

Existence Results on General Integrodifferential Evolution Equations in Banach Space

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ABSTRACT

In this paper we prove the existence of mild solutions of a general class of nonlinear evolution integrodifferential equation in Banach spaces. Based on the resolvent operator and the Schaefer fixed point theorem, a sufficient condition for the existence of general integrodifferential evolution equations is established.

Keywords: Existence; Nonlinear Integrodifferential Evolution Equation; Resolvent Operator; Schaefer's Fixed Point Theorem

1. Introduction

Pazy [1] has discussed the existence and uniqueness of mild, strong and classical solutions of semilinear evolution equations by using semigroup theory. The nonlocal Cauchy problem for the same equation has been studied by Byszewskii [2,3]. Balachandran and Chandrasekaran [4] investigated the nonlocal Cauchy problem for semilinear integrodifferential equation with deviating argument. Balachandran and Park [5] have discussed about the existence of solutions and controllability of nonlinear integrodifferential systems in Banach spaces. Grimmer [6] obtained the representation of solutions of integrodifferential equations by using resolvent operators in a Banach space. Liu [7] discussed the Cauchy problem for integrodifferential evolution equations in abstract spaces and also in [8] he discussed nonautonomous integrodifferential equations.

Lin and Liu [9] studied the nonlocal Cauchy problem for semilinear integrodifferential equations by using resolvent operators. Liu and Ezzinbi [10] investigated nonautonomous integrodifferential equations with nonlocal conditions. Byszewskii and Acka [11] studied the classical solution of nonlinear functional differential equation with time varying delays. There are several papers appeared on the existence of differential and integrodifferential equations in Banach spaces [12,13]. The purpose of this paper is to prove the existence of mild solutions for time varying delay integrodifferential evolution equations with the help of Schaefer's fixed point theorem. The results generalize the results of [14].

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The paper is organized as follows: In Section 2, we give the necessary definition and gave a description of the idea of the proof of the main results formulated and proved in Section 3. Moreover in Section 3, we prove the existence of solution of general integrodifferential evolution equation with nonlocal condition.

2. Preliminaries

Consider the nonlinear delay integrodifferential evolution equation with nonlocal condition of the form

$$\begin{aligned} & \frac{d}{dt} \left[x(t) - G \left(t, x(t), \int_0^t g(t, s, x(s)) ds \right) \right] \\ &= A(t)x(t) + \int_0^t B(t, s)x(s) ds \\ &+ F \left[t, x(t), \int_0^t f_1(t, s, x(s)) ds, \right. \end{aligned} \tag{1}$$

$$\begin{aligned} & \left. \int_0^t f_2(t, s, x(s)) ds, \dots, \int_0^t f_n(t, s, x(s)) ds \right] \\ & t \in J \\ & x(0) + h(x) = x_0 \end{aligned} \tag{2}$$

where $A(t)$ and $B(t,s)$ are closed linear operators on a Banach space X with dense domain $D(A)$ which is independent of t ,

$f_i : J \times J \times X \rightarrow X$, $g : J \times J \times X \rightarrow X$,
 $h : C(J, X) \rightarrow X$, $F : J \times X^{n+1} \rightarrow X$ and
 $G : J \times X \times X \rightarrow X$ are given functions. Here $J = [0, T]$.

We shall make the following conditions:

(H_1) $A(t)$ generates a strongly continuous semigroup of evolution operators.

(H₂) Suppose Y is a Banach space formed from $D(A)$ with the graph norm. $A(t)$ and $B(t,s)$ are closed operators it follows that $A(t)$ and $B(t,s)$ are in the set of bounded linear operators from Y to $X, B(Y,X)$, for $0 \leq t \leq T$ and $0 \leq s \leq t \leq T$, respectively. $A(t)$ and $B(t,s)$ are continuous on $0 \leq t \leq T$ and $0 \leq s \leq t \leq T$, respectively, into $B(Y,X)$.

Definition 2.1. A resolvent operator for (1) and (2) is a bounded operator valued function $R(t,s) \in B(X)$, $0 \leq s \leq t \leq T$, the space of bounded linear operators on X , having the following properties.

(i) $R(t,s)$ is strongly continuous in s and t . $R(t,t)=I$, the identity operator on X . $\|R(t,s)\| \leq Me^{\beta(t-s)}$ $t, s \in J$ and M, β are constants.

(ii) $R(t,s)Y \subset Y, R(t,s)$ is strongly continuous in s and t on Y .

(iii) For $y \in Y, R(t,s)y$ is continuously differentiable in s and t , and for $0 \leq s \leq t \leq T$,

$$\begin{aligned} x(t) &= R(t,0)[x_0 - h(x)] - R(t,0)[G(0,x(0),0)] + G\left(t,x(t), \int_0^t g(t,s,x(s))ds\right) \\ &+ \int_0^t R(t,s)F\left(s,x(s), \int_0^s f_1(s,\tau,x(\tau))d\tau, \dots, \int_0^s f_n(s,\tau,x(\tau))d\tau\right)ds \\ &- \int_0^t R(t,s)A(s)G\left(s,x(s), \int_0^s g(s,\tau,x(\tau))d\tau\right)ds \\ &- \int_0^t \left[\int_s^t R(t,\tau)B(\tau,s)G\left(\tau,x(\tau), \int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau \right] ds \end{aligned}$$

is satisfied.

Schaefer's Theorem [15]. Let E be a normed linear space. Let $F : E \rightarrow E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let

$$\zeta(F) = \{x \in E : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Assume that the following conditions hold:

(H₃) There exists a resolvent operator $R(t,s)$ which is compact and continuous in the uniform operator topology for $t > s$. Further, there exists a constant $M_1 > 0$ such that

$$\|R(t,s)\| \leq M_1.$$

(H₄) The function $h : C(J, X) \rightarrow X$ is continuous and there exists a constant $M_2 > 0$ such that $\|h(x)\| \leq M_2$ for any $x \in X$.

(H₅) For each $t \in J$, the function $F(t, \cdot) : X^{n+1} \rightarrow X$ is continuous and for each $(x_0, x_1, \dots, x_n) \in X^{n+1}$ the function $F(\cdot, x_0, x_1, \dots, x_n) : J \rightarrow X$ is strongly measurable.

(H₆) There exists an integrable function $m_i : J \rightarrow [0, \infty)$ such that $\|f_i(t,s,x)\| \leq m_i(t,s)\Omega_i(\|x\|)$, for any $t, s \in J, x \in X$,

$$\begin{aligned} \frac{\partial}{\partial t} R(t,s)y &= A(t)R(t,s)y \\ &+ \int_s^t B(t,r)R(r,s)ydr, \\ \frac{\partial}{\partial s} R(t,s)y &= -R(t,s)A(s)y \\ &- \int_s^t R(t,r)B(r,s)ydr, \end{aligned}$$

with $\frac{\partial}{\partial t} R(t,s)y$ and $\frac{\partial}{\partial s} R(t,s)y$ are strongly continuous on $0 \leq s \leq t \leq T$. Here $R(t,s)$ can be extracted from the evolution operator of the generator $A(t)$. The resolvent operator is similar to the evolution operator for nonautonomous differential equations in Banach spaces.

Definition 2.2. A continuous function $x(t)$ is said to be a mild solution of the nonlocal Cauchy problems (1) and (2), if

for any $t \in J, x \in X$. where $\Omega_i : [0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function.

(H₇) There exists an integrable function $m_0 : J \times J \rightarrow [0, \infty)$ such that

$$\|g(t,s,x)\| \leq m_0(t,s)\Omega_0(\|x\|), \text{ for any } t, s \in J, x \in X,$$

where $\Omega_0 : [0, \infty) \rightarrow [0, \infty)$ is a continuous nondecreasing function.

(H₈) The function $G : C(J, X) \rightarrow X$ is completely continuous and there exists a constant $0 \leq c_1 \leq 1$ such that

$$\|G(t,x(t),y(t))\| \leq c_1 [\|x(t)\| + \|y(t)\|], \text{ for any } t \in J$$

and is equicontinuous in (J,X)

(H₉) The function $G : C(J, X) \rightarrow X$ is completely continuous and there exists a constant $0 \leq c_2 \leq 1$ such that

$$\|F(t,x(t),y(t))\| \leq c_2 [\|x(t)\| + \|y(t)\|], \text{ for any } t \in J$$

and is equicontinuous in (J,X)

(H₁₀) There are function $H_1(\cdot), H_2(\cdot), \mu_1(\cdot), \mu_2(\cdot) : J \rightarrow [0, \infty]$ such that

$$\|A(t)R(t,s)\| \leq H_1(t)\mu_1(s)$$

$$\|B(t,s)R(t,s)\| \leq H_2(t)\mu_2(s).$$

(H_{11}) The function

$$\hat{m}(t) = \max \left\{ \frac{1}{1-c_1} M_1 c_2, \frac{M_1 c_2}{1-c_1} \sum_{i=1}^n m_i(t, s), \right. \\ \left. \frac{c_1}{1-c_1} m_0(t, s), \frac{c_1}{1-c_1} H_1(t) \mu_1(t), \right. \\ \left. \frac{c_1}{1-c_1} H_1(t) \mu_1(t) m_0(t, s), \right. \\ \left. \frac{c_1}{1-c_1} \int_s^t H_2(s) \mu_2(s) ds, \right. \\ \left. \frac{c_1}{1-c_1} \int_s^t \int_0^s H_1(t) \mu_1(t) m_0(t, s) dt ds \right\} \\ \int_0^T \hat{m}(s) ds < \int_c^\infty \frac{ds}{3s + 2\Omega_0(s) + \sum_{i=1}^n \Omega_i(s)}$$

where $c = \frac{1}{1-c_1} M_1 [\|x_0\| + M_2 + G(0, x(0), 0)]$

3. Existence of Mild Solutions

The main result is as follows.

Theorem 3.1. If the assumptions (H_1)-(H_{11}) are satisfied then the problems (1) and (2) has a mild solution on J .

Proof: Consider the Banach space $Z = C(J, X)$. We establish the existence of a mild solution of the problems (1) and (2) by applying the Schaefer’s fixed point theorem.

First we obtain *a priori* bounds for the operator equation

$$x(t) = \lambda \Phi x(t), \quad 0 < \lambda < 1, \tag{3}$$

where $\Phi : Z \rightarrow Z$ is defined as

$$(\Phi x)(t) = R(t, 0) [x_0 - h(x) - G(0, x(0), 0)] + \int_0^t R(t, s) F \left(s, x(s), \int_0^s f_1(s, \tau, x(\tau)) d\tau, \dots, \int_0^s f_n(s, \tau, x(\tau)) d\tau \right) ds \\ + G \left(t, x(t), \int_0^t g(t, s, x(s)) ds \right) - \int_0^t R(t, s) A(s) G \left(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau \right) ds \\ - \int_0^t \left[\int_s^t R(t, \tau) B(\tau, s) G \left(\tau, x(\tau), \int_0^\tau g(\tau, \zeta, x(\zeta)) d\zeta \right) d\tau \right] ds \tag{4}$$

Then froms (3) and (4) we have

$$\|x(t)\| \leq M_1 [\|x_0\| + M_2 + G(0, x(0), 0)] + M_1 c_2 \int_0^t [\|x(t)\| + \sum_{i=1}^n m_i(t, s) \Omega_i(\|x(s)\|)] ds \\ + c_1 [\|x(t)\| + \int_0^t m_0(t, s) \Omega_0(\|x(s)\|) ds] + c_1 \int_0^t H_1(s) \mu_1(s) [\|x(s)\| + \int_0^s m_0(s, \tau) \Omega_0(\|x(\tau)\|) d\tau] ds \\ + c_1 \int_0^t \left[\int_s^t H_2(\tau) \mu_2(\tau) [\|x(s)\| + \int_0^\tau m_0(\tau, \zeta) \Omega_0(\|x(\zeta)\|) d\zeta] d\tau \right] ds \\ (1-c_1) \|x(t)\| \leq M_1 [\|x_0\| + M_2 + G(0, x(0), 0)] + c_1 \int_0^t m_0(t, s) \Omega_0(\|x(s)\|) ds \\ + M_1 c_2 \int_0^t [\|x(t)\| + \sum_{i=1}^n m_i(t, s) \Omega_i(\|x(s)\|)] ds \\ + c_1 \int_0^t H_1(s) \mu_1(s) [\|x(s)\| + \int_0^s m_0(s, \tau) \Omega_0(\|x(\tau)\|) d\tau] ds \\ + c_1 \int_0^t \left[\int_s^t H_2(\tau) \mu_2(\tau) [\|x(s)\| + \int_0^\tau m_0(\tau, \zeta) \Omega_0(\|x(\zeta)\|) d\zeta] d\tau \right] ds \\ \|x(t)\| \leq \frac{1}{(1-c_1)} \left\{ M_1 [\|x_0\| + M_2 + G(0, x(0), 0)] + c_1 \int_0^t m_0(t, s) \Omega_0(\|x(s)\|) ds \right. \\ + M_1 c_2 \int_0^t [\|x(t)\| + \sum_{i=1}^n m_i(t, s) \Omega_i(\|x(s)\|)] ds \\ + c_1 \int_0^t H_1(s) \mu_1(s) [\|x(s)\| + \int_0^s m_0(s, \tau) \Omega_0(\|x(\tau)\|) d\tau] ds \\ \left. + c_1 \int_0^t \left[\int_s^t H_2(\tau) \mu_2(\tau) [\|x(s)\| + \int_0^\tau m_0(\tau, \zeta) \Omega_0(\|x(\zeta)\|) d\zeta] d\tau \right] ds \right\}$$

Denoting the right hand side of the above inequality as $v(t)$. Then $\|x(t)\| \leq v(t)$ and

$$v(0) = c = \frac{1}{1-c_1} M_1 [\|x_0\| + M_2 + G(0, x(0), 0)].$$

$$\begin{aligned} v'(t) &= \frac{1}{1-c_1} \left\{ c_1 m_0(t, s) \Omega_0(v(t)) \right. \\ &+ M_1 c_2 \left[v(t) + \sum_{i=1}^n m_i(t, s) \Omega_i(v(t)) \right] \\ &+ c_1 H_1(t) \mu_1(t) \left[v(t) + \int_0^t m_0(t, s) \Omega_0(v(s)) ds \right] \\ &+ c_1 \left[\int_s^t H_2(s) \mu_2(s) \left[v(t) + \int_0^s m_0(s, \tau) \Omega_0(v(\tau)) d\tau \right] ds \right] \left. \right\} \\ &\leq \hat{m}(t) \left[\Omega_0(v(t)) + v(t) + \sum_{i=1}^n \Omega_i(v(t)) \right], \end{aligned}$$

This implies

$$\begin{aligned} \int_{v(0)}^{v(t)} \frac{ds}{3s + \sum_{i=1}^n \Omega_i(s) + 2\Omega_0(s)} &\leq \int_0^T \hat{m}(s) ds \\ &< \int_c^\infty \frac{ds}{3s + \sum_{i=1}^n \Omega_i(s) + 2\Omega_0(s)}, \quad 0 \leq t \leq T. \end{aligned} \tag{5}$$

where $c = \frac{1}{1-c_1} M_1 [\|x_0\| + M_2 + G(0, x(0), 0)]$

Inequality (5) implies that there is a constant K such that $v(t) \leq K, t \in J$ and hence we have

$\|x\| = \sup\{\|x(t)\| : t \in J\} \leq K$, where K depends only on T and on the functions \hat{m}, Ω_0 and Ω_i .

We shall now prove that the operator $\Phi : Z \rightarrow Z$ is a completely continuous operator. Let

$B_k = \{x \in Z : \|x\| \leq k\}$ for some $k \geq 1$. We first show that Φ maps B_k into an equicontinuous family.

Let $x \in B_k$ and $t_1, t_2 \in [0, T]$. Then if $0 < t_1 < t_2 < T$

$$\begin{aligned} \|(\Phi x)(t_1) - (\Phi x)(t_2)\| &\leq \left\| (R(t_1, 0) - R(t_2, 0)) [x_0 - h(x) - G(0, x(0), 0)] \right\| \\ &+ \left\| \int_0^{t_1} [R(t_1, s) - R(t_2, s)] F\left(s, x(s), \int_0^s f_1(s, \tau, x(\tau)) d\tau, \dots, \int_0^s f_n(s, \tau, x(\tau)) d\tau\right) ds \right\| \\ &+ \left\| G\left(t_1, x(t_1), \int_0^{t_1} g(t_1, s, x(s)) ds\right) - G\left(t_2, x(t_2), \int_0^{t_2} g(t_2, s, x(s)) ds\right) \right\| \\ &+ \left\| \int_0^{t_1} [(R(t_1, s) - R(t_2, s)) A(s) G(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau)] ds \right\| \\ &+ \left\| \int_0^{t_1} \left[\int_s^{t_1} (\mathbf{R}(t_1, \tau) - \mathbf{R}(t_2, \tau)) \mathbf{B}(\tau, s) \mathbf{G}(t, \mathbf{x}(\tau), \int_0^\tau \mathbf{g}(\tau, \zeta, \mathbf{x}(\zeta)) d\zeta) d\tau \right] ds \right\| \\ &+ \left\| \int_{t_1}^{t_2} [R(t_2, s) F\left(s, x(s), \int_0^s f_1(s, \tau, x(\tau)) d\tau, \dots, \int_0^s f_n(s, \tau, x(\tau)) d\tau\right)] ds \right\| \\ &+ \left\| \int_{t_1}^{t_2} [R(t_2, s) A(s) G(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau)] ds \right\| \\ &+ \left\| \int_{t_1}^{t_2} \left[\int_s^{t_2} (\mathbf{R}(t_2, s) \mathbf{B}(\tau, s) \mathbf{G}(t, \mathbf{x}(\tau), \int_0^\tau \mathbf{g}(\tau, \zeta, \mathbf{x}(\zeta)) d\zeta) \right] d\tau \right] ds \right\| \\ &\leq \left\| (R(t_1, 0) - R(t_2, 0)) [x_0 - h(x) - G(0, x(0), 0)] \right\| \\ &+ \left\| \int_0^{t_1} [R(t_1, s) - R(t_2, s)] F\left(s, x(s), \int_0^s f_1(s, \tau, x(\tau)) d\tau, \dots, \int_0^s f_n(s, \tau, x(\tau)) d\tau\right) ds \right\| \\ &+ \left\| G\left(t_1, x(t_1), \int_0^{t_1} g(t_1, s, x(s)) ds\right) - G\left(t_2, x(t_2), \int_0^{t_2} g(t_2, s, x(s)) ds\right) \right\| \\ &+ \left\| \int_0^{t_1} [(R(t_1, s) - R(t_2, s)) A(s) G(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau)] ds \right\| \\ &+ \left\| \int_0^{t_1} \left[\int_s^{t_1} (\mathbf{R}(t_1, \tau) - \mathbf{R}(t_2, \tau)) \mathbf{B}(\tau, s) \mathbf{G}(t, \mathbf{x}(\tau), \int_0^\tau \mathbf{g}(\tau, \zeta, \mathbf{x}(\zeta)) d\zeta) d\tau \right] ds \right\| \\ &+ M_1 c_2 \int_{t_1}^{t_2} \sum_{i=1}^n [k + m_i(t, s) \Omega_i(k)] ds - M_1 c_1 \int_{t_1}^{t_2} H_1(s) \mu_1(s) \left[k + \int_0^s m_0(s, \tau) \Omega_0(k) d\tau \right] ds \\ &+ M_1 c_1 \int_{t_1}^{t_2} \left[\int_s^{t_2} \mathbf{H}_1(\tau) \mu_1(\tau) \left[k + \int_0^s m_0(s, \zeta) \Omega_0(k) d\zeta \right] d\tau \right] ds \end{aligned}$$

The right hand side is independent of $x \in B_k$ and tends to zero as $t_2 - t_1 \rightarrow 0$, since f is completely continuous and by (H_3) , $R(t, s)$ for $t > s$ is continuous in the uniform operator topology. Thus Φ maps B_k into an equicontinuous family of functions.

It is easy to see that ΦB_k is uniformly bounded. Next,

we show $\overline{\Phi B_k}$ is compact. Since we have shown ΦB_k is equicontinuous collection, by the Arzela-Ascoli theorem it suffices to show that Φ maps B_k into a precompact set in X .

Let $0 < t \leq T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$\begin{aligned}
 (\Phi_\epsilon x)(t) &= R(t, 0) \left[x_0 - h(x) - G(0, x(0), 0) \right] + G \left(t, x(t), \int_0^{t-\epsilon} g(t, s, x(s)) ds \right) \\
 &\quad + \int_0^{t-\epsilon} R(t, s) F \left(s, x(s), \int_0^s f_1(s, \tau, x(\tau)) d\tau, \dots, \int_0^s f_n(s, \tau, x(\tau)) d\tau \right) ds \\
 &\quad - \int_0^{t-\epsilon} \left[\int_s^{t-\epsilon} R(t, \tau) B(\tau, s) G \left(\tau, x(\tau), \int_0^\tau g(\tau, \zeta, x(\zeta)) d\zeta \right) d\tau \right] ds \\
 &\quad - \int_0^{t-\epsilon} \left[R(t, s) A(s) G \left(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau \right) \right] ds
 \end{aligned} \tag{6}$$

Since $R(t, s)$ is a compact operator, the set $Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in B_k\}$ is precompact in X for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $x \in B_k$ we have

$$\begin{aligned}
 \|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| &\leq \int_{t-\epsilon}^t R(t, s) F \left(s, x(s), \int_0^s f_1(s, \tau, x(\tau)) d\tau, \dots, \int_0^s f_n(s, \tau, x(\tau)) d\tau \right) ds \\
 &\quad + \int_{t-\epsilon}^t \left[\int_s^{t-\epsilon} \mathbf{R}(t, \tau) \mathbf{B}(\tau, s) \left[\mathbf{G} \left(\tau, x(\tau), \int_0^\tau \mathbf{g}(s, \zeta, x(\zeta)) d\zeta \right) \right] d\tau \right] ds \\
 &\quad + \int_{t-\epsilon}^t \left[R(t, s) A(s) G \left(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau \right) \right] ds \\
 &\leq M_1 c_2 \int_{t-\epsilon}^t \sum_{i=1}^n [k + m_i(t, s) \Omega_i(k) ds] \\
 &\quad + M_1 c_1 \int_{t-\epsilon}^t H_1(s) \mu_1(s) \left[k + \int_0^s m_0(s, \tau) \Omega_0(k) d\tau \right] ds \\
 &\quad + M_1 c_1 \int_{t-\epsilon}^t \left[\int_s^{t-\epsilon} H_1(\tau) \mu_1(\tau) \left[\mathbf{k} + \int_0^\tau \mathbf{m}_0(\tau, \zeta) \mathbf{\Omega}_0(\mathbf{k}) d\zeta \right] d\tau \right] ds
 \end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_k\}$.

Hence, the set $\{(\Phi x)(t) : x \in B_k\}$ is precompact in X .

It remains to show that $\Phi : Z \rightarrow Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \rightarrow x$ in Z . Then there is an integer q such that $\|x_n(t)\| \leq q$ for all n and $t \in J$, so $x_n \in B_q$ and $x \in B_q$. By (H_4) ,

$$\begin{aligned}
 &F \left(t, x_n(t), \int_0^t f_1(t, s, x_n(s)) ds, \int_0^t f_2(t, s, x_n(s)) ds, \dots, \int_0^t f_n(t, s, x_n(s)) ds \right) \\
 &\rightarrow F \left(t, x(t), \int_0^t f_1(t, s, x(s)) ds, \int_0^t f_2(t, s, x(s)) ds, \dots, \int_0^t f_n(t, s, x(s)) ds \right)
 \end{aligned}$$

for each $t \in J$ and since

$$\begin{aligned}
 &\left\| F \left(t, x_n(t), \int_0^t f_1(t, \tau, x_n(\tau)) d\tau, \int_0^t f_2(t, \tau, x_n(\tau)) d\tau, \dots, \int_0^t f_n(t, \tau, x_n(\tau)) d\tau \right) \right. \\
 &\quad \left. - F \left(t, x(t), \int_0^t f_1(t, \tau, x(\tau)) d\tau, \int_0^t f_2(t, \tau, x(\tau)) d\tau, \dots, \int_0^t f_n(t, \tau, x(\tau)) d\tau \right) \right\| \\
 &\leq 2c_2 m_i(t, \tau) \Omega_i(q) \|g(t, \tau, x_n(\tau)) - g(t, \tau, x(\tau))\| \leq 2m_0(t, \tau) \Omega_0(q) \\
 &\left\| A(\tau) R(t, \tau) G \left(\tau, x_n(\tau), \int_0^\tau g(t, \tau, x_n(\tau)) d\tau \right) - A(\tau) R(t, \tau) G \left(\tau, x(\tau), \int_0^\tau g(t, \tau, x(\tau)) d\tau \right) \right\| \\
 &\leq 2c_1 \mu_1(t) H_1(t) q
 \end{aligned}$$

and

$$\left\| B(t, \tau)R(t, \tau)G\left(\tau, x_n(\tau), \int_0^\tau g(t, \tau, x_n(\tau))d\tau\right) - B(t, \tau)R(t, \tau)G\left(\tau, x(\tau), \int_0^\tau g(t, \tau, x(\tau))d\tau\right) \right\| \\ \leq 2c_1\mu_2(t)H_2(t)q \quad 0 \leq c_1 \leq 1, 0 \leq c_2 \leq 1.$$

we have by dominated convergence theorem

$$\left\| \Phi x_n - \Phi x \right\| \leq \int_0^t \left\| R(t, s) \left[F\left(t, x_n(t), \int_0^t f_1(t, \tau, x_n(\tau))d\tau, \dots, \int_0^t f_n(t, \tau, x_n(\tau))d\tau\right) \right. \right. \\ \left. \left. - F\left(t, x(t), \int_0^t f_1(t, \tau, x(\tau))d\tau, \dots, \int_0^t f_n(t, \tau, x(\tau))d\tau\right) \right] \right\| ds \\ + \int_0^t \left\| R(t, s) \left[g(t, \tau, x_n(\tau)) - g(t, \tau, x(\tau)) \right] \right\| ds \\ + \int_0^t \left\| R(t, s) \left\| A(\tau)R(t, \tau)G\left(\tau, x_n(\tau), \int_0^\tau g(t, \tau, x_n(\tau))d\tau\right) - A(\tau)R(t, \tau)G\left(\tau, x(\tau), \int_0^\tau g(t, \tau, x(\tau))d\tau\right) \right\| \right\| ds \\ + \int_0^t \left[\int_0^s \left\| B(t, \tau)R(t, \tau)G\left(\tau, x_n(\tau), \int_0^\tau g(t, \zeta, x_n(\zeta))d\zeta\right) - B(t, \tau)R(t, \tau)G\left(\tau, x(\tau), \int_0^\tau g(t, \zeta, x(\zeta))d\zeta\right) \right\| d\zeta \right] ds \rightarrow 0$$

as $n \rightarrow \infty$

Thus Φ is continuous. This completes the proof that Φ is completely continuous.

Finally the set $\zeta(\Phi) = \{x \in Z : x = \lambda\Phi x, \lambda \in (0, 1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer's theorem, the operator Φ has a fixed point in Z . This means that any fixed point of Φ is a mild solution of (1) and (2) on J satisfying $(\Phi x)(t) = x(t)$.

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