s = 0$, and they do not coexist. Furthermore, $F(s,\mu) = 0$ has explicitly a sufficiently small positive solution $s = -\delta^{-1}w_{33}^{-1}w_{31}M_1\mu + h.o.t.$ if $w_{31}w_{33}M_1\mu < 0$. On the other hand, it has a solution $s = 0$ when $\mu \in H^1 \triangleq \{\mu : M_1\mu + h.o.t. = 0\}$, so we have

Theorem 3.1 *Suppose that* $M_1 \neq 0$ *and* $w_{33} \neq 0$ *hold, then system* (1.1) *has at most one* 1*-periodic orbit or one* 1*-homoclinic orbit in the neighborhood of* Γ*. Moreover an* 1*-periodic orbit exists* (*resp. does not exist*) *as* μ *in the region defined by* $w_{31}w_{33}M_1\mu < 0$ (*resp.* > 0) *and an* 1*-homoclinic orbit exists as* $\mu \in H^1$*, but they do not coexist.*

In the following stage, we try to look for bifurcations according to the case $2\lambda_1 > \lambda_2 > \rho_2$ for $w_{33} = 0$.

To begin with we divide (3.1) into two parts:

$$P(s,\mu) \triangleq w_{44}w_{42}^{-1}\delta^{-1}M_4\mu s - w_{14}\delta^{-1}M_1\mu + h.o.t.,$$
$$Q(s,\mu) \triangleq s^{\frac{\rho_2}{\lambda_1}} + w_{14}\delta^{-1}M_3\mu s^{\frac{\lambda_2}{\lambda_1}} + h.o.t..$$

Therefore $F(s,\mu) = w_{14}^{-1}\delta\left(Q(s,\mu) - P(s,\mu)\right)$, where $W = P(s,\mu)$ is a line and $W = Q(s,\mu)$ is a curve with $Q(0,\mu) = 0$ according to the variable s.

Theorem 3.2 *Suppose that* $\operatorname{Rank}(M_1,M_4) = 2, 2\lambda_1 > \lambda_2 > \rho_2, w_{33} = 0, w_{14}M_1\mu > 0$ *and* $w_{42}w_{44}M_4\mu > 0$*, system* (1.1) *then has a unique double* 1*-periodic orbit near* Γ*, and two* (*resp. not any*) 1*-periodic orbits near* Γ *when* μ *lies on the side of* SN^1 *which points to the direction* $-(\operatorname{sgn} w_{14})M_1$ (*resp. in the opposite direction of* SN^1)*. The corresponding double* 1*-periodic orbit bifurcation surface* SN^1 *is*

$$w_{14}M_1\mu = \frac{\rho_2 - \lambda_1}{\lambda_1}\delta\left(\frac{\lambda_1 w_{44}M_4\mu}{\rho_2\delta w_{42}}\right)^{\frac{\rho_2}{\rho_2-\lambda_1}} + h.o.t.$$

with the normal vector M_1 *at* $\mu = 0$.

Proof Consider equations $P(s,\mu) = Q(s,\mu)$, $P'(s,\mu) = Q'(s,\mu)$ and $P''(s,\mu) \neq Q''(s,\mu)$, that is,

$$\begin{aligned} &w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s - M_1\mu = w_{14}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} + M_3\mu s^{\frac{\lambda_2}{\lambda_1}} + h.o.t., \\ &w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu = \rho_2\lambda_1^{-1}w_{14}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}-1} \\ &\quad + \lambda_2\lambda_1^{-1}M_3\mu s^{\frac{\lambda_2}{\lambda_1}-1} + h.o.t., \\ &0 \neq \rho_2\left(\rho_2 - \lambda_1\right)w_{14}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}-2} \\ &\quad + \lambda_2\left(\lambda_2 - \lambda_1\right)M_3\mu s^{\frac{\lambda_2}{\lambda_1}-2} + h.o.t.. \end{aligned} \tag{3.2}$$

The second equation permits a solution

$$s_* = \left(\frac{\lambda_1 w_{44} M_4 \mu}{\rho_2 \delta w_{42}}\right)^{\frac{\lambda_1}{\rho_2-\lambda_1}} + h.o.t.$$

as

$$w_{42} w_{44} M_4 \mu > 0 .$$

Substituting it into the first equation of (3.2), we obtain the tangency condition, which corresponds to the existence of the double periodic orbit bifurcation surface SN^1 situated in the region $w_{42} w_{44} M_4 \mu > 0$ and $w_{14} M_1 \mu > 0$. Notice that, when the tangency takes place, the line $W = P(s,\mu)$ lies under the curve $W = Q(s,\mu)$. So if $-w_{14} M_1 \mu$ increases (resp. decreases), the line must intersects the curve at two (resp. no) sufficiently small positive points. Now the proof is complete.

Theorem 3.3 *Suppose that* $2\lambda_1 > \lambda_2 > \rho_2$ *and* $w_{33} = 0$ *are true, then there exists two codimension-one hypersurfaces*

$$H^1 = \left\{\mu : P(0,\mu) = 0, P'(0,\mu) \neq 0, |\mu| \ll 1\right\}$$

and

$$\Sigma = \left\{\mu : P'(0,\mu) = 0, P(0,\mu) \neq 0, |\mu| \ll 1\right\},$$

such that

System (1.1) has only one 1-homoclinic orbit near Γ as $\mu \in H^1$ and $w_{42} w_{44} M_4 \mu < 0$;

System (1.1) has only one 1-periodic orbit near Γ as $\mu \in \Sigma$ and $w_{14} M_1 \mu < 0$;

System (1.1) has exactly one 1-homoclinic orbit and one 1-periodic orbit near Γ as $\mu \in H^1$ and $w_{42} w_{44} M_4 \mu > 0$;

System (1.1) has not any 1-periodic orbit or 1-homoclinic orbit as $\mu \in \Sigma$ and $w_{14} M_1 \mu > 0$.

Proof When $\mu \in H^1$, we have at once

$$F(0,\mu) = -P(0,\mu) = 0$$

and $M_4 \mu \neq 0$, therefore

$$F(s,\mu) = s\left(-w_{44} w_{42}^{-1} w_{14}^{-1} M_4 \mu + w_{14}^{-1} \delta s^{\frac{\rho_2-\lambda_1}{\lambda_1}} + M_3 \mu s^{\frac{\lambda_2-\lambda_1}{\lambda_1}} + h.o.t.\right) = 0$$

has always two nonnegative solutions $s_1 = 0$ and

$$s_2 = \left(\frac{w_{44} M_4 \mu}{w_{42} \delta}\right)^{\frac{\lambda_1}{\rho_2-\lambda_1}} + h.o.t.$$

for $w_{44} w_{42} M_4 \mu > 0$ or has only a zero solution $s_1 = 0$ for $w_{44} w_{42} M_4 \mu < 0$. If $\mu \in \Sigma$, there is, on the contrary, $M_4 \mu = 0$ but $P(0,\mu) \neq 0$, apparently the line $W = P(s,\mu)$ is horizontal. So $F(s,\mu) = 0$ has a solution

$$s_0 = \left(-\delta^{-1} w_{14} M_1 \mu\right)^{\frac{\lambda_1}{\rho_2}} + h.o.t.$$

if and only if $w_{14} M_1 \mu < 0$. The proof is complete.

From the above proof, we see that if the line $W = P(s,\mu)$ has a small positive section with the W-axis or small positive slope, then there exists a small positive $\tilde{s}$ such that $P(\tilde{s},\mu) = Q(\tilde{s},\mu)$. Thus the following corollary is valid, which is a complement of Theorem 3.2.

Corollary 3.4 *Assume that the hypotheses of Theorem* 3.2 *are valid, system* (1.1) *then has a unique* 1*-periodic orbit near* Γ *as* μ *is situated in the region defined by* $w_{14} M_1 \mu < 0$ *and* $w_{42} w_{44} M_4 \mu < 0$ *or* $w_{14} M_1 \mu < 0$, $w_{42} w_{44} M_4 \mu > 0$ *and* $0 \ll |\mu| \ll 1$*; has not any* 1*-periodic orbit as* $w_{14} M_1 \mu > 0$ *and* $w_{42} w_{44} M_4 \mu < 0$.

Notice that in Theorem 3.3, system (1.1) has a codimension-1 1-homoclinic orbit, see **Figure 2(a)**, that is the existing homoclinic orbit has no longer orbit flip. But an orbit flip homoclinic orbit could still exist if

$$y_0 = M_4 \mu + h.o.t. = 0,$$

see **Figure 2(b)**.

Corollary 3.5 *Suppose that* $2\lambda_1 > \lambda_2 > \rho_2$ *and* $w_{33} = 0$ *hold, system* (1.1) *has a codimension*-2 *orbit-flip homoclinic orbit as*

$$\mu \in \left\{\mu : F(0,\mu) = M_1 \mu + h.o.t. = 0, M_4 \mu + h.o.t. = 0\right\}.$$

Now we turn to study the homoclinic doubling bifurcations. To begin with we look for the 2-homoclinic orbit and 2-periodic orbit bifurcation surfaces. Reset τ_1 and τ_2 be the time going from $q_0(x_0,y_0,u_0,v_0) \in S_0$ to $q_1(x_1,y_1,u_1,v_1) \in S_1$ and from $q_2(x_2,y_2,u_2,v_2) \in S_0$ to $q_3(x_3,y_3,u_3,v_3) \in S_1$ respectively, $s_1 = \mathrm{e}^{-\lambda_1 \tau_1}, s_2 = \mathrm{e}^{-\lambda_1 \tau_2}$, and

$$F_0(q_0) = q_1, F_1(q_1) = q_2, F_0(q_2) = q_3, F_1(q_3) = q_4 = q_0.$$

Then recall the process of the establishment of (2.10), similarly we may get the associated second returning successor function $F \circ F(q_0) - q_0$ expressed as

$$G^2(s_1,s_2,u_1,u_3,y_0,y_2) = \left(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4\right):$$

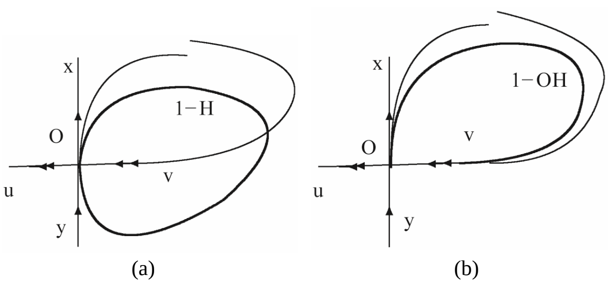

Figure 2. 1-homoclinic orbit (1-H) and (1-OH). (a) $\mu \in \Sigma 1$; (b) $F(0, \mu) = 0, y_0 = M_4\mu + h.o.t. = 0$.

$$G_1^1 = w_{14}^{-1}\delta s_1^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}s_1^{\frac{\rho_1}{\lambda_1}}y_0 - u_3 s_2^{\frac{\lambda_2}{\lambda_1}} + w_{33}w_{31}^{-1}\delta s_2 + M_1\mu + h.o.t.,$$

$$G_1^3 = u_1 - w_{13}w_{14}^{-1}\delta s_1^{\frac{\rho_2}{\lambda_1}} - w_{31}^{-1}\delta s_2 + \left(w_{13}w_{44}w_{14}^{-1} - w_{43}\right)w_{42}^{-1}s_1^{\frac{\rho_1}{\lambda_1}}y_0 + M_3\mu + h.o.t.,$$

$$G_1^4 = w_{42}^{-1}s_1^{\frac{\rho_1}{\lambda_1}}y_0 - y_2 + w_{12}s_2^{\frac{\lambda_2}{\lambda_1}}u_3 + \left(w_{32} - w_{12}w_{33}\right)w_{31}^{-1}\delta s_2 + M_4\mu + h.o.t..$$

$$G_2^1 = w_{14}^{-1}\delta s_2^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}s_2^{\frac{\rho_1}{\lambda_1}}y_2 - u_1 s_1^{\frac{\lambda_2}{\lambda_1}} + w_{33}w_{31}^{-1}\delta s_1 + M_1\mu + h.o.t.,$$

$$G_2^3 = u_3 - w_{13}w_{14}^{-1}\delta s_2^{\frac{\rho_2}{\lambda_1}} - w_{31}^{-1}\delta s_1 + \left(w_{13}w_{44}w_{14}^{-1} - w_{43}\right)w_{42}^{-1}s_2^{\frac{\rho_1}{\lambda_1}}y_2 + M_3\mu + h.o.t.,$$

$$G_2^4 = w_{42}^{-1}s_2^{\frac{\rho_1}{\lambda_1}}y_2 - y_0 + w_{12}s_1^{\frac{\lambda_2}{\lambda_1}}u_1 + \left(w_{32} - w_{12}w_{33}\right)w_{31}^{-1}\delta s_1 + M_4\mu + h.o.t..$$

Eliminating again y_0, u_1, y_2 and u_3 from $G_i^j = 0, i = 1,2$ and $j = 3,4$, and assuming

$$w_{33} = 0, 2\lambda_1 > \lambda_2 > \rho_2$$

we obtain

$$w_{14}^{-1}\delta s_1^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_1 - w_{31}^{-1}\delta s_1 s_2^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_2^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + h.o.t. = 0, \tag{3.3}$$

$$w_{14}^{-1}\delta s_2^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_2 - w_{31}^{-1}\delta s_2 s_1^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_1^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + h.o.t. = 0. \tag{3.4}$$

We know that a 2-homoclinic orbit Γ^2 corresponds to the solution $s_1 = 0$ and $s_2 > 0$ or $s_1 > 0$ and $s_2 = 0$ of (3.3) and (3.4), that means an orbit returns once nearby the singular point in limit time and twice in limitless time. So it is sufficient to seek the small solutions of $s_1 = 0$ and $s_2 > 0$ by symmetry of G^2. Therefore

$$M_3\mu s_2^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + h.o.t. = 0, \tag{3.5}$$

$$s_2^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{42}^{-1}\delta^{-1}M_4\mu s_2 + \delta^{-1}w_{14}M_1\mu + h.o.t. = 0. \tag{3.6}$$

Clearly (3.5) yields

$$s_2 = \left(-\frac{M_1\mu}{M_3\mu}\right)^{\frac{\lambda_1}{\lambda_2}} + h.o.t.$$

for $M_1\mu M_3\mu < 0$, and $|M_1\mu|/|M_3\mu|$ sufficiently small. With this, Equation (3.6) determines a 2-homoclinic orbit bifurcation surface

$$H^2 : |M_1\mu| = \left[-w_{14}\delta^{-1}\operatorname{sgn}(M_1\mu)|M_1\mu|^{1-\frac{\lambda_1}{\lambda_2}}|M_3\mu|^{\frac{\rho_2}{\lambda_2}} + w_{44}w_{42}^{-1}\delta^{-1}M_4\mu|M_3\mu|^{\frac{\rho_2-\lambda_1}{\lambda_2}}\right]^{\frac{\lambda_2}{\rho_2-\lambda_1}} + h.o.t.$$

for $M_1\mu M_3\mu < 0$ and $|M_1\mu| \ll |M_3\mu|$, which has a normal vector M_1 at $\mu = 0$.

Continually, differentiating both sides of (3.3) and (3.4) with respect to μ and for $\mu \in H^2$, we obtain

$$-w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_{1\mu} + \lambda_1^{-1}\lambda_2 M_3\mu s_2^{\frac{\lambda_2}{\lambda_1}-1}s_{2\mu} - w_{31}^{-1}\delta s_{1\mu}s_2^{\frac{\lambda_2}{\lambda_1}} + M_1 + h.o.t. = 0,$$

$$-w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_{2\mu} + \rho_2\lambda_1^{-1}w_{14}^{-1}\delta s_2^{\frac{\rho_2}{\lambda_1}-1}s_{2\mu} + M_1 + h.o.t. = 0.$$

In the region defined by

$$w_{42}w_{44}M_4\mu > 0 \text{ and } |M_1\mu|^{1-\frac{\lambda_1}{\lambda_2}}|M_3\mu|^{\frac{\lambda_1}{\lambda_2}} \ll |M_4\mu|,$$

the 2-homoclinic orbit bifurcation surface H^2 is simplified to be

$$M_1\mu = -\left(\frac{w_{44}M_4\mu}{w_{42}\delta}\right)^{\frac{\lambda_2}{\rho_2-\lambda_1}} M_3\mu + h.o.t..$$

Accordingly

$$s_2 = \left(\frac{w_{44}M_4\mu}{w_{42}\delta}\right)^{\frac{\lambda_1}{\rho_2-\lambda_1}} + h.o.t..$$

Then one may derive

$$s_{1\mu} = \frac{w_{14}w_{42}M_1}{w_{44}M_4\mu} + h.o.t.,$$

$$s_{2\mu} = -\frac{\lambda_1 w_{14}w_{42}M_1}{(\rho_2 - \lambda_1)w_{44}M_4\mu} + h.o.t.,$$

which informs that s_1 increases (resp. decreases) as μ moves along the direction $w_{14}M_1$ (resp. the opposite direction) such that a 2-periodic orbit bifurcates from the 2-homoclinic orbit Γ^2 as μ leaves H^2 for the side pointed by $w_{14}M_1$.

Notice that confined on the surface H^1, (3.3) and (3.4) has a unique positive solution, meanwhile Theorem 3.3 indicates exactly the existence of one 1-periodic orbit when $\mu \in H^1$ for $w_{42}w_{44}M_4\mu > 0$, so there does not exist any 2-periodic orbit when μ is near H^1. Therefore in the region bounded by the surfaces H^2 to H^1, there must exist another bifurcation surface which merges the 1-periodic orbit and the 2-periodic orbit into a new 1-periodic orbit with the different stability from the original one. We call this surface the period-doubling bifurcation surface and denote it by P^2.

The above reasonings can repeat itself many times to find the 2^n-homoclinic orbit bifurcation surface

$$H^{2^n}: M_1\mu = O\left(\left|M_1\mu\right|^{\frac{\lambda_2-\lambda_1}{\rho_2-\lambda_1}}\left|M_3\mu\right|^{\frac{\rho_2}{\rho_2-\lambda_1}} + \left|M_4\mu\right|^{\frac{\lambda_2}{\rho_2-\lambda_1}}\left|M_3\mu\right|\right)$$

in the same region of μ space and simultaneously the presence of period-doubling bifurcation surface P^{2^n} of 2^{n-1}-periodic orbit.

In short, we conclude that:

Theorem 3.6 *Suppose that*

$$\operatorname{Rank}\left(M_1, M_3, M_4\right) = 3, 2\lambda_1 > \lambda_2 > \rho_2$$

and $w_{33} = 0$ *hold, then for*

$$M_1\mu M_3\mu < 0, w_{14}w_{42}w_{44}M_1\mu M_4\mu > 0$$

and $\left|M_1\mu\right| \ll \left|M_3\mu\right|$, *there exists a* 2^n*-homoclinic orbit bifurcation surface* H^{2^n} *with the normal vector* M_1 *at* $\mu = 0$ *and the period-doubling bifurcation surface* P^{2^n} *of* 2^{n-1}*-periodic orbit in the small neighborhood of the origin of* μ *space. Moreover system* (1.1) *has exactly a* 2^n*-homoclinic orbit as* $\mu \in H^{2^n}$ *and a* 2^n*-periodic orbit as* μ *moves away to the side of* H^{2^n} *pointing to the direction* $\left(\operatorname{sgn} w_{14}\right)M_1$ *and none on the other side.*

To well illustrate our results, a bifurcation diagram is drawn in **Figure 3**, where p^k represents a k-periodic orbit.

4. Conclusion

Homoclinic orbits generically occur as a codimension-one phenomenon, while if the genericity conditions are broken, some high codimension instance including the resonant and flips cases, concomitant usually with chaotic behavior, may take place. Homburg and Oldeman studied two kinds of resonant homoclinic flips in [8,9] with unfolding techniques and numerical methods respectively. Zhang in [10,11] continued to research on these problems and gave some theoretical proofs of the existence of n-periodic orbit and n-homoclinic orbit and also their existence regions via the method initially established in [18]. Besides these the flip heterodimensional cycles have also attracted attentions nowadays, see [16]. In this paper, we extend the method to fit a higher codimension case of 3 flips with resonant. With the delicate analysis, the existence of 1-periodic orbit, 1-homoclinic orbit, and double periodic orbit are proven and also the 2^n-homoclinic orbit and their corresponding bifurcation surfaces. With the work, we find the extensive existence of the double periodic orbit bifurcation and the homoclinic-doubling bifurcation, which efficiently advance the development of the flips homoclinic study.

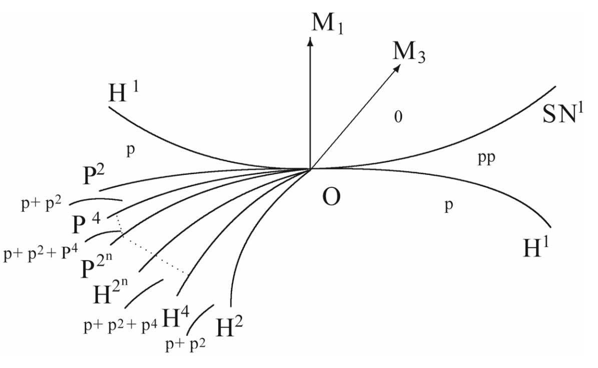

Figure 3. Location of bifurcation surfaces for rank (M_1, M_3, M_4) = 3, w_{33} = 0, $2\lambda_1 > \lambda_2 > \rho_2$.

REFERENCES

[1] L. Silnikov, "A Case of the Existence of a Denumerable Set of Periodic Motions," *Soviet Mathematics Doklady*, Vol. 6, 1965, pp. 163-166.

[2] S. Chow, B. Deng and B. Fiedler, "Homoclinic Bifurcation at Resonant Eigenvalues," *Journal of Dynamics and Differential Equations*, Vol. 2, No. 2, 1990, pp. 177-244. doi:10.1007/BF01057418

[3] B. Sandstede, "Constructing Dynamical Systems Having Homoclinic Bifurcation Points of Codimension Two," *Journal of Dynamics and Differential Equations*, Vol. 9, No. 2, 1997, pp. 269-288. doi:10.1007/BF02219223

[4] C. Morales and M. Pacifico, "Inclination-Flip Homoclinic Orbits Arising from Orbit-Flip," *Nonlinearity*, Vol. 14, No. 2, 2001, pp. 379-393. doi:10.1088/0951-7715/14/2/311

[5] M. Kisaka, H. Kokubu and H. Oka, "Bifurcations to N-Homoclinic Orbits and N-Periodic Orbits in Vector Fields," *Journal of Dynamics and Differential Equations*, Vol. 5, No. 2, 1993, pp. 305-357. doi:10.1007/BF01053164

[6] V. Naudot, "Bifurcations Homoclines des Champes de Vecteurs en Dimension Trois," PH.D. Thesis, l'Université de Bourgogne, Dijon, 1996.

[7] A. Homburg, H. Kokubu and V. Naudot, "Homoclinic-Doubling Cascades," *Archive for Rational Mechanics and Analysis*, Vol. 160, No. 3, 2001, pp. 195-243. doi:10.1007/s002050100159

[8] A. Homburg and B. Krauskopf, "Resonant Homoclinic Flip Bifurcations," *Journal of Dynamics and Differential Equations*, Vol. 12, No. 4, 2000, pp. 807-850. doi:10.1023/A:1009046621861

[9] B. Oldeman, B. Krauskopf and A. Champneys, "Numerical Unfoldings of Codimension-Three Resonant Homoclinic Flip Bifurcations," *Nonlinearity*, Vol. 14, No. 3, 2001,

pp. 597-621. doi:10.1088/0951-7715/14/3/309

[10] T. Zhang and D. Zhu, "Homoclinic Bifurcation of Orbit Flip with Resonant Principal Eigenvalues," *Acta Mathematica Sinica*, Vol. 22, No. 3, 2006, pp. 855-864.

[11] T. Zhang and D. Zhu, "Bifurcations of Homoclinic Orbit Connecting Two Nonleading Eigendirections," *International Journal of Bifurcation and Chaos*, Vol. 17, No. 3, 2007, pp. 823-836. doi:10.1142/S0218127407017574

[12] M. Krupa, B. Sandstede and P. Szmolyan, "Fast and Slow Waves in the FitzHugh-Nagumoequation," *Journal of Differential Equations*, Vol. 133, No. 1, 1997, pp. 49-97. doi:10.1006/jdeq.1996.3198

[13] U. Feudel, A. Neiman, X. Pei, W. Wojtenek, H. Braun, M. Huber and F. Moss, "Homoclinic Bifurcation in a Hodgkin-Huxley Model of Thermally Sensitive Neurons," *Chaos*, Vol. 10, No. 1, 2000, pp. 231-239. doi:10.1063/1.166488

[14] A. Champneys, Kuznetsov and A. Yu, "Numerical Detection and Continuation of Codimension-Two Homoclinic Bifurcation," *International Journal of Bifurcation and Chaos*, Vol. 4, No. 4, 1994, pp. 785-822. doi:10.1142/S0218127494000587

[15] F. Geng and D. Zhu, "Bifurcations of Generic Heteroclinic Loop Accompanied by Transcritical Bifurcation," *International Journal of Bifurcation and Chaos*, Vol. 18, 2008, pp. 1069-1083. doi:10.1142/S0218127408020847

[16] Q. Lu, Z. Qiao, T. Zhang and D. Zhu, "Heterodimensional Cycle Bifurcation with Orbit-Filp," *International Journal of Bifurcation and Chaos*, Vol. 20, No. 2, 2010, pp.491-508. doi:10.1142/S0218127410025569

[17] X. Liu, "Homoclinic Flip Bifurcations Accompanied by Transcritical Bifurcation," *Chinese Annals of Mathematics*, *Series B*, Vol. 32, No. 6, 2011, pp. 905-916. doi:10.1007/s11401-011-0675-y

[18] D. Zhu and Z. Xia, "Bifurcation of Heteroclinic Loops," *Science in China*, *Series A*, Vol. 41, No. 8, 1998, pp. 837-848. doi:10.1007/BF02871667

