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Solving a Nonlinear Multi-Order Fractional Differential Equation Using Legendre Pseudo-Spectral Method

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ABSTRACT

In this paper, we apply the Legendre spectral-collocation method to obtain approximate solutions of nonlinear multi-order fractional differential equations (M-FDEs). The fractional derivative is described in the Caputo sense. The study is conducted through illustrative example to demonstrate the validity and applicability of the presented method. The results reveal that the proposed method is very effective and simple. Moreover, only a small number of shifted Legendre polynomials are needed to obtain a satisfactory result.

Keywords: Legendre Pseudo-Spectral Method; Multi-Order Fractional Differential Equations; Caputo Derivative

1. Introduction

Many phenomena in engineering physics, chemistry, and other sciences can be described very successfully by models that use mathematical tools of fractional calculus, *i.e*. the theory of derivatives and integrals of non-integer order [1-3]. For example, they have been successfully used in modeling frequency dependent damping behavior of many viscoelastic materials. There are numerous research which demonstrate the applications of fractional derivatives in the areas of electrochemical processes, dielectric polarization, colored noise, and chaos.

The numerical solution of differential equations of integer order has been a hot topic in numerical and computational mathematics for a long time. The solution of fractional differential equations has been recently studied by numerous authors. However, the state of the art is far less advanced for general fractional order differential equations. Moreover, to the best of the authors knowledge, very few algorithms for the numerical solution of Multi-order Fractional Differential Equations (M-FDEs) have been suggested [4-6], particularly algorithms for analytical solutions and approximate solutions of nonlinear M-FDEs.

As we know, the fractional derivatives are global dependence problems (they are definite by the integral in [0, t]), from this point, the global methods—spectral methods maybe are more suit to solve the FDEs. As we know the standard spectral methods possess infinite order of accuracy for the equations with well regularity solutions, while fail to many complicated problems with singular solutions. So, it is attracted considerable interest that how to enlarge the adaptability of spectral methods, and construct certain simple approximation schemes without loss of any accuracy to even more complicated problems. It is well known that Legendre polynomials are well known family of orthogonal polynomials on the interval [−1, 1]. They are widely used because of their good properties in the approximation of functions [7-9]. For multi-order fractional differential equation, a operational matrix is studied [4], Chebyshev wavelets is considered in [10], adomian decomposition is considered in [11], a variational iteration method is considered in [12], predictor-corrector method is studied in [13], operator splitting method is considered in [14], and Adams method is researched in [15]. In this paper, we introduce the Legendre pseudo-spectral method to solve multi-order arbitrary differential equations, which include the linear and nonlinear differential equations.

The outline of this paper is as follows. In Section 2, we review the basic definitions and the properties of the fractional calculus. In Section 3, the approximation of fractional derivative by Legendre Polynomials is obtained. In Section 4, we present the application of the Legendre pseudo-spectral method to multi-order fractional differential equation. Some numerical examples are provided in Section 5. Also a conclusion is given in Section 6.

2. Fractional Calculus

In this section, we first review the basic definitions and the operational properties of fractional integral and derivative for the purpose of acquainting with sufficient fractional calculus theory. Many definitions and studies of fractional calculus have been proposed in the past two centuries. These definitions include Riemann-Liouville, Reize, Caputo and Grnwald-Letnikov fractional operators. The two most commonly used definitions are the Riemann-Liouville operator and the Caputo operator. We give some definitions and properties of the fractional calculus.

${}^cD^\alpha$ denotes Caputo fractional derivative of order α is defined

$$\begin{aligned} {}^cD^\alpha y(t) &= I^{n-\alpha}D^n y(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\tau^n}y(\tau)\mathrm{d}\tau, \quad (1) \\ & n-1<\alpha<n, \end{aligned}$$

where I^α denotes the Riemann-Liouville fractional integral of order order α defined as

$$\begin{aligned} I^\alpha y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} y(\tau)\mathrm{d}\tau \\ I^0 y(t) &= y(t) \end{aligned} \quad (2)$$

As we all know, there are some different definitions of fractional operator except the Caputo fractional derivative. From a theoretical point of view the most natural approach is the Riemann-Liouville definition defined as

$$\begin{aligned} {}^RD_t^\alpha y(t) &= D^n\left(I^{n-\alpha}y(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t}\int_0^t (t-\tau)^{n-\alpha-1} y(\tau)\mathrm{d}\tau, \\ & n-1<\alpha<n, \end{aligned}$$

The relationship between the Caputo definition and the Riemann-Liouville definition can be given by the following formula

$$\begin{aligned} {}^RD_t^\alpha y(t) &= {}^cD^\alpha y(t) + \sum_{k=0}^{n-1}\psi_{k-\alpha+1}(t) f^{(k)}(0), \\ & n-1<\alpha<n, \end{aligned}$$

where

$$\psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operation:

$$\begin{aligned} & {}^cD^\alpha\left(\lambda f(t)+\mu g(t)\right) \\ & = \lambda^cD^\alpha f(t) + \mu^cD^\alpha g(t) \end{aligned}$$

For the Caputo and Riemann-Liouville's derivative we have:

$${}^cD^\alpha C = 0, \quad C \text{ is a constant},$$

$$\begin{aligned} & {}^cD^\alpha t^n \\ & = \begin{cases} 0, & \text{for } n\in N_0 \text{ and } n<\lceil\alpha\rceil, \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}, & \text{for } n\in N_0 \text{ and } n\ge\lceil\alpha\rceil. \end{cases} \end{aligned} \quad (3)$$

Amongst a variety of definition for fractional order derivatives, Caputo fractional derivative has been used because it allows physically interpretable initial conditions.

3. Evaluation of the Fractional Derivative Using Legendre Polynomials

The well known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula:

$$\begin{aligned} & L_{i+1}(z) = \frac{2i+1}{i+1}zL_i(z) - \frac{i}{i+1}L_{i-1}(z), \\ & i=1,2,\cdots, \end{aligned} \quad (4)$$

where $L_0(z)=1$ and $L_1(z)=z$. In order to use these polynomials on the interval $t\in[0,L]$ we define the so called shifted Legendre polynomials by introducing the change of variable $z=(2t-L)/L$. Let the shifted Legendre polynomials $L_i\left(\frac{2t-L}{L}\right)$ be denoted by $P_i(t)$. Then $P_i(t)$ can be obtained as follows:

$$\begin{aligned} & P_{i+1}(t) = \frac{(2i+1)(2t-L)}{(i+1)L}P_i(t) - \frac{i}{i+1}P_{i-1}(t), \\ & i=1,2,\cdots, \end{aligned} \quad (5)$$

where $P_0(t)=1$ and $P_1(t)=(2t-1)/L$. The analytic form of the shifted Legendre polynomials $P_i(t)$ of degree i given by:

$$P_i(t) = \sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!t^k}{(i-k)!(k!)^2L^k}. \quad (6)$$

Note that $P_i(0)=(-1)^i$ and $P_i(L)=1$. The orthogonality condition is:

$$\int_0^1 P_i(t)P_j(t)\mathrm{d}t = \begin{cases} \dfrac{L}{2i+1}, & \text{for } i=j; \\ 0, & \text{for } i\ne j \end{cases} \quad (7)$$

The function $y(t)$, square integrable in $[0,L]$, may be expressed in terms of shifted Legendre polynomials as:

$$y(t) = \sum_{i=0}^{\infty} y_i P_i(t), \quad (8)$$

where the coefficients y_i are given by:

$$y_i = \frac{2i+1}{L}\int_0^1 y(t)P_i(t)\mathrm{d}t, \quad i = 1,2,\cdots. \tag{9}$$

In practice, only the first $(m+1)$-terms shifted Legendre polynomials are considered. Then we have:

$$y_m(t) = \sum_{i=0}^{m} y_i P_i(t), \tag{10}$$

In the following theorem we introduce an approximate formula of the fractional derivative of $y(t)$.

Theorem 1 *Let* $y(t)$ *be approximated by shifted Legendre polynomials as* (10) *and also suppose* $\alpha > 0$ *then*:

$$D^{\alpha}\left(y_m(t)\right) = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} u_i w_{i,k}^{(\alpha)} t^{k-\alpha}, \tag{11}$$

where $w_{i,k}^{(\alpha)}$ is given by:

$$w_{i,k}^{(\alpha)} = \frac{(-1)^{(i+k)}(i+k)!}{(i-k)!(k)!\Gamma(k+1-\alpha)L^k}. \tag{12}$$

Proof. Since the Caputo's fractional differentiation is a linear operation we have:

$$D^{\alpha}\left(y_m(t)\right) = \sum_{i=0}^{m} y_i D^{\alpha}\left(P_i(t)\right).$$

Employing Equations (3) in Equation (6) we have:

$$D^{\alpha}P_i(t) = 0, \quad i = 0,1,\cdots,\lceil\alpha\rceil - 1, \quad \alpha > 0.$$

Also, for $i = \lceil\alpha\rceil,\cdots,m$, by using Equations (3) in Equation (6) we get:

$$D^{\alpha}P_i(t) = \sum_{i=0}^{m}\frac{(-1)^{(i+k)}(i+k)!}{(i-k)!(k)^2!L^k}D^{\alpha}\left(t^k\right)$$
$$= \sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{(i+k)}(i+k)!}{(i-k)!(k)!\Gamma(k+1-\alpha)L^k}t^{k-\alpha}.$$

4. Solution of Multi-Order Fractional Differential Equations

Consider the multi-order fractional differential equations of type given in (13):

$$D^{\alpha}y(t) = F\left(t, y(t), D^{\beta_1}y(t),\cdots,D^{\beta_m}y(t)\right), \quad t \in [0,L) \tag{13}$$

with initial conditions:

$$y^{(k)}(0) = d_k, \quad k = 0,1,\cdots,n, \tag{14}$$

where $n < \alpha \le n+1 = \lceil\alpha\rceil$, $0 < \beta_1 < \beta_2 < \cdots < \beta_m < \alpha$ and D^a denotes the Caputo fractional derivative of order *a*. It should be noted that *F* can be nonlinear in general. To solve problems (13) and (14) we approximate $y(t)$, $D^{\alpha}y(t)$ and (t), for $i = 1,2,\cdots,m$ by the shifted Legendre polynomials.

In order to use Legendre pseudo-spectral method, we first approximate $y(t)$ as:

$$y_m(t) = \sum_{i=0}^{m} y_i P_i(t), \tag{15}$$

From (13), (15) and Theorem 1, we have:

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} y_i w_{i,k}^{(\alpha)} t^{k-\alpha} = F\left(t, \sum_{i=0}^{m} y_i P_i(t), \sum_{i=\lceil\beta_1\rceil}^{m}\sum_{k=\lceil\beta_1\rceil}^{i} y_i w_{i,k}^{(\beta_1)} t^{k-\beta_1},\cdots, \sum_{i=\lceil\beta_m\rceil}^{m}\sum_{k=\lceil\beta_m\rceil}^{i} y_i w_{i,k}^{(\beta_m)} t^{k-\beta_m}\right) \tag{16}$$

we now collocate (16) at $m+1-\lceil\alpha\rceil$ points t_p as:

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} y_i w_{i,k}^{(\alpha)} t_p^{k-\alpha} = F\left(t_p, \sum_{i=0}^{m} y_i P_i(t_p), \sum_{i=\lceil\beta_1\rceil}^{m}\sum_{k=\lceil\beta_1\rceil}^{i} y_i w_{i,k}^{(\beta_1)} t_p^{k-\beta_1},\cdots, \sum_{i=\lceil\beta_m\rceil}^{m}\sum_{k=\lceil\beta_m\rceil}^{i} y_i w_{i,k}^{(\beta_m)} t_p^{k-\beta_m}\right) \tag{17}$$
$$p = 0,1,\cdots,m-\lceil\alpha\rceil.$$

For suitable collocation points t_p, we use $m+1-\lceil\alpha\rceil$ roots of shifted Legendre polynomial $P_{m+1-\lceil\alpha\rceil}(x)$.

Also, by substituting (15) in the initial conditions (14) we can obtain $\lceil\alpha\rceil$ equations as follows:

$$\sum_{i=0}^{m} y_i P_i^{(k)}(0) = d_k, \quad k = 0,1,\cdots,n. \tag{18}$$

$m+1-\lceil\alpha\rceil$ equations in (17) together with $\lceil\alpha\rceil$ equations of the boundary conditions (18), generate $(m+1)$ equations which can be solved using Newton's iterative method for the unknown $y_i, i = 0,1,\cdots,m$. Consequently $y(t)$ can be calculated.

5. Numerical Examples

In this section, we will use the Legendre collocation method to solve nonlinear fractional (arbitrary) order differential equation. These examples are considered because closed form solutions are available for them, or they have also been solved using other numerical schemes. This allows one to compare the results obtained using this scheme with the analytical solution or the solutions obtained using other schemes.

Example 1. *Consider the following initial value problem*

$$\begin{cases} D^{\alpha}u(t)+t\cdot D^{\beta}u(t)+u(t)=\dfrac{6}{\Gamma(4-\alpha)}t^{3-\alpha} \\ +\dfrac{6}{\Gamma(4-\beta)}t^{4-\beta}+t^3+t, \alpha\in(2,3), \beta\in[1,2] \\ u(0)=0, \quad u'(0)=0, \quad u''(0)=0. \end{cases} \quad (19)$$

If we take $\alpha = 2.5$, $\beta = 1.9$, the exact solution of (19) is $y(t) = t^3 + t$.

To solve the above problem, by applying the technique described in last section with $m = 4$, we get

$$y_0 = 0.75, \quad y_1 = 0.95, \quad y_2 = 0.25,$$
$$y_3 = 0.05, \quad y_4 = 0.0000.$$

Then the approximate solution will be

$$\sum_{i=0}^{4} y_i P_i(t) = 0.75 + 0.95(2t-1) + 0.25(6t^2+6t+1) + 0.05(20t^3+30t^2+12t-1) = t^3 + t.$$

It is clear that the approximate solution coincides with the analytic solution.

Example 2. *Consider the following nonlinear differential equation*:

$$\begin{cases} D^4 u(t) + D^{\frac{7}{2}} u(t) + u^3(t) = t^9, \\ u(0) = u'(0) = u''(0) = 0, \quad u^{(3)} = 6. \end{cases} \quad (20)$$

The exact solution of (20) is $y(t) = t^3$.

We use the Legendre pseudo-spectral method with $m = 4$ to obtain 4

$$y_0 = 0.2506, \quad y_1 = 0.4511, \quad y_2 = 0.2508,$$
$$y_3 = 0.0503, \quad y_4 = 0.0000.$$

Then the approximate solution will be

$$\sum_{i=0}^{4} y_i P_i(t)$$
$$= 0.2506 + 0.4511(2t-1)$$
$$+ 0.2508(6t^2+6t+1) + 0.0503(20t^3+30t^2+12t-1)$$
$$= 1.006t^3 - 0.0042t^2 + 0.001t \approx t^3.$$

My result is in good agreement with exact solution. This demonstrates the importance of my numerical scheme in solving nonlinear multi-order fractional differential equations.

Example 3. *Following Odibat and Momani* [6], *we consider fractional Riccati equation*

$$D^{\alpha} y(t) = 2y(t) - y(t)^2 + 1, \quad 0 < \alpha \le 1, \quad 0 \le t < 2 \quad (21)$$

subject to the initial state $y(0) = 0$, which is studied by Odibat [6] by using the modified homotopy perturbation method and Li [10] by using the Chebyshev wavelet operational matrices method. Here we use the Legendre pseudo-spectral method to solve it.

This is a nonlinear system of algebraic equations. The numerical solution, for $m = 8$, is shown in **Figure 1**. The exact solution of this problem, when $\alpha = 1$, is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$$

and we can observe that, as $t \to \infty$, $y(t) \to 1 + \sqrt{2}$. From **Figure 1** we can see the numerical solution is very good agreement with the exact solution when $\alpha = 1$. When $\alpha = 0.5$ and $\alpha = 0.75$, the numerical solution is very good agreement with the result in Ref. [10]. Therefore, we hold that the solution for $\alpha = 0.5$ and $\alpha = 0.75$ is also credible. Numerical results with comparison to Ref. [6] and [10] are given in **Table 1** on the interval $[0, 1]$.

The difference between our result or in Ref. [10] and the result in Ref. [6] is obvious. Because only the fourth-order term of the homotopy perturbation solution were used in evaluating the approximate solutions in Ref. [6]. We hold that our results are better for $\alpha = 0.5$ and $\alpha = 0.75$. Compared to the Chebyshev wavelet operational matrices method in Ref. [10] with 192 degrees of freedom, our method reached the same accuracy used 9 degrees of freedom only.

6. Conclusion

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. For that reason we need a reliable and efficient technique for the solution of

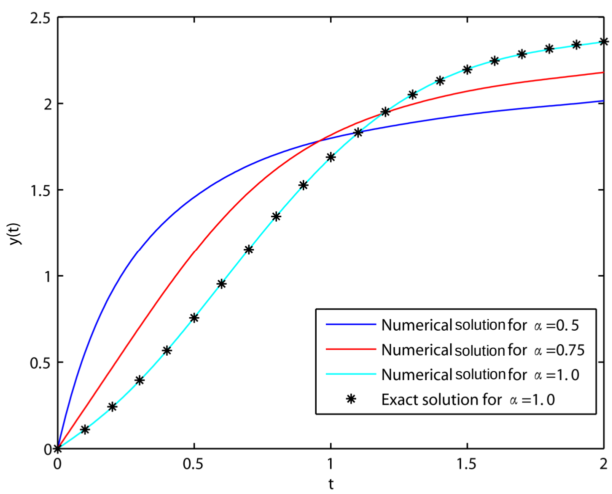

Figure 1. The behavior of the exact and approximate solution of example with $m = 8$ of Example 3.

Table 1. Numerical results with comparison to Ref. [6] and [10] for Example 3.

	$\alpha = 0.5$			$\alpha = 0.75$			$\alpha = 1.0$			
t	Ref. [6]	Ref. [10]	Ours	Ref. [6]	Ref. [10]	Ours	Ref. [6]	Ref. [10]	Ours	Exact
0.1	0.3217	0.5957	0.5550	0.2168	0.3107	0.2343	0.1102	0.1103	0.1113	0.1103
0.2	0.6296	0.9331	0.9120	0.4288	0.5843	0.4692	0.2419	0.2419	0.2401	0.2420
0.3	0.9409	1.1739	1.1533	0.6546	0.8221	0.7031	0.3951	0.3951	0.3928	0.3951
0.4	0.2507	1.3466	1.3265	0.8914	1.0249	0.9289	0.5681	0.5678	0.5674	0.5678
0.5	1.5494	1.4738	1.4575	1.1327	1.1986	1.1383	0.7575	0.7560	0.7579	0.7560
0.6	1.8254	1.5705	1.5600	1.3702	1.3491	1.3250	0.9582	0.9535	0.9563	0.9536
0.7	2.0665	1.6461	1.6412	1.5942	1.4814	1.4855	1.1634	1.1529	1.1547	1.1529
0.8	2.2606	1.7068	1.7058	1.7948	1.5992	1.6194	1.3652	1.3463	1.3460	1.3464
0.9	2.3968	1.7566	1.7571	1.9622	1.7053	1.7287	1.5549	1.5269	1.5247	1.5269
1.0	2.4660	1.7982	1.7986	2.0873	1.8017	1.8170	1.7281	1.6894	1.6866	1.6895

fractional differential equations. This paper deals with the approximate solution of a class of multi-order fractional differential equations. The fractional derivatives are described in the Caputo sense. Our main aim is to evaluation of fractional derivative using Legendre polynomials and implementing it to solve the nonlinear multi-order fractional differential equations. The main characteristic behind the approach using this technique is that it reduces such problems to those of solving a system of algebraic equations thus greatly simplifying the problem. Illustrative example is included to demonstrate the validity and applicability of the presented technique. The comparison certifies that our method gives good results.

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