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An Upper Bound for Conditional Second Moment of the Solution of a SDE

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ABSTRACT

Let $\mathbb{F}=\left(\mathcal{F}(t),t\in\mathbb{R}_+\right)$ be a filtration on some probability space and let $\mathcal{K}$ denote the class of all $\mathbb{F}$-adapted $\mathbb{R}^d$-valued stochastic processes M such that $\mathrm{E}\left(\left|M(t)\right|^2\middle|\mathcal{F}(0)\right)<\infty,\mathrm{E}\left(M(t)\middle|\mathcal{F}(s)\right)=M(s)$ for all $t>s\geq 0$ and the process $\mathrm{E}\left(\left|M(\cdot)\right|^2\middle|\mathcal{F}(0)\right)$ is continuous (the conditional expectations are extended, so we do not demand that $\mathrm{E}\left|M(t)\right|^2<\infty$). It is shown that each $M\in\mathcal{K}$ is a locally square integrable martingale w. r. t. $\mathbb{F}$. Let X be the strong solution of the equation $X(t)=\int_0^t Q(s,X(s))\mathrm{d}\iota(s)+M(t)$, where $M\in\mathcal{K}$, ι is a continuous increasing process with $\mathcal{F}(0)$-measurable values at all times, and Q is an $\mathbb{R}^d$-valued random function on $\mathbb{R}_+\times\mathbb{R}^d$, continuous in $x\in\mathbb{R}^d$ and $\mathbb{F}$-progressive at fixed *x*. Suppose also that there exists an $\mathcal{F}(0)\otimes\mathcal{B}_+$-measurable in (ω,t) non-negative random process ψ such that, for all t,x, $x^{\mathrm{T}}Q(t,x)\leq-\psi(t)|x|^2$ and $\int_0^t\psi(s)\mathrm{d}\iota(s)<\infty$. Then $\mathrm{E}^0\left|X(t)\right|^2\leq\mathrm{e}^{-\Psi(t)}\left|M(0)\right|^2+\mathrm{e}^{-\Psi(t)}\int_0^t\mathrm{e}^{\Psi(s)}\mathrm{d}\mathrm{E}^0\,\mathrm{tr}\langle M\rangle(s)$, where $\Psi(t)=2\int_0^t\psi(\tau)\mathrm{d}\iota(\tau)$.

Keywords: Conditional Expectation; Martingale; Stochastic Equation

1. Introduction

The random processes under consideration are assumed, firstly, given on a common probability space $(\Omega,\mathcal{F},\mathrm{P})$ (without any exception) and, secondly, càdlàg (the exceptions will be stipulated). Let $\mathcal{F}^0$ be a sub-σ-algebra of $\mathcal{F}$. We introduce the notation: $\mathrm{E}^0=\mathrm{E}\left(\cdots\middle|\mathcal{F}^0\right)$, $\mathrm{P}^0=\mathrm{P}\left\{\cdots\middle|\mathcal{F}^0\right\}$; $\mathcal{V}_0^+$—the class of all increasing from zero numeral random processes whose values at all times are $\mathcal{F}^0$-measurable random variables. If, besides, a filtration $\mathbb{F}=\left(\mathcal{F}(t),t\in\mathbb{R}_+\right)$ is given, then we identify $\mathcal{F}^0$ with $\mathcal{F}(0)$. By $\bar{\mathcal{M}}_2$ we denote, following [1], the class of all $\mathbb{R}^d$-valued (d will be determined by the context, if matters) $\mathbb{F}$-martingales M such that for every t $\mathrm{E}\left|M(t)\right|^2<\infty$; $\ell\mathcal{M}_2$ signifies (see ibid.) the class of all locally square integrable martingales w. r. t. $\mathbb{F}$.

The definition of conditional expectation, in particular E^0, adopted in this article is due to Meyer (see [2]). It admits existence of the conditional expectation of a random variable with infinite first absolute moment. Thus generalized conditional expectation inherits most of the familiar properties (listed, for example, in [2]) of the classical one, but in this case new proofs are required. They are gathered in Section 2.

Let X be the solution of a stochastic differential equation of the kind

$$X(t)=\int_0^t Q(s,X(s))\mathrm{d}\iota(s)+M(t),$$

where ι is a continuous process from $\mathcal{V}_0^+$ and M is chosen from some subclass of $\ell\mathcal{M}_2$ which is constructed and studied in Section 3. The goal of this article is to find an upper bound, much more exact than that provided by the Gronwall—Bellman lemma, for $\mathrm{E}^0\left|X(t)\right|^2$. This is done in Section 4 containing the only final result of the article. The reader inclined to accept that result in less generality, when M is a quasicontinuous process from $\bar{\mathcal{M}}_2$ and $\mathcal{F}^0=\{\varnothing,\Omega\}$ (so that $\mathrm{E}^0=\mathrm{E}$), may skip all the preceding material. But for the approach underlying the derivations in Section 4 such a confinement is unnatural. That is a reason why 3/4 of the article's volume are allocated to ancillary results. Another reason is that those results may prove useful beyond the context of this article.

Upper bounds for $\mathrm{E}\left|X(t)\right|^p$ are usually obtained with the aid of Lyapunov's functions (see, e.g., [3,4]). Our alternative approach is based on a "comparison theorem" (Corollary 4.2) allowing both to weaken the assumptions and to refine the conclusion (cf. our Theorem 4.3 with Theorem I.4.2 in [3]).

All vectors are thought of, unless otherwise stated, as columns; $\int_a^b$ means $\int_{]a,b]}$. The space of all d-dimensional row vectors with real components is denoted $\mathbb{R}^{d*}$. The words "almost surely" are tacitly implied in relations between random variables, including the convergence relation, unless it is explicitly written as the convergence in probability. Indicators are denoted by I with two possible modes of writing the set: I_B or $I\{\cdots\}$.

The reference books for the notions and results of stochastic analysis used in this paper are [1,5,6].

2. Extended Conditional Expectations

Denote $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ and, for $a \in \mathbb{R},\ \ a_+ = a \vee 0, a_- = -(a \wedge 0)$, so that $a = a_+ - a_-$. In what follows, "nonnegative" means "$\mathbb{R}_+$-valued" (the value ∞ is not admitted). The Borel σ-algebra in $\mathbb{R}_+$ will be denoted $\mathcal{B}_+$.

Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. The conditional given $\mathcal{G}$ expectation of an $\overline{\mathbb{R}}_+$-valued random variable γ is defined, according to [2], as the $\mathcal{G}$-measurable $\overline{\mathbb{R}}_+$-valued random variable $\mathrm{E}(\gamma|\mathcal{G})$ such that $\mathrm{E}\gamma I_G = \mathrm{E}\left(\mathrm{E}(\gamma|\mathcal{G}) I_G\right)$ for every $G \in \mathcal{G}$. For an $\mathbb{R}$-valued random variable γ such that $\mathrm{P}\left\{\mathrm{E}(\gamma_+|\mathcal{G}) = \infty = \mathrm{E}(\gamma_-|\mathcal{G})\right\} = 0$ we set by definition $\mathrm{E}(\gamma|\mathcal{G}) = \mathrm{E}(\gamma_+|\mathcal{G}) - \mathrm{E}(\gamma_-|\mathcal{G})$. Further the conditional expectation of a $\mathbb{C}^d$-valued random variable is defined in the obvious way. Thus defined conditional expectation will be called *extended*. Unlike the classical conditional expectation (defined only for $\gamma \in \mathrm{L}_1(\Omega,\mathcal{F},\mathrm{P})$) it does not possess, generally speaking, the property

$$\mathcal{G}_1 \subset \mathcal{G}_2 \Rightarrow \mathrm{E}\left(\mathrm{E}(\gamma|\mathcal{G}_2)|\mathcal{G}_1\right) = \mathrm{E}(\gamma|\mathcal{G}_1). \tag{1}$$

But for an $\overline{\mathbb{R}}_+$-valued γ this property remains valid—with the same proof as for $\gamma \in \mathrm{L}_1$.

Obviously, the extended conditional expectation of $\beta \in \mathrm{L}_1(\Omega,\mathcal{F},\mathrm{P})$ coincides with the classical one and therefore

$$\mathrm{E}(\beta+\gamma|\mathcal{G}) = \mathrm{E}(\beta|\mathcal{G}) + \mathrm{E}(\gamma|\mathcal{G}), \tag{2}$$

$$\mathrm{E}(c\gamma|\mathcal{G}) = c\mathrm{E}(\gamma|\mathcal{G}) \tag{3}$$

for every $\beta,\gamma \in \mathrm{L}_1(\Omega,\mathcal{F},\mathrm{P})$ and $c \in \mathbb{C}$. Equality (2) holds for $\overline{\mathbb{R}}_+$-valued β and γ, as well, which is immediate from the definition of extended conditional expectation. In particular,

$$\mathrm{E}(|\gamma||\mathcal{G}) = \mathrm{E}(\gamma_+|\mathcal{G}) + \mathrm{E}(\gamma_-|\mathcal{G}) \tag{4}$$

for every $\mathbb{R}$-valued random variable γ.

The next two statements are immediate from the definition of extended conditional expectation.

Lemma 2.1. *Let γ be an $\mathbb{R}$-valued random variable such that $\mathrm{E}(\gamma|\mathcal{G})$ exists. Then Equality* (3) *holds for every $c \in \mathbb{R}$.*

Lemma 2.2. *Let Ξ be an $\overline{\mathbb{R}}_+$-valued random variable. Then for any $S \in \mathcal{G}$ $\mathrm{E}(\Xi I_S|\mathcal{G}) = \mathrm{E}(\Xi|\mathcal{G}) I_S$.*

Lemma 2.3. *Let β and γ be nonnegative random variables such that $\beta \le \gamma$. Then $\mathrm{E}(\beta|\mathcal{G}) \le \mathrm{E}(\gamma|\mathcal{G})$.*

Proof. Denote $\chi = I\left\{\mathrm{E}(\gamma|\mathcal{G}) < \infty\right\}, \Gamma = \left(\mathrm{E}(\gamma|\mathcal{G}) - \mathrm{E}(\beta|\mathcal{G})\right)\chi$. The assumption $\beta \le \gamma$ and the definition of extended conditional expectation yield $\mathrm{E}\Gamma I_G \ge 0$ for every $G \in \mathcal{G}$. Consequently $\Gamma I\{\Gamma < 0\} = 0$. □

Lemma 2.4. *Let Ξ be an $\overline{\mathbb{R}}_+$-valued random variable. Then for any $\varepsilon > 0$ and $a > 0$*

$$\mathrm{P}\{\Xi > \varepsilon\} \le a/\varepsilon + \mathrm{P}\left\{\mathrm{E}^0\Xi > a\right\}.$$

Proof. By Formula (1) $\mathrm{P}\left\{\Xi > \varepsilon, \mathrm{E}^0\Xi \le a\right\} = \mathrm{E}\mathrm{E}^0 I\left\{\Xi > \varepsilon, \mathrm{E}^0\Xi \le a\right\}$. By Lemma 2.2 $\mathrm{E}^0 I\left\{\Xi > \varepsilon, \mathrm{E}^0\Xi \le a\right\} = I\left\{\mathrm{E}^0\Xi \le a\right\}\mathrm{P}^0\{\Xi > \varepsilon\}$. By Lemmas 2.3 and 2.1 $\mathrm{E}^0\Xi \ge \varepsilon \mathrm{P}^0\{\Xi > \varepsilon\}$ and therefore $I\left\{\mathrm{E}^0\Xi \le a\right\}\mathrm{P}^0\{\Xi > \varepsilon\} \le a/\varepsilon$. It remains to write the evident inclusion

$$\{\Xi > \varepsilon\} \subset \left\{\Xi > \varepsilon, \mathrm{E}^0\Xi \le a\right\} \cup \left\{\mathrm{E}^0\Xi > a\right\}. \ □$$

Corollary 2.5. *Let $\mathcal{F}^0 \subset \mathcal{G} \subset \mathcal{F}$ and let γ be a nonnegative random variable such that $\mathrm{E}^0\gamma < \infty$. Then $\mathrm{E}(\gamma|\mathcal{G}) < \infty$.*

Proof. Lemma 2.4 and Formula (1) yield for arbitrary $N > 0$ and $a > 0$

$$\mathrm{P}\left\{\mathrm{E}(\gamma|\mathcal{G}) > N\right\} \le a/N + \mathrm{P}\left\{\mathrm{E}^0\gamma > a\right\}.$$

Passing in this inequality to the limit at first as $N \to \infty$ and hereafter as $a \to \infty$, we get

$$\lim_{N\to\infty} \mathrm{P}\left\{\mathrm{E}(\gamma|\mathcal{G}) > N\right\} = 0. □$$

Lemma 2.6. *Let (β_n) be an increasing sequence of $\overline{\mathbb{R}}_+$-valued random variables. Then $\mathrm{E}\lim\beta_n = \lim\mathrm{E}\beta_n$.*

Proof. In case the r.h.s is finite this is the Beppo Levi theorem. Having written $\mathrm{E}\lim\beta_m \ge \mathrm{E}\beta_n$, we obtain the same equality when $\mathrm{E}\beta_n \to \infty$. □

Lemma 2.7. *Let (Γ_n) be an increasing sequence of $\overline{\mathbb{R}}_+$-valued random variables. Then*

$$\mathrm{E}(\lim\Gamma_n|\mathcal{G}) = \lim\mathrm{E}(\Gamma_n|\mathcal{G}).$$

Proof. Denote $\Gamma = \lim\Gamma_n, \phi_n = \mathrm{E}(\Gamma_n|\mathcal{G}), \phi = \lim\phi_n$. By construction ϕ is $\mathcal{G}$-measurable. Lemma 2.6 and the definition of conditional expectation yield, for arbitrary $G \in \mathcal{G}$,
$\mathrm{E}\phi I_G = \lim \mathrm{E}\phi_n I_G$, $\mathrm{E}\phi_n I_G = \mathrm{E}\Gamma_n I_G$, $\lim \mathrm{E}\Gamma_n I_G = \mathrm{E}\Gamma I_G$.
So $\mathrm{E}\phi I_G = \mathrm{E}\Gamma I_G$, which in view of $\mathcal{G}$-measurability of ϕ proves the lemma. □

Corollary 2.8. *For every sequence* (γ_n) *of nonnegative random variables the inequality* $\mathrm{E}(\underline{\lim}\gamma_n|\mathcal{G}) \le \underline{\lim}\mathrm{E}(\gamma_n|\mathcal{G})$ *is valid.*

Proof. Denote $\Gamma_n = \inf_{k\ge n}\gamma_k, \Gamma = \underline{\lim}\gamma_n$. By Lemma 2.3 $\mathrm{E}(\Gamma_n|\mathcal{G}) \le \inf_{k\ge n}\mathrm{E}(\gamma_k|\mathcal{G})$. Herein $\Gamma_n \nearrow \Gamma$, whence by Lemma 2.7 $\mathrm{E}(\Gamma_n|\mathcal{G}) \to \mathrm{E}(\Gamma|\mathcal{G})$. □

Lemma 2.9. *Let* (Γ_n) *be a decreasing sequence of nonnegative random variables such that* $\mathrm{E}(\Gamma_1|\mathcal{G}) < \infty$. *Then* $\mathrm{E}(\lim\Gamma_n|\mathcal{G}) = \lim\mathrm{E}(\Gamma_n|\mathcal{G})$.

Proof. Retaining the notation of the proof of Lemma 2.7, we denote additionally
$\Lambda = \mathrm{E}(\Gamma_1|\mathcal{G}), A^N = \{\Lambda \le N\} (\in \mathcal{G})$. Then from the definition of conditional expectation we have

$$\mathrm{E}\phi_n I_{A^N} I_G = \mathrm{E}\Gamma_n I_{A^N} I_G \tag{5}$$

for arbitrary $N > 0$ and $G \in \mathcal{G}$. By condition $\phi_n \le \Lambda$, so $0 \le \mathrm{E}\phi_n I_{A^N} I_G \le N, 0 \le \mathrm{E}\Gamma_n I_{A^N} I_G \le N$, whence, taking to account monotonicity of (Γ_n) (and therefore of (ϕ_n)) we conclude by the Beppo Levi theorem that $\mathrm{E}\Gamma I_{A^N} I_G = \lim \mathrm{E}\Gamma_n I_{A^N} I_G$, $\mathrm{E}\phi I_{A^N} I_G = \lim \mathrm{E}\phi_n I_{A^N} I_G$.
Juxtaposing these two equalities with (5), we see that

$$\mathrm{E}\phi I_G I_{A^N} = \mathrm{E}\Gamma I_G I_{A^N} \tag{6}$$

for any $N > 0$. Herein $I_{A^N} \to 1$ as $N \to \infty$, since by assumption $\mathrm{P}\{\Lambda < \infty\} = 1$. Then from (6) we get by Lemma 2.6 $\mathrm{E}\phi I_G = \mathrm{E}\Gamma I_G$. □

Theorem 2.10. *Let* (ρ_n) *be a sequence of* $\mathbb{R}^d$*-valued random variables almost surely converging to a random variable* ρ *and such that*

$$\mathrm{E}\left(\sup_n|\rho_n|\,\middle|\,\mathcal{G}\right) < \infty. \tag{7}$$

Then $\mathrm{E}(|\rho||\mathcal{G}) < \infty$ and $\mathrm{E}(\rho_n|\mathcal{G}) \to \mathrm{E}(\rho|\mathcal{G})$.

Proof. Let first the ρ_n's be nonnegative. Denote $\gamma_n = \inf_{k\ge n}\rho_k, \Gamma_n = \sup_{k\ge n}\rho_k$. Then

$$\gamma_n \le \rho_n \le \Gamma_n, \tag{8}$$

$\gamma_n \nearrow \rho,\ \Gamma_n \searrow \rho.$ From the second relation we have by Corollary 2.8 $\mathrm{E}(\rho|\mathcal{G}) \le \underline{\lim}\mathrm{E}(\gamma_n|\mathcal{G})$; the third relation together with (7) yields by Lemma 2.9 $\mathrm{E}(\rho|\mathcal{G}) = \lim\mathrm{E}(\Gamma_n|\mathcal{G})$. Comparing these two conclusions with (8), we get $\mathrm{E}(\rho|\mathcal{G}) = \lim\mathrm{E}(\rho_n|\mathcal{G})$. Thus we have proved the theorem for nonnegative random variables. The transition to the general case is trivial. □

Lemma 2.11. *Let* β *and* γ *be* $\mathbb{R}^d$*-valued random variables such that the conditional expectations* $\mathrm{E}(\beta|\mathcal{G})$ *and* $\mathrm{E}(\gamma|\mathcal{G})$ *exist and are component-wise finite. Then* $\mathrm{E}(\beta+\gamma|\mathcal{G})$ *exists and Equality* (2) *holds.*

Proof. The assumptions of the lemma together with Equality (4) imply that

$$\mathrm{E}(|\beta||\mathcal{G}) < \infty, \quad \mathrm{E}(|\gamma||\mathcal{G}) < \infty. \tag{9}$$

For nonnegative random variables Equality (2) ensues, as was pointed out above, directly from the definition of extended conditional expectation, so Inequalities (9) yield

$$\mathrm{E}(|\beta|+|\gamma||\mathcal{G}) < \infty. \tag{10}$$

Denote, for each $n \in \mathbb{N}$,

$$\beta_n = \frac{n\beta}{n \vee |\beta|}, \quad \gamma_n = \frac{n\gamma}{n \vee |\gamma|},$$

$\rho_n = \beta_n + \gamma_n$. By construction $|\beta_n| \le n, |\gamma_n| \le n$ and therefore $\beta_n, \gamma_n \in \mathrm{L}_1(\Omega, \mathcal{F}, \mathrm{P})$. Consequently, $\mathrm{E}(\rho_n|\mathcal{G}) = \mathrm{E}(\beta_n|\mathcal{G}) + \mathrm{E}(\gamma_n|\mathcal{G})$.

Obviously, $\beta_n \to \beta, \gamma_n \to \gamma, \rho_n \to \beta+\gamma.$ Herein by construction $|\beta_n| \le |\beta|, |\gamma_n| \le |\gamma|, |\rho_n| \le |\beta|+|\gamma|$, which together with (10) and (9) implies (7) and the same for (β_n) and (γ_n). Hence and from the above asymptotic relations we get by Theorem 2.10

$$\mathrm{E}(\beta_n|\mathcal{G}) \to \mathrm{E}(\beta|\mathcal{G}), \mathrm{E}(\gamma_n|\mathcal{G}) \to \mathrm{E}(\gamma|\mathcal{G}),$$
$$\mathrm{E}(\rho_n|\mathcal{G}) \to \mathrm{E}(\beta+\gamma|\mathcal{G}).$$ □

Lemma 2.12. *Let* $\mathcal{F}^0 \subset \mathcal{G} \subset \mathcal{F}$ *and* γ *be a* $\mathbb{R}^d$*-valued random variable such that* $\mathrm{E}^0|\gamma| < \infty$. *Then* $\mathrm{E}^0\gamma = \mathrm{E}^0\mathrm{E}(\gamma|\mathcal{G})$.

Proof. It suffices to consider the case $d = 1$. Then the last assumption of the lemma amounts to $\mathrm{E}^0\gamma_\pm < \infty$. Denote $\Gamma_1 = \mathrm{E}(\gamma_+|\mathcal{G}), \Gamma_2 = \mathrm{E}(\gamma_-|\mathcal{G})$. By Formula (1) $\mathrm{E}^0\Gamma_{1,2} = \mathrm{E}^0\gamma_\pm$ and therefore $\mathrm{E}^0\Gamma_{1,2} < \infty$. Then by Lemma 2.11 $\mathrm{E}^0(\Gamma_1 - \Gamma_2) = \mathrm{E}^0\Gamma_1 - \mathrm{E}^0\Gamma_2$, which together with the previous inequality and the definition of extended conditional expectation yields $\mathrm{E}^0(\Gamma_1 - \Gamma_2) = \mathrm{E}^0\gamma$. The inequalities $\mathrm{E}^0\Gamma_{1,2} < \infty$ imply, by Corollary 2.5, that $\Gamma_{1,2} < \infty$, whence by the definitions of Γ_i and extended conditional expectation we have $\Gamma_1 - \Gamma_2 = \mathrm{E}(\gamma|\mathcal{G})$. □

Lemma 2.13. *Let* Υ *and* Ξ *be nonnegative random variables*, Υ *be* $\mathcal{G}$*-measurable. Then* $\mathrm{E}(\Upsilon\Xi|\mathcal{G}) = \Upsilon\mathrm{E}(\Xi|\mathcal{G})$.

Proof. Denote $S_{nk} = \{k2^{-n} < \Upsilon \le (k+1)2^{-n}\}$ ($\in \mathcal{G}$ due to $\mathcal{G}$-measurability of Υ), $J_{nk} = I_{S_{nk}}$,

$$\Upsilon_{nm} = 2^{-n}\sum_{k=1}^{m} kJ_{nk}, \Upsilon_n = 2^{-n}\sum_{k=1}^{\infty} kJ_{nk}.$$

Formula (2) (for nonnegative random variables), Lemma 2.1 and the definition of Υ_{nm} yield

$\mathrm{E}\left(\Upsilon_{nm}\Xi|\mathcal{G}\right)=2^{-n}\sum_{k=1}^{m}k\mathrm{E}\left(J_{nk}\Xi|\mathcal{G}\right)$. Noting that $\mathrm{E}\left(J_{nk}\Xi|\mathcal{G}\right)=\mathrm{E}\left(\Xi|\mathcal{G}\right)J_{nk}$ by Lemma 2.2, we convert this equality to $\mathrm{E}\left(\Upsilon_{nm}\Xi|\mathcal{G}\right)=\Upsilon_{nm}\mathrm{E}\left(\Xi|\mathcal{G}\right)$. Obviously, $\Upsilon_{nm}\nearrow\Upsilon_n$ as $m\to\infty$. Then by Lemma 2.7 $\mathrm{E}\left(\Upsilon_{nm}|\mathcal{G}\right)\nearrow\mathrm{E}\left(\Upsilon_n|\mathcal{G}\right)$ as $m\to\infty$, which together with the last equality yields $\mathrm{E}\left(\Upsilon_n\Xi|\mathcal{G}\right)=\Upsilon_n\mathrm{E}\left(\Xi|\mathcal{G}\right)$. It remains to let $n\to\infty$ and again make use of Lemma 2.7. □

Lemma 2.14. *Let Υ and Ξ be random variables with values in $\mathbb{R}^d$ and $\mathbb{R}^p$, respectively. Suppose that Υ is $\mathcal{G}$-measurable and $\mathrm{E}\left(|\Xi|\,|\mathcal{G}\right)<\infty$. Then* $\mathrm{E}\left(\Upsilon\Xi^{\mathrm{T}}|\mathcal{G}\right)=\Upsilon\mathrm{E}\left(\Xi^{\mathrm{T}}|\mathcal{G}\right)$.

Proof. It suffices to consider the case $d=1,p=1$. Writing, for arbitrary $a,b\in\mathbb{R}$, the evident equalities $(ab)_+=a_+b_++a_-b_-,(ab)_-=a_+b_-+a_-b_+$, we get from Lemma 2.13

$$\mathrm{E}\left((\Upsilon\Xi)_+|\mathcal{G}\right)=\Upsilon_+\mathrm{E}\left(\Xi_+|\mathcal{G}\right)+\Upsilon_-\mathrm{E}\left(\Xi_-|\mathcal{G}\right),$$
$$\mathrm{E}\left((\Upsilon\Xi)_-|\mathcal{G}\right)=\Upsilon_+\mathrm{E}\left(\Xi_-|\mathcal{G}\right)+\Upsilon_-\mathrm{E}\left(\Xi_+|\mathcal{G}\right).$$

The assumption $\mathrm{E}\left(|\Xi|\,|\mathcal{G}\right)<\infty$ implies finiteness of the right-hand sides of both equalities. Consequently, the left-hand sides are finite, too. Then by the definition of extended conditional expectation $\mathrm{E}\left(\Upsilon\Xi|\mathcal{G}\right)=\mathrm{E}\left((\Upsilon\Xi)_+|\mathcal{G}\right)-\mathrm{E}\left((\Upsilon\Xi)_-|\mathcal{G}\right)$, which together with the two preceding equalities completes the proof. □

Lemma 2.15. *Let $\mathcal{F}^0\subset\mathcal{G}\subset\mathcal{F}$, and let Υ and Ξ be random variables with values in $\mathbb{R}^d$ and $\mathbb{R}^p$, respectively, such that:* $\mathrm{E}^0|\Upsilon||\Xi|<\infty,\mathrm{E}\left(|\Xi|\,|\mathcal{G}\right)<\infty,\mathrm{E}\left(\Xi|\mathcal{G}\right)=0$ *and Υ is $\mathcal{G}$-measurable. Then $\mathrm{E}^0\Upsilon\Xi^{\mathrm{T}}=O$ (the null matrix).*

Proof. From the last three assumptions we get by Lemma 2.14 $\mathrm{E}\left(\Upsilon\Xi|\mathcal{G}\right)=O$; the first assumption implies, according to Lemma 2.12, the equality $\mathrm{E}^0\Upsilon\Xi^{\mathrm{T}}=\mathrm{E}^0\mathrm{E}\left(\Upsilon\Xi^{\mathrm{T}}|\mathcal{G}\right)$.□

Lemma 2.16. *Let (Ξ_n) be a converging in probability to zero sequence of nonnegative random variables such that for some increasing unbounded function $F:\mathbb{R}_+\to\mathbb{R}_+$ the sequence $\left(\mathrm{E}\left(\Xi_nF(\Xi_n)|\mathcal{G}\right)\right)$ is stochastically bounded. Then $\mathrm{E}\left(\Xi_n|\mathcal{G}\right)\xrightarrow{\mathrm{P}}0$.*

Proof. From the first assumption we have $\mathrm{E}\left(\Xi_nI\{\Xi_n\le N\}|\mathcal{G}\right)\xrightarrow{\mathrm{P}}0$ for every $N>0$, so it suffices to show that for any $\varepsilon>0$

$$\lim_{N\to\infty}\overline{\lim_{n\to\infty}}\mathrm{P}\left\{\mathrm{E}\left(\Xi_nI\{\Xi_n>N\}|\mathcal{G}\right)>\varepsilon\right\}=0. \tag{11}$$

Since F increases to infinity, we shall have $\Xi_nI\{\Xi_n>N\}\le F(N)^{-1}\Xi_nF(\Xi_n)$ for sufficiently large N (such that $F(N)>0$). Then by Lemma 2.3

$$\overline{\lim_{n\to\infty}}\mathrm{P}\left\{\mathrm{E}\left(\Xi_nI\{\Xi_n>N\}|\mathcal{G}\right)>\varepsilon\right\}$$
$$\le\overline{\lim_{n\to\infty}}\mathrm{P}\left\{\mathrm{E}\left(\Xi_nF(\Xi_n)|\mathcal{G}\right)>\varepsilon F(N)\right\}$$

for those N. Letting here $N\to\infty$, we deduce (11) from the last assumption of the lemma and unbounded growth of F.□

Lemma 2.17. *Let φ be an $\mathbb{R}_+$-valued measurable random process. Then for any $\mathcal{F}^0$-measurable random variable τ we have*

$$\mathrm{E}^0\varphi(\tau)=\mathrm{E}^0\varphi(s)\big|_{s=\tau}. \tag{12}$$

Proof. Denote $\mathcal{C}=\left\{C\in\mathcal{F}\otimes\mathcal{B}_+:\mathrm{E}^0I_C\left(\cdot,\tau(\cdot)\right)=\mathrm{E}^0I_C\left(\cdot,s\right)\big|_{s=\tau(\cdot)}\right\}$. Let $Q\in\mathcal{F},B\in\mathcal{B}_+$. Then $I_{Q\times B}\left(\omega,\tau(\omega)\right)=I_Q(\omega)I_{\tau^{-1}(B)}(\omega)$, whence by the assumption about τ and by Lemma 2.2 we have

$$\mathrm{E}^0I_{Q\times B}(\tau)=I_{\tau^{-1}(B)}\mathrm{E}^0I_Q\equiv\mathrm{E}^0\left(I_QI_B(s)\right)\big|_{s=\tau}.$$

Thus $\mathcal{C}$ contains all sets of the kind $Q\times B$, where $Q\in\mathcal{F},B\in\mathcal{B}_+$ ("measurable rectangles"). Then it follows from (2) (for nonnegative random variables) that $\mathcal{C}$ contains also all possible finite unions of pairwise disjoint measurable rectangles. According to Lemma 2.6 $\mathcal{C}$ contains the union of every increasing sequence of its members. Consequently, it contains the σ-algebra generated by measurable rectangles, *i.e.* Equality (12) holds for $\varphi=I_C,C\in\mathcal{F}\otimes\mathcal{B}_+$.

Passing to the general case, we denote $C_{nk}=\left\{k2^{-n}<\varphi\le(k+1)2^{-n}\right\}$ ($\in\mathcal{F}\otimes\mathcal{B}_+$ due to measurability of φ), $\chi_{nk}=I_{C_{nk}},\varphi_n=2^{-n}\sum_{k=1}^{\infty}k\chi_{nk}$. By construction $\varphi_n(s)\nearrow\varphi(s)$ for all ω and s and therefore $\varphi_n(\tau)\nearrow\varphi(\tau)$. From these relations we get by Lemma 2.7

$$\mathrm{E}^0\varphi_n(s)\nearrow\mathrm{E}^0\varphi(s),\quad\mathrm{E}^0\varphi_n(\tau)\nearrow\mathrm{E}^0\varphi(\tau) \tag{13}$$

As was shown (in another notation) in the proof of Lemma 2.13, $\mathrm{E}^0\varphi_n(\tau)=2^{-n}\sum_kk\mathrm{E}^0\chi_{nk}(\tau)$. By what was proved $\mathrm{E}^0\chi_{nk}(\tau)=\mathrm{E}^0\chi_{nk}(s)\big|_{s=\tau}$, which together with the previous equality yields $\mathrm{E}^0\varphi_n(\tau)=\mathrm{E}^0\varphi_n(s)\big|_{s=\tau}$. Juxtaposing this with (13), we arrive at (12). □

In the next two statements, the process φ need not be càdlàg.

Lemma 2.18. *Let $H\in\mathcal{V}_0^+$ and φ be a bounded measurable random process on $[a,b]\subset\mathbb{R}_+$. Then*

$$\mathrm{E}^0\int_a^b\varphi(s)\mathrm{d}H(s)=\int_a^b\mathrm{E}^0\varphi(s)\mathrm{d}H(s). \tag{14}$$

Proof. 1) Lemma 2.11 allows to consider, without loss of generality, that ϕ is $\mathbb{R}_+$-valued. Then the boundedness assumption together with Lemma 2.1 allows to consider that $0\le\varphi\le1$.

Let at first $\varphi(s)=\gamma f(s)$, where γ and f are a random variable and a Borel function, respectively. Then

Equality (14) follows from Lemma 2.13.

2) Let for all $s\in[a,b]$ $\varphi_n(s)\nearrow\varphi(s)$, where (φ_n) is an increasing sequence of [0,1]-valued random processes such that for each n

$$\mathrm{E}^0\int_a^b\varphi_n(s)\,\mathrm{d}H(s)=\int_a^b\mathrm{E}^0\varphi_n(s)\,\mathrm{d}H(s). \quad (15)$$

Then: for any s the sequence $(\mathrm{E}^0\varphi_n(s))$ increases by Lemma 2.3 and $\mathrm{E}^0\varphi_n(s)\to\mathrm{E}^0\varphi(s)$ by Lemma 2.13; $\int_a^b\varphi_n(s)\,\mathrm{d}H(s)\nearrow\int_a^b\varphi(s)\,\mathrm{d}H(s)$ by the Beppo Levi theorem. By the same theorem we get from the first relation $\int_a^b\mathrm{E}^0\varphi_n(s)\mathrm{d}H(s)\to\int_a^b\mathrm{E}^0\varphi(s)\mathrm{d}H(s)$. The second relation jointly with Lemma 2.13 yields

$\mathrm{E}^0\int_a^b\varphi_n(s)\mathrm{d}H(s)\to\mathrm{E}^0\int_a^b\varphi(s)\mathrm{d}H(s)$. Comparing these two conclusions with (15), we obtain (14).

3) Let $\mathcal{C}$ denote the class of all $C\in\mathcal{F}\otimes\left(\mathcal{B}_+\cap 2^{[a,b]}\right)$ such that Equality (14) holds for $\varphi=I_C$. According to item 1) $\mathcal{C}$ contains the algebra generated by measurable triangles. Then it follows from item 2) that $\mathcal{C}\supset\mathcal{F}\otimes\left(\mathcal{B}_+\cap 2^{[a,b]}\right)$.

4) Let us define the sequence (φ_n) by $\varphi_n=2^{-n}\sum_{k=0}^{2^n-1}k\chi_{nk}$, where the χ_{nk}'s are the same as in the proof of Lemma 2.17. Item 3), Lemma 2.11 and Lemma 2.1 imply together (15) for each *n*. Herein by construction $\varphi_n\nearrow\varphi$. It remains to refer to item 2). □

Theorem 2.19. *Let* $H\in\mathcal{V}_0^+$ *and* φ *be a nonnegative measurable random process on* $[a,b]\subset\mathbb{R}_+$. *Then Equality* (14) *holds with possible value* ∞ *of both sides.*

Proof. By Lemma 2.18 for any $n\in\mathbb{N}$

$$\mathrm{E}^0\int_a^b(\varphi(s)\wedge n)\mathrm{d}H(s)=\int_a^b\mathrm{E}^0(\varphi(s)\wedge n)\mathrm{d}H(s). \quad (16)$$

By Lemma 2.6 for any s

$$\mathrm{E}^0(\varphi(s)\wedge n)\to\mathrm{E}^0\varphi(s). \quad (17)$$

By the same argument as in the proof of that lemma,

$$\int_a^b(\varphi(s)\wedge n)\mathrm{d}H(s)\to\int_a^b\varphi(s)\mathrm{d}H(s) \quad (18)$$

and, in view of (17),

$$\int_a^b\mathrm{E}^0(\varphi(s)\wedge n)\mathrm{d}H(s)\to\int_a^b\mathrm{E}^0\varphi(s)\mathrm{d}H(s) \quad (19)$$

From (18) we have by Lemma 2.6 $\mathrm{E}^0\int_a^b(\varphi(s)\wedge n)\mathrm{d}H(s)\to\int_a^b\mathrm{E}^0\varphi(s)\mathrm{d}H(s)$ which together with (16) and (19) proves (14). □

3. A Subclass of the Class of Locally Square Integrable Martingales

The stochastic integral $\int_0^t\zeta(s)\mathrm{d}X(s)$ w.r.t. a local martingale X will be written, following [5,6], as $\zeta\cdot X(t)$. The designation of this section is to find the least restrictive extra assumptions providing the properties

$\mathrm{E}^0|\zeta\cdot X(t)|^2<\infty$,

$\mathrm{E}(\zeta\cdot X(t\vee s)|\mathcal{F}(t\wedge s))=\zeta\cdot X(t\wedge s)$

of $\zeta\cdot X$ underlying the derivations in Section 4. Herein we do not demand that $\mathrm{E}|\zeta\cdot X(t)|<\infty$, so the conditional expectations in these properties are not classical but extended.

The following statement differs from Doob's optional theorem for nonnegative discrete-time submartingales only with the absence of the demand $\mathrm{E}\gamma_k<\infty$ falling out of the proof if one uses the extended expectation instead of the ordinary one.

Lemma 3.1. *Let* $(\gamma_k,k\in\mathbb{Z}_+)$ *be a sequence of nonnegative random variables adapted to a flow* $(\mathcal{G}_k,\,k\in\mathbb{Z}_+)$ *and such that* $\mathrm{E}(\gamma_k|\mathcal{G}_{k-1})\ge\gamma_{k-1},\ k\in\mathbb{N}$. *Then the inequality* $\mathrm{E}(\gamma_\tau|\mathcal{G}_\sigma)\ge\gamma_\sigma$ *holds for any bounded stopping times* (*w.r.t. the same flow*) τ *and* $\sigma\le\tau$.

This result leads in the standard way to Doob's inequality asserted by the following lemma.

Lemma 3.2. *Under the assumptions of Lemma* 3.1, $\mathrm{E}\left(\max_{k\le n}\gamma_k^2\middle|\mathcal{G}_0\right)\le 4\mathrm{E}(\gamma_n^2|\mathcal{G}_0),\ n\in\mathbb{N}$.

Let $\underline{\mathcal{K}}$ denote the class of all $\mathbb{F}$-adapted $\mathbb{R}^d$-valued (d will be determined by the context, if matters) random processes M satisfying the conditions:

M1. For all t $\mathrm{E}^0|M(t)|^2<\infty$.

M2. For all $t\ge s\ge 0$ $\mathrm{E}(M(t)|\mathcal{F}(s))=M(s)$.

Lemma 3.3. *Let* $M\in\underline{\mathcal{K}}$. *Then* $\mathrm{E}^0(M(t)-M(s))\Upsilon^{\mathrm{T}}=O$ *for every* $t>s\ge0,p\in\mathbb{N}$ *and* $\mathcal{F}(s)$*-measurable* $\mathbb{R}^p$*-valued random variable* Υ *such that* $\mathrm{E}^0|M(t)-M(s)||\Upsilon|<\infty$.

Proof. Denote $\Xi=M(t)-M(s)$. Then: $\mathrm{E}(\Xi|\mathcal{F}(s))=0$ by condition **M2** and the assumption that M is $\mathbb{F}$-adapted; $\mathrm{E}^0(|\Xi||\mathcal{F}(s))<\infty$ by condition **M1**. It remains to refer to Lemma 2.15. □

Corollary 3.4. (from Lemmas 3.3 and 2.3) *Let* $M\in\underline{\mathcal{K}}$. *Then for all* $t>s\ge0$

$$\mathrm{E}^0(M(t)-M(s))M(s)^{\mathrm{T}}=O.$$

Hence and from the identity $|x|^2=\operatorname{tr}xx^{\mathrm{T}}$ $(x\in\mathbb{R}^d)$ we get

Corollary 3.5. *Let* $M\in\underline{\mathcal{K}}$. *Then for all* $t>s\ge0$ $\mathrm{E}^0|M(t)-M(s)|^2=\mathrm{E}^0|M(t)|^2-\mathrm{E}^0|M(s)|^2$.

Lemma 3.6. *Let* $M\in\underline{\mathcal{K}}$. *Then for any* $t_1>t_0\ge0$

$$\mathrm{E}\left(\sup_{t_0\le t\le t_1}|M(t)|^2\middle|\mathcal{F}(t_0)\right)\le 4\mathrm{E}\left(|M(t)|^2\middle|\mathcal{F}(t_0)\right).$$

Proof. Denote $t_{nk} = t_0 + k2^{-n}(t_1 - t_0)$, $\gamma_{nk} = |M(t_{nk})|$, $\Gamma_n = \max_{0\le k\le 2^n} \gamma_{nk}^2$. By construction and condition **M1** $\mathrm{E}(\gamma_{nk} | \mathcal{F}(t_{nk-1})) \ge |\mathrm{E}(M(t_{nk}) | \mathcal{F}(t_{nk-1}))| = \gamma_{nk-1}$, whence by Lemma 3.2 $\mathrm{E}(\Gamma_n | \mathcal{F}(t_0)) \le 4\mathrm{E}(\gamma_{n2^n} | \mathcal{F}(t_0)) \equiv 4\mathrm{E}(|M(t_1)|^2 | \mathcal{F}(t_0))$. Herein M is càdlàg (see the first sentence of the article), so $\Gamma_n \nearrow \sup_{t_0\le t\le t_1} |M(t)|^2$. It remains to make use of Lemma 2.7. □

Henceforth "stopping time" means "stopping time w.r.t. the flow $\mathbb{F}$".

Lemma 3.7. *Let* $M \in \underline{\mathcal{K}}$. *Then the equality* $\mathrm{E}(M(\sigma) | \mathcal{F}(s)) = M(\sigma \wedge s)$ *holds for every* $s \in \mathbb{R}_+$ *and bounded stopping time* σ.

Proof. We consider, without loss of generality, $\mathbb{R}$-valued processes. Writing $M(\sigma)\, I\{\sigma \le s\} = M(\sigma \wedge s)\, I\{\sigma \le s\}$ and noting that the r.h.s. of the equality is $\mathcal{F}(s)$-measurable, we get $\mathrm{E}(M(\sigma)\, I\{\sigma \le s\} | \mathcal{F}(s)) = M(\sigma \wedge s)\, I\{\sigma \le s\}$. So it suffices, in view of Lemma 2.11, to show that

$$\begin{aligned}&\mathrm{E}(M(\sigma) I\{\sigma > s\} | \mathcal{F}(s)) \\ &= M(s) I\{\sigma > s\} (\equiv M(\sigma \wedge s) I\{\sigma > s\}).\end{aligned} \tag{20}$$

By assumption there exists a number C such that $\sigma \le C$. We will prove Equality (20) for $s < C$ (otherwise it is trivial). Denote $N = 2^n$, $s_{n0} = s$, $s_{nk} = s + k2^{-n}(C - s)$, $I_{nk} = I\{s_{nk-1} < \sigma \le s_{nk}\}$, $\sigma_n = \sum_{k=1}^N s_{nk} I_{nk}$, $\Xi_n = (M(\sigma_n) - M(\sigma)) I\{\sigma > s\}$. By construction σ_n is a stopping time and $\sigma_n \searrow \sigma$ for all $\omega \in \Omega$. From the last relation and right-continuity of M we have $\Xi_n \to 0$. Herein $\Xi_n^2 \le 4\sup_{t\le C} M(t)^2$, whence by Lemma 3.6 $\mathrm{E}^0 \Xi_n^2 \le 16\mathrm{E}^0 M(t)^2$, which in view of **M1** proves stochastic boundedness of the sequence (Ξ_n^2). Then by Lemma 2.16

$$\mathrm{E}(\Xi_n | \mathcal{F}(s)) \xrightarrow{\mathrm{P}} 0. \tag{21}$$

Denote $\mu_{nk} = M(s_{nk})$, $\iota_{nk} = I\{\sigma \le s_{nk}\}$, $k = 0, \cdots, N$. From **M1** we have by Corollary 2.5 $\mathrm{E}(|\mu_{nk}| | \mathcal{F}(s)) < \infty$. On the strength of **M2** $\mathrm{E}(\mu_{nk+1} - \mu_{nk} | \mathcal{F}(s)) = M(s) - M(s) = 0$, which together with the previous relation results, by Lemma 2.14, in

$$\mathrm{E}((\mu_{nk+1} - \mu_{nk}) \iota_{nk} I\{\sigma > s\} | \mathcal{F}(s)) = 0. \tag{22}$$

By the same lemma and property **M2** of M

$$\mathrm{E}(\mu_{nN} I\{\sigma > s\} | \mathcal{F}(s)) = M(s) I\{\sigma > s\}. \tag{23}$$

By the construction of σ_n

$$\begin{aligned}M(\sigma_n) &= \sum_{k=1}^N \mu_{nk} I_{nk} \\ &\equiv \mu_{nN}\iota_{nN} - \mu_{n0}\iota_{n0} - \sum_{k=1}^{N-1} (\mu_{nk+1} - \mu_{nk}) \iota_{nk}.\end{aligned} \tag{24}$$

Herein $\iota_{n0} I\{\sigma > s\} = I\{s < \sigma \le s\} = 0$ and $\iota_{nN} = I\{\sigma \le b\} = 1$, which together with (24)-(22) and Lemma 2.11 yields $\mathrm{E}(M(\sigma_n) I\{\sigma > s\} | \mathcal{F}(s)) = M(s) I\{\sigma > s\}$. This equality jointly with (21) proves (20). □

The class of all random processes $M \in \underline{\mathcal{K}}$ such that the process $\mathrm{E}^0 |M|^2$ is continuous will be denoted $\mathcal{K}$.

Lemma 3.8. $\mathcal{K}$ *contains the sum of every two its elements.*

Proof. Let $Z = X + Y$, where $X, Y \in \mathcal{K}$. Property **M1** of Z ensues from Lemma 2.11. It follows from Lemma 2.3 that $\mathrm{E}^0 |Z(t)|^2 \le \mathrm{E}^0 |X(t)|^2 + \mathrm{E}^0 |Y(t)|^2$,

$$\mathrm{E}^0 |Z(t) - Z(s)|^2 \le \mathrm{E}^0 |X(t) - X(s)|^2 + \mathrm{E}^0 |Y(t) - Y(s)|^2$$

for all t and s. Hence property **M2** and, with account of Corollary 3.5, continuity of $\mathrm{E}^0 |Z|^2$ emerge. □

Theorem 3.9. $\mathcal{K} \subset \ell\mathcal{M}_2$.

Proof. Let $M \in \mathcal{K}$. Denote $U = \mathrm{E}^0 |M|^2$, $\tau_n = \inf\{s : U(s) \ge n\}$, $M_n(t) = M(t \wedge \tau_n)$. By construction all τ_n's are $\mathcal{F}(0)$-measurable random variables (and therefore stopping times) and $\tau_n \nearrow \infty$. The process $|M|^2$ is $\mathbb{F}$-adapted and right-continuous and therefore, by Theorem 2.1.1 [1], $\mathbb{F}$-progressive and all the more measurable. Then Lemma 2.17 applied to $\varphi = |M|^2$ and $\sigma = t \wedge \tau_n$ yields $\mathrm{E}^0 |M_n(t)|^2 = U(t \wedge \tau_n)$. By Corollary 3.5 U is an increasing process and therefore $U(t \wedge \tau_n) \le U(\tau_n)$. By the choice of M the process U is continuous, so $U(\tau_n) I\{\tau_n < \infty\} = n, U(\tau_n) \le n$. Consequently, $\mathrm{E}^0 |M_n(t)|^2 \le n$ and therefore $\mathrm{E} |M_n(t)|^2 \le n$. Herein by Lemma 3.7 $\mathrm{E}(M_n(t) | \mathcal{F}(s)) = M(t \wedge \tau_n \wedge s)$ ($\equiv M_n(s)$ as $t > s$). Thus $\sup_t \mathrm{E} |M_n(t)|^2 < \infty$ and M_n is a martingale. This means, since (τ_n) is an increasing to infinity sequence of stopping times, that $M \in \ell\mathcal{M}_2$. □

The quadratic variation of a semimartingale S and the quadratic characteristic of a locally square integrable martingale M will be denoted $[S]$ and $\langle M \rangle$, respectively.

The following statement is immediate from Theorem 1.8.1 in [5] and the definition of quadratic characteristic.

Lemma 3.10. *Let* M *be an* $\mathbb{R}$*-valued locally square integrable martingale. Then for any stopping time* τ $\mathrm{E}^0 (M(\tau) - M(0))^2 = \mathrm{E}^0 [M](\tau) = \mathrm{E}^0 \langle M \rangle(\tau)$.

Corollary 3.11. *Let* M *be an* $\mathbb{R}^d$*-valued locally square integrable martingale. Then for any stopping time* τ $\mathrm{E}^0 (M(\tau) - M(0))^2 = \mathrm{E}^0 \mathrm{tr} [M](\tau) = \mathrm{E}^0 \mathrm{tr} \langle M \rangle(\tau)$.

Note that all the random variables $\mathrm{E}^0\cdots$ in the above two statements are, generally speaking, $\overline{\mathbb{R}}_+$-valued.

The Lebesgue - Stieltjes integral $\int_0^t f(s)\,\mathrm{d}A(s)$, where A is a random process of locally bounded variation, will be written shortly as $f\circ A(t)$.

In the next statement, the process $\mathfrak{z}$ need not be right-continuous and even may have second-kind discontinuities.

Lemma 3.12. *Let W be an $\mathbb{R}^d$-valued process of class $\mathcal{K}$ and $\mathfrak{z}$ be an $\mathbb{R}^{d*}$-valued $\mathbb{F}$-predictable random process such that*

$$\mathrm{E}^0\left(|\mathfrak{z}|^2\circ\mathrm{tr}\langle W\rangle(t)\right)<\infty,\quad t\in\mathbb{R}_+,\tag{25}$$

and the process $\mathrm{E}^0\left(|\mathfrak{z}|^2\circ\mathrm{tr}\langle W\rangle\right)$ is continuous. Then $\mathfrak{z}\cdot W\in\mathcal{K}$.

Proof. Lemma 3.8 allows us to confine ourselves to the case $d=1$.

The assumptions of the lemma imply by Theorem I.4.40 [6] existence of the process $M\equiv\mathfrak{z}\cdot W$. The same theorem asserts that $M\in\ell\mathcal{M}_2$ and $\langle M\rangle=\mathfrak{z}^2\circ\langle W\rangle$, From the last equality we also have by Corollary 3.11 $\mathrm{E}^0M^2=\mathrm{E}^0\left(\mathfrak{z}^2\circ\langle W\rangle\right)$, which together with (25) proves property **M1** of M and continuity of E^0M^2.

The relation $M\in\ell\mathcal{M}_2$ implies existence of an increasing to infinity sequence (σ_n) of stopping times such that for all $n\in\mathbb{N}, t>s\geq 0$

$$\mathrm{E}\left(M(t\wedge\sigma_n)\middle|\mathcal{F}(s)\right)=M(s\wedge\sigma_n).\tag{26}$$

Setting in Lemma 3.10 at first $\tau=t\wedge\sigma_n$ and then $\tau=t$ and taking to account that $\langle M\rangle$ is an increasing process, we get with account of Lemma 2.3 $\mathrm{E}^0M(t\wedge\sigma_n)^2\leq\mathrm{E}^0M(t)^2$, which together with **M1** entails stochastic boundedness of the sequences $\left(\mathrm{E}^0\left(M(t\wedge\sigma_n)^2\right),n\in\mathbb{N}\right)$ and (in view of Lemma 2.4) $\left(\mathrm{E}\left(M(t\wedge\sigma_n)^2\middle|\mathcal{F}(s)\right),n\in\mathbb{N}\right)$. So Lemma 2.16 asserts that $\mathrm{E}\left(M(t\wedge\sigma_n)\middle|\mathcal{F}(s)\right)\xrightarrow{\mathrm{P}}\mathrm{E}\left(M(t)\middle|\mathcal{F}(s)\right)$. Thus, letting $n\to\infty$ in (26), we obtain **M2**. □

4. The Main Result

Lemma 4.1. *Let Λ be a continuous increasing function, b and H be bounded in each interval Borel functions and U be a function satisfying, for all $t\in\mathbb{R}_+$, the equality*

$$U(t)=\int_0^t q(s)\,\mathrm{d}\Lambda(s)+H(t),\tag{27}$$

where q is a Borel function with values in $\mathbb{R}\cup\{-\infty\}$ such that $q\leq -bU$. Suppose also that

$$\int_0^t|b(s)|\,\mathrm{d}\Lambda(s)<\infty\tag{28}$$

for all t. Then $U\leq T$, where T is the solution of the equation

$$T=-(bT)\circ\Lambda+H.\tag{29}$$

Proof. By condition (28) and the assumptions about Λ the integral $b\circ\Lambda$ exists on $\mathbb{R}_+$ and is a function of locally bounded variation. Equality (27) and the assumptions about Λ and H show that U is a Borel function. So $(bU)\circ\Lambda=U\circ(b\circ\Lambda)$. The assumptions of the lemma imply existence of the integral $q\circ\Lambda\,(=U-H$ because of (27)$)$, as well (so that $q>-\infty$ almost everywhere w.r.t. the measure with distribution function Λ). This entitles us to define the function h by $h=(q+bU)\circ\Lambda$. It decreases, since, by assumption, $q+bU\leq 0$ and Λ increases. Also, it is continuous, since so is Λ.

Denoting $y=U-T$ and subtracting (27) from (29), we get the equation $y=-y\circ(b\circ\Lambda)+h$. Hence, taking to account that h is continuous and starts from zero, we find

$$y(t)=\mathrm{e}^{-b\circ\Lambda(t)}\int_0^t\mathrm{e}^{b\circ\Lambda(s)}\mathrm{d}h(s).$$

The function h being decreasing, the r.h.s. is nonpositive. □

Corollary 4.2. *Let Λ be a continuous increasing $\mathbb{F}$-adapted random process, b be an $\mathbb{F}$-progressive random process with values in $\mathbb{R}\cup\{-\infty\}$ satisfying, for all t, condition* (28), *H be an $\mathbb{F}$-semimartingale and U be a random process satisfying, for all t, equality* (27), *where q is a measurable random process such that $q\leq -bU$. Then for all t*

$$U(t)\leq\mathrm{e}^{-R(t)}H(0)+\mathrm{e}^{-R(t)}\int_0^t\mathrm{e}^{R(s)}\mathrm{d}H(s),\tag{30}$$

where $R=b\circ\Lambda$.

Proof. Denote $V=\mathrm{e}^{-R}$. Noting that $(bT)\circ\Lambda=T\circ R$ and taking to account continuity of Λ, we write down the solution of (29):

$$T(t)=H(t)+\mathrm{e}^{R(t)}\int_0^t\mathrm{e}^{-R(s)}H(s)\,\mathrm{d}R(s).\tag{31}$$

By construction R is a continuous process of locally bounded variation, so $\mathrm{e}^{-R}\mathrm{d}R=-\mathrm{d}V$. By Proposition I.4.49d [6] the covariation of any such process and a semimartingale equals zero, so the integration-by-parts formula yields $H\circ V=HV-H(0)V(0)-V\cdot H$. Thus $\left(\mathrm{e}^{-R}H\right)\circ R=H(0)-HV+V\cdot H$, which turns (31) into

$$T(t)=\mathrm{e}^{-R(t)}H(0)+\mathrm{e}^{-R(t)}\int_0^t\mathrm{e}^{R(s)}\mathrm{d}H(s).$$

Now, (30) follows from Lemma 4.1. □

The main result of this article concerns equations of the kind

$$X(t)=\int_0^t Q(s,X(s))\,\mathrm{d}\iota(s)+Y(t),\tag{32}$$

and relies on the assumption

S. For every $\mathbb{R}^d$-valued random process $Y \in \mathcal{K}$ equation (32) has a unique strong solution.

As usually, S^c signifies the continuous martingale constituent (see [1,5,6]) of a semimartingale S.

Theorem 4.3. *Let* $\iota \in \mathcal{V}_0^{+,c}, M$ *be an* $\mathbb{R}^d$*-valued process of class* $\mathcal{K}$ *and* Q *be an* $\mathbb{R}^d$*-valued random function on* $\mathbb{R}_+ \times \mathbb{R}^d$, *continuous in* $x \in \mathbb{R}^d$ *and* $\mathbb{F}$*-progressive in* $(\omega, t) \in \Omega \times \mathbb{R}_+$. *Suppose also that condition* **S** *is fulfilled and there exists an* $\mathcal{F}(0) \otimes \mathcal{B}_+$*-measurable in* (ω, t) *nonnegative random process* ψ *such that* $x^{\mathrm{T}} Q(t, x) \le -\psi(t)|x|^2$ *and*

$$\int_0^t \psi(s)\,\mathrm{d}\iota(s) < \infty, \quad t \in \mathbb{R}_+. \tag{33}$$

Then the strong solution of the equation

$$X(t) = \int_0^t Q(s, X(s))\,\mathrm{d}\iota(s) + M(t) \tag{34}$$

satisfies, for all t, the inequality

$$\mathrm{E}^0 |X(t)|^2 \le \mathrm{e}^{-\Psi(t)} |M(0)|^2 + \mathrm{e}^{-\Psi(t)} \int_0^t \mathrm{e}^{\Psi(s)} \mathrm{d}\mathrm{E}^0 \mathrm{tr}\langle M\rangle(s),$$

where $\Psi = 2\psi \circ \iota$.

Proof. Denote $\tau_n = \inf\{s : |X(s)| \ge n\}$, $M_n(t) = M(t \wedge \tau_n)$, so that τ_n is a stopping time, $M_n \in \ell\mathcal{M}_2$ and

$$|X(s-)| \le k \quad \text{as} \quad s \le \tau_n. \tag{35}$$

Let further X_n denote the solution of the equation

$$X_n(t) = \int_0^t Q(s, X_n(s))\,\mathrm{d}\iota(s) + M_n(t) \tag{36}$$

(this definition of X_n is correct due to condition **S**). Then $X_n(s-) = X(s-)$ as $s \le \tau_n$ (because $M_n = M$ for these s). Consequently,

$$\left(X_n^{\mathrm{T}}\right)^- \cdot M_n = \left(\left(X^{\mathrm{T}}\right)^- I_{[0,\tau_n]}\right) \cdot M. \tag{37}$$

By the choice of M and by Corollary 3.11 and Theorem 3.9 $\mathrm{E}^0 \mathrm{tr}\langle M\rangle(t) < \infty$ for all t and the process $\mathrm{E}^0 \mathrm{tr}\langle M\rangle$ is continuous. Then because of (35) $\mathrm{E}^0\left(\left|X^-\right|^2 I_{[0,\tau_n]}\right) \circ \mathrm{tr}\langle M\rangle(t) < \infty$ for all t. Obviously, the process $\mathrm{E}^0\left(\left|X^-\right|^2 I_{[0,\tau_n]}\right) \circ \mathrm{tr}\langle M\rangle$ is continuous, too. Thus Lemma 3.12 asserts that $\left(\left(X^{\mathrm{T}}\right)^- I_{[0,\tau_n]}\right) \cdot M \in \mathcal{K}$, whence in view of (37)

$$\mathrm{E}^0 \left(X_n^{\mathrm{T}}\right)^- \cdot M_n = 0. \tag{38}$$

Denote

$$\varphi_n(\tau) = X_n(s)^{\mathrm{T}} Q(s, X_n(s)),$$

$$D_n(t) = 2\int_0^t \varphi_n(s)\,\mathrm{d}\iota(s), \tag{39}$$

$H_n = |M_n(0)|^2 + \mathrm{E}^0 \mathrm{tr}[M_n]$. From (36) we have by the assumptions about ι and M

$$X_n^c = M_n^c, \quad \Delta X_n = \Delta M_n, \quad X_n(0) = 0. \tag{40}$$

By Theorem 2.4.6 in [1] (or, the same, Theorem I.4.47 in [6])

$$[M_n](t) = \left\langle M_n^c\right\rangle(t) + \sum_{0<s\le t} \Delta M_n(s) \Delta M_n(s)^{\mathrm{T}}. \tag{41}$$

Writing Itô's formula for $f(X_n(t))$ and putting $f(x) = |x|^2$, so that $f'(x) = 2x^{\mathrm{T}}, (1/2) f'' = \mathbf{1}$ (a twice covariant tensor), $f(x+y) - f(x) - f'(x) y = |y|^2$, we get with account of (36), (40) and (41), continuity of ι and the identity $|x|^2 = \mathrm{tr} xx^{\mathrm{T}}$

$$[M_n](t) = \left\langle M_n^c\right\rangle(t) + \sum_{0<s\le t} \Delta M_n(s) \Delta M_n(s)^{\mathrm{T}}.$$

By Theorem 3.9 and Corollary 3.11 $\mathrm{E}^0 \mathrm{tr}[M](t) < \infty$ for all t since $M \in \mathcal{K}$. Hence and from the evident inequality $\mathrm{tr}[M_n] \le \mathrm{tr}[M]$ we have $H_n(t) < \infty$, which together with (38) yields, by Lemma 2.11,

$$\mathrm{E}^0\left(2\left(X_n^{\mathrm{T}}\right)^- \cdot M_n + \mathrm{tr}[M_n]\right) = H_n.$$

By construction and the assumptions about Q and $\iota D_n \le 0$, whence by Formula (2) for nonnegative random variables $\mathrm{E}^0\left(|X_n|^2 - D_n\right) = \mathrm{E}^0 |X_n|^2 - \mathrm{E}^0 D_n$. The last three equalities together with Lemmas 2.11 and 2.13 imply that

$$\mathrm{E}^0 |X_n|^2 = |M_n(0)|^2 + \mathrm{E}^0 D_n + H_n. \tag{42}$$

By construction and the assumption on Q the process φ_n is càdlàg and non-positive. Then from (39) we have by the choice of ι and by Theorem 2.19

$$\mathrm{E}^0 D_n = q_n \circ \Lambda, \tag{43}$$

where $q_n(t) = \mathrm{E}^0\left(X_n(t)^{\mathrm{T}} Q(t, X_n(t))\right), \Lambda = 2\iota$. Then equality (42), whose l.h.s. is, evidently, an $\overline{\mathbb{R}}_+$-valued process, together with established above finiteness of H_n shows that $q_n \circ \Lambda(t) > -\infty$ for all t (though q_n may take the value $-\infty$ with positive probability).

By the construction of q_n, the assumption on Q and by Lemma 2.3 $q_n \le -\mathrm{E}^0\left(\psi |X_n|^2\right)$. The process ι was assumed increasing and therefore Λ increases, too; the process ψ was assumed nonnegative, so $q_n \le 0$ by Lemma 2.3. Thus $q_n \circ \Lambda \le 0$, which together with (42), (43) and finiteness of H_n yields $\mathrm{E}^0 |X_n|^2 < \infty$. Then from $\mathcal{F}(0)$-measurability of $\psi(t), t \in \mathbb{R}_+$, we have by Lemma 2.15 $\mathrm{E}^0\left(\psi |X_n|^2\right) = \psi \mathrm{E}^0 |X_n|^2$ and therefore $q_n \le -\psi \mathrm{E}^0 |X_n|^2$. From this inequality and (33), (42), (43) we get by Corollary 4.2

$$\mathrm{E}^0 |X_n(t)|^2 \le \mathrm{e}^{-\Psi(t)} |M_n(0)|^2 + \mathrm{e}^{-\Psi(t)} \int_0^t \mathrm{e}^{\Psi(s)} \mathrm{d}H_n(s)$$

and all the more

$$\mathrm{E}^0 \left|X_n(t)\right|^2 \le \mathrm{e}^{-\Psi(t)} \left|M_n(0)\right|^2 + \mathrm{e}^{-\Psi(t)} \int_0^t \mathrm{e}^{\Psi(s)} \mathrm{d}\mathrm{E}^0 \mathrm{tr}\left[M\right](s).$$

Obviously, $X_n(t) \to X(t), M_n(0) \to M(0)$ as $n \to \infty$. Then Corollary 2.8 asserts that $\mathrm{E}^0 \left|X(t)\right|^2 \le \underline{\lim}\mathrm{E}^0 \left|X_n(t)\right|^2$. It remains to note that $\mathrm{E}^0 \mathrm{tr}\left[M\right] = \mathrm{E}^0 \mathrm{tr}\langle M \rangle$ by Corollary 3.11. □

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