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Approximate Method of Riemann-Hilbert Problem for Elliptic Complex Equations of First Order in Multiply Connected Unbounded Domains

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ABSTRACT

In this article, we discuss the approximate method of solving the Riemann-Hilbert boundary value problem for nonlinear uniformly elliptic complex equation of first order

$$w_{\bar{z}} = F(z, w, w_z) \text{ in } D, \tag{0.1}$$

with the boundary conditions

$$\operatorname{Re}\left[\overline{\lambda(t)}w(t)\right] = r(t) \text{ on } \Gamma, \tag{0.2}$$

in a multiply connected unbounded domain *D*, the above boundary value problem will be called Problem A. If the complex Equation (0.1) satisfies the conditions similar to Condition C of (1.1), and the boundary condition (0.2) satisfies the conditions similar to (1.5), then we can obtain approximate solutions of the boundary value problems (0.1) and (0.2). Moreover the error estimates of approximate solutions for the boundary value problem is also given. The boundary value problem possesses many applications in mechanics and physics etc., for instance from (5.114) and (5.115), Chapter VI, [1], we see that Problem A of (0.1) possesses the important application to the shell and elasticity.

Keywords: Approximate Method; Riemann-Hilbert Problem; Nonlinear Elliptic Complex Equations; Multiply Connected Unbounded Domains

1. Formulation of Elliptic Equations and Boundary Value Problem

Let D be an $(N+1)$-connected domain including the infinite point with the boundary $\Gamma = \bigcup_{j=0}^{N} \Gamma_j$ in $\mathbb{C}$, where $\Gamma \in C_\mu^2$ $(0 < \mu < 1)$. Without loss of generality, we assume that D is a circular domain in $|z| > 1$, where the boundary consists of $N+1$ circles $\Gamma_0 = \Gamma_{N+1} = \{|z| = 1\}$, $\Gamma_j = \{|z - z_j| = r_j\}, j = 1, \cdots, N$ and $z = \infty \in D$. In this article, the notations are as the same in References [1-6]. We discuss the nonlinear uniformly elliptic complex equation of first order

$$\begin{cases} w_{\bar{z}} = F(z, w, w_z), \\ F = Q_1 w_z + Q_2 \bar{w}_{\bar{z}} + A_1 w + A_2 \bar{w} + A_3, \\ Q_j = Q_j(z, w, w_z), j = 1, 2, \\ A_j = A_j(z, w), j = 1, 2, 3, \end{cases} \tag{1.1}$$

which is the complex form of the real nonlinear elliptic system of first order equations

$$\Phi_j(x, y, u, v, u_x, u_y, v_x, v_y) = 0, j = 1, 2 \tag{1.2}$$

under certain conditions (see [3]). Suppose that the complex Equation (1.1) satisfies the following conditions, namely

Condition C: 1) $Q_j(z, w, U)$ $(j = 1, 2)$, $A_j(z, w)$ $(j = 1, 2, 3)$ are measurable in $z \in D$ for all continuous functions $w(z)$ on $\bar{D}$ and all measurable functions $U(z) \in L_{p_0,2}(\bar{D})$, and satisfy

$$L_{p,2}\left[A_j, \bar{D}\right] \le k_0, j = 1, 2, \; L_{p,2}\left[A_3, \bar{D}\right] \le k_1, \tag{1.3}$$

where $p_0, p(2 < p_0 \le p), k_0, k_1$ are non-negative constants.

2) The above functions are continuous in $w \in \mathbb{C}$ for almost every point $z \in D, U \in \mathbb{C}$ and $Q_j = 0(j = 1, 2), A_j = 0(j = 1, 2, 3)$ for $z \notin D$.

3) The complex Equation (1.1) satisfies the uniform ellipticity condition, *i.e.* for any $U_1, U_2 \in \mathbb{C}$, the following inequality in almost every point $z \in D$ holds:

$$\left|F(z, w, U_1) - F(z, w, U_2)\right| \le q_0 \left|U_1 - U_2\right|, \tag{1.4}$$

in which $q_0(<1)$ is a non-negative constant.

Problem A: The Riemann-Hilbert boundary value problem for the complex Equation (1.1) may be formulated as follows: Find a continuous solution $w(z)$ of (1.1) on $\bar{D}$ satisfying the boundary condition

$$\operatorname{Re}\left[\overline{\lambda(z)}w(z)\right]=r(z),\ z\in\Gamma, \tag{1.5}$$

where $|\lambda(z)|=1, z\in\Gamma$, and $\lambda(z), r(z)$ satisfy the conditions

$$C_\alpha\left[\lambda(z),\Gamma\right]\le k_0,\ C_\alpha\left[r(z),\Gamma\right]\le k_2, \tag{1.6}$$

in which $\alpha(0<\alpha<1)$, k_0, k_2 are non-negative constants.

This boundary value problem for (1.1) with $A_3(z,w)=0, z\in D, w\in\mathbb{C}$ and $r(z)=0, z\in\Gamma$ will be called Problem A_0. The integer

$$K=\frac{1}{2\pi}\Delta_\Gamma \arg\lambda(z)$$

is called the index of Problem A and Problem A_0.

Due to when the index $K<0$, Problem A may not be solvable, when $K\ge 0$, the solution of Problem A is not necessarily unique. Hence we put forward some well posednesses of Problem A with modified boundary conditions.

Problem B₁: Find a continuous solution $w(z)$ of the complex Equation (1.1) in $\bar{D}$ satisfying the boundary condition

$$\operatorname{Re}\left[\overline{\lambda(z)}w(z)\right]=r(z)+h(z),\ z\in\Gamma, \tag{1.7}$$

where

$$h(z)=\begin{cases}0, z\in\Gamma, & \text{if } K\ge N,\\ \left.\begin{array}{l}h_j, z\in\Gamma_j, j=1,\cdots,N-K,\\ 0, z\in\Gamma_j, j=N-K+1,\cdots,N+1,\end{array}\right\} & \text{if } 0\le K<N,\\ \left.\begin{array}{l}h_j, z\in\Gamma_j, j=1,\cdots,N,\\ h_0+\operatorname{Re}\sum_{m=1}^{-K-1}\left(h_m^+ + ih_m^-\right)z^m, z\in\Gamma_0,\end{array}\right\} & \text{if } K<0,\end{cases} \tag{1.8}$$

in which $h_j(j=0,1,\cdots,N), h_m^\pm(m=1,\cdots,-K-1, K<0)$ are unknown real constants to be determined appropriately. In addition, we may assume that the solution $w(z)$ satisfies the following side conditions (point conditions)

$$\operatorname{Im}\left[\overline{\lambda(a_j)}w(a_j)\right]=b_j,\quad j\in J=\begin{cases}1,\cdots,2K-N+1, & \text{if } K\ge N,\\ N-K+1,\cdots,N+1, & \text{if } 0\le K<N,\end{cases} \tag{1.9}$$

where

$$a_j\in\Gamma_j(j=1,\cdots,N),$$
$$a_j\in\Gamma_0(j=N+1,\cdots,2K-N+1, K\ge N)$$

are distinct points, and $b_j(j\in J)$ are all real constants satisfying the conditions

$$|b_j|\le k_3,\ j\in J, \tag{1.10}$$

herein k_3 is a nonnegative constant.

Now, we give the second well posed-ness of Problem A.

Problem B₂: If the point condition (1.9) in Problem B_1 is replaced by the integral conditions

$$L_j(\bar{\lambda}w)=\begin{cases}\operatorname{Im}\int_{\Gamma_j}\overline{\lambda(z)}w(z)\mathrm{d}s=B_j,\\ j=\begin{cases}N-K+1,\cdots,N+1 & \text{if } 0\le K<N,\\ 1,\cdots,N+1 & \text{if } K\ge N,\end{cases}\\ \left.\begin{array}{l}\operatorname{Re}\int_{\Gamma_0}z^{j-N-1}\overline{\lambda(z)}w(z)\mathrm{d}s=B_j,\\ j=N+2,\cdots,K+1\\ \operatorname{Im}\int_{\Gamma_0}z^{j-K-1}\overline{\lambda(z)}w(z)\mathrm{d}s=B_j,\\ j=K+2,\cdots,2K-N+1\end{array}\right\}\ \text{if } K\ge N,\end{cases} \tag{1.11}$$

respectively, where $B_j(j\in J)$ are real constants satisfying the conditions

$$|B_j|\le k_3,\ j\in J, \tag{1.12}$$

in which k_3 is a nonnegative real constant.

For convenience, we sometimes will subsume the integral conditions or the point conditions under boundary conditions.

2. *A Priori* Estimates of Solutions of Boundary Value Problem

First of all, we give a representation theorem of solutions for Problem B_1 and for Problem B_2.

Theorem 2.1. Suppose that the complex Equation (1.1) satisfies Condition C, and $w(z)$ is any solution of Problem B_1 (or Problem B_2) for (1.1). Then $w(z)$ is representable by

$$w(z)=\Phi\left[\zeta(z)\right]e^{\phi(z)}+\psi(z), \tag{2.1}$$

where $\zeta(z)$ is a homeomorphism on $\bar{D}$, which quasiconformally maps D onto an $(N+1)$-connected circular domain G with boundary $L=\bigcup_{j=0}^N L_j$ where the $L_j=\zeta(\Gamma_j)$ $(j=1,\cdots,N)$ are located in $|\zeta|>1$ by $L_0=\zeta(\Gamma_0)=\{|\zeta|=1\}$, and $\zeta(\infty)=\infty$, $\Phi(\zeta)$ is an analytic function in G, $\psi(z),\phi(z),\zeta(z)$ and its inverse function $z(\zeta)$ satisfy the estimates

$$C_\beta\left[\tau,\bar{D}\right]\le k_4,\ \tau=\psi(z),\varphi(z),\zeta(z)/z,\quad C_\beta\left[z(\zeta)/\zeta,\bar{G}\right]\le k_4, \tag{2.2}$$

$$L_{p_0,2}\left[|\psi_{\bar z}|+|\psi_z|,\bar{D}\right]\le k_4,\quad L_{p_0,2}\left[|\phi_{\bar\zeta}|+|\phi_\zeta|,\bar{G}\right]\le k_4, \tag{2.3}$$

$$L_{p_0,2}\left[|\zeta_{\bar z}|+|\zeta_z|,\bar{D}\right]\le k_5,\quad L_{p_0,2}\left[|z_{\bar\zeta}|+|z_\zeta|,\bar{G}\right]\le k_5, \tag{2.4}$$

in which $\beta=\min(\alpha,1-2/p_0),\ p_0\,(2<p_0\le p)$, k_4,k_5 are non-negative constants, $k_j=k_j(q_0,p_0,k_0,k_1,D),\ j=4,5$.

Proof. Similarly to Theorem 2.4, Chapter 2 in [3], we substitute the solution $w(z)$ of Problem B_1 (or Problem B_2) into the coefficients of the complex Equation (1.1) and consider the following system

$$\psi_{\bar z}=Q\psi_z+A_1\psi+A_2\bar\psi+A_3,\quad Q=\begin{cases}Q_1+Q_2\,\overline{w_{\bar z}}/w_z & \text{for } w_z\ne 0,\\ 0 & \text{for } w_z=0 \text{ or } z\notin D,\end{cases} \tag{2.5}$$

$$\phi_{\bar z}=Q\phi_z+A,\quad A=\begin{cases}A_1+A_2\,\bar w/w & \text{for } w(z)\ne 0,\\ 0 & \text{for } w(z)=0 \text{ or } z\notin D,\end{cases} \tag{2.6}$$

$$W_{\bar z}=QW_z,\ W(z)=\Phi\left[\zeta(z)\right]. \tag{2.7}$$

By using the continuity method and the principle of contracting mappings, we can find the solutions

$$\psi(z)=\hat{T}f,\phi=\hat{T}g,\zeta(z)=\Psi\left[\chi(z)\right],\quad \chi(z)=1/z+\hat{T}h, \tag{2.8}$$

where $f(z),\ g(z),\ h(z)\in L_{p_0,2}(\bar D),\ 2<p_0\le p,\ \chi(z)$ is a homeomorphism on $\bar D,\Psi(\chi)$ is a univalent analytic function, which conformally maps $E=\chi(D)$ onto an $(N+1)$-connected circular domain G, and $\Phi(\zeta)$ is an analytic function in G. We can verify that $\psi(z),\phi(z)$ satisfy the estimates (2.2) and (2.3). Moreover noting that $\zeta(z)$ is a homeomorphic solution of the Beltrami complex Equation (2.7), which maps the circular domain D onto the circular domain G with the condition $\infty=\zeta(\infty)$ and $L_0=\zeta(\Gamma_0)=\{|\zeta|=1\}$, in accordance with the result in Lemma 2.1, Chapter 2, [3], we see that the estimate (2.4) is true.

Now, we derive a priori estimates of solutions for Problem B_1 and for Problem B_2 for the complex Equation (1.1).

Theorem 2.2. Under the same conditions as in Theorem 2.1, any solution $w(z)$ of Problem B_1 (or Problem B_2) for (1.1) satisfies the estimates

$$C_\eta\left[w(z),\bar D\right]\le M_1=M_1(q_0,p_0,k,\alpha,K,D), \tag{2.9}$$

$$L_{p_0,2}\left[|w_{\bar z}|+|w_z|,\tilde D\right]\le M_2=M_2\left(q_0,p_0,k,\alpha,K,\tilde D\right), \tag{2.10}$$

where $k=k(k_0,k_1,k_2,k_3),\eta=\alpha\beta^2,2<p_0\le p,M_j\,(j=1,2)$ are non-negative constants only dependent on q_0,p_0,k,α,K,D and $q_0,p_0,k,\alpha,K,\tilde D$ respectively.

Proof. On the basis of Theorem 2.1, the solution $w(z)$ of Problem B_1 (or Problem B_2) can be expressed the formula as in (2.1), hence the boundary value problem B_1 can be transformed into the boundary value problem (Problem $\tilde B$) for analytic functions

$$\mathrm{Re}\left[\overline{\Lambda(\zeta)}\Phi(\zeta)\right]=R(\zeta)+H(\zeta),\ \zeta\in L, \tag{2.11}$$

$$H(\zeta)=\begin{cases}0,\zeta\in L, & \text{if } K\ge N\\ \left.\begin{array}{l}h_j,\zeta\in L_j,\ j=1,\cdots,N-K,\\ 0,\zeta\in L_j,\ j=N-K+1,\cdots,N+1,\end{array}\right\} & \text{if } 0\le K<N\\ \left.\begin{array}{l}h_j,\zeta\in L_j,\ j=1,\cdots,N,\\ h_0+\mathrm{Re}\sum_{m=1}^{-K-1}\left(h_m^++\mathrm{i}h_m^-\right)\zeta^m,z\in L_0,\end{array}\right\} & \text{if } K<0\end{cases} \tag{2.12}$$

$$\mathrm{Im}\left[\overline{\Lambda(a_j')}\Phi(a_j')\right]=b_j',\ j\in J, \tag{2.13}$$

where

$$\overline{\Lambda(\zeta)}=\overline{\lambda\left[z(\zeta)\right]}e^{\phi[z(\zeta)]},$$
$$R(\zeta)=r\left[z(\zeta)\right]-\mathrm{Re}\left\{\overline{\lambda\left[z(\zeta)\right]}\psi\left[z(\zeta)\right]\right\},$$
$$a_j'=\zeta(a_j),b_j'=b_j-\mathrm{Im}\left[\overline{\lambda(a_j)}\psi(a_j)\right],j\in J.$$

By (2.2)-(2.4), it can be seen that $\Lambda(\zeta),R(\zeta),b_j'\,(j\in J)$ satisfy the conditions

$$C_{\alpha\beta}\left[\Lambda(\zeta),L\right]\le M_3,C_{\alpha\beta}\left[R(\zeta),L\right]\le M_3,\quad |b_j'|\le M_3,j\in J, \tag{2.14}$$

where $M_3=M_3(q_0,p_0,k,\alpha,K,D)$. If we can prove that the solution $\Phi(\zeta)$ of Problem $\tilde B$ satisfies the estimate

$$C_{\alpha\beta}\left[\Phi(\zeta),\bar G\right]\le M_4,\ C_{\alpha\beta}\left[\Phi'(\zeta),\tilde G\right]\le M_5, \tag{2.15}$$

in which $\tilde D=\{z\in D\,|\,\mathrm{dist}(z,\Gamma)\ge\varepsilon>0\}$, $\tilde G=\zeta(\tilde D)$, $M_4=M_4(q_0,p_0,k,\alpha,K,D)$, $M_5=M_5\left(q_0,p_0,k,\alpha,K,\tilde D\right)$, then from the representation (2.1) of the solution $w(z)$ and the estimates (2.2)-(2.4) and (2.15), the estimates (2.9) and (2.10) can be derived.

It remains to prove that (2.15) holds. For this, we first verify the boundedness of $\Phi(\zeta)$, *i.e.*

$$C\left[\Phi(\zeta),\bar{G}\right]\le M_6 = M_6\left(q_0,p_0,k,\alpha,K,D\right). \tag{2.16}$$

Suppose that (2.16) is not true. Then there exist sequences of functions $\{\Lambda_n(\zeta)\}$, $\{R_n(\zeta)\},\{b'_{jn}\}$ satisfying the same conditions as $\Lambda(\zeta),R(\zeta),b'_j$, which uniformly converge to $\Lambda_0(\zeta),R_0(\zeta),b'_{j0}(j\in J)$ on L respectively. For the solution $\Phi_n(\zeta)$ of the boundary value problem (Problem B_n) corresponding to $\Lambda_n(\zeta),R_n(\zeta),b'_{jn}(j\in J)$, we have $I_n=C\left[\Phi_n(\zeta),\bar{G}\right]\to\infty$ as $n\to\infty$. There is no harm in assuming that $I_n\ge 1,n=1,2,\cdots$. Obviously $\tilde{\Phi}_n(\zeta)=\Phi_n(\zeta)/I_n$ satisfies the boundary conditions

$$\operatorname{Re}\left[\overline{\Lambda_n(\zeta)}\Phi_n(\zeta)\right] = \left[R_n(\zeta)+H(\zeta)\right]/I_n,\zeta\in L, \tag{2.17}$$

$$\operatorname{Im}\left[\overline{\Lambda_n(a'_n)}\tilde{\Phi}_n(a'_n)\right]=b'_{jn}/I_n,\ j\in J. \tag{2.18}$$

Applying the Schwarz formula, the Cauchy formula and the method of symmetric extension (see Theorem 1.4, Chapter 1, [3]), the estimates

$$C_{\alpha\beta}\left[\tilde{\Phi}_n(\zeta),\bar{G}\right]\le M_7,\ C\left[\tilde{\Phi}'_n(\zeta),\bar{G}\right]\le M_8 \tag{2.19}$$

can be obtained, where $M_7=M_7\left(q_0,p_0,k,\alpha,K,D\right)$, $M_8=M_8\left(q_0,p_0,k,\alpha,K,\tilde{D}\right)$. Thus we can select a subsequence of $\{\tilde{\Phi}_n(\zeta)\}$, which uniformly converge to an analytic function $\tilde{\Phi}_0(\zeta)$ in G, and $\tilde{\Phi}_0(\zeta)$ satisfies the homogeneous boundary conditions

$$\operatorname{Re}\left[\overline{\Lambda_0(\zeta)}\Phi_0(\zeta)\right]=H(\zeta),\zeta\in L, \tag{2.20}$$

$$\operatorname{Im}\left[\overline{\Lambda_0(a'_j)}\tilde{\Phi}_0(a'_j)\right]=0,j\in J. \tag{2.21}$$

On the basis of the uniqueness theorem (see Theorem 2.4), we conclude that $\tilde{\Phi}_0(\zeta)=0,\ \zeta\in\bar{G}$. However, from $C\left[\tilde{\Phi}_n(\zeta),\bar{G}\right]=1$, it follows that there exists a point $\zeta_*\in\bar{G}$, such that $\left|\tilde{\Phi}_0(\zeta_*)\right|=1$. This contradiction proves that (2.16) holds. Afterwards using the method which leads from $C\left[\tilde{\Phi}_n(\zeta),\bar{G}\right]=1$ to (2.19), the estimate (2.15) can be derived.

Similarly, we can verify that any solution $w(z)$ of Problem B_2 satisfies the estimates (2.9) and (2.10).

Theorem 2.3. Under the same conditions as in Theorem 2.1, any solution $w(z)$ of Problem B_1 (or Problem B_2) for (1.1) satisfies

$$C_\eta\left[w(z),\bar{D}\right]\le M_9k_*,\quad L_{p_0,2}\left[|w_{\bar{z}}|+|w_z|,\tilde{D}\right]\le M_{10}k_*, \tag{2.22}$$

where $\eta,p_0,\tilde{D}$ are as stated in Theorem 2.2,

$k_*=k_1+k_2+k_3, M_j=M_j\left(q_0,p_0,k_0,\alpha,K,D\right), j=9,10.$

Proof. If $k_*=0$, *i.e.* $k_1=k_2=k_3=0$, from Theorem 2.4, it follows that $w(z)=0,z\in D$. If $k_*>0$, it is easy to see that $W(z)=w(z)/k_*$ satisfies the complex equation and boundary conditions

$$W_{\bar{z}}-Q_1W_z-Q_2\bar{W}_{\bar{z}}-A_1W-A_2\bar{W}=A_3/k_*,z\in D, \tag{2.23}$$

$$\operatorname{Re}\left[\overline{\lambda(z)}W(z)\right]=\left[r(z)+h(z)\right]/k_*,z\in\Gamma, \tag{2.24}$$

$$\operatorname{Im}\left[\overline{\lambda(a_j)}W(a_j)\right]=b_j/k_*,\quad \left(\text{or } L_j\bar{\lambda}W=B_j/k_*\right),j\in J, \tag{2.25}$$

Noting that

$$L_{p,2}\left[A_3/k_*,\bar{D}\right]\le 1,C_\alpha\left[r(z)/k_*,\Gamma\right]\le 1,$$
$$\left|b_j/k_*\right|\le 1\left(\text{or }\left|B_j/k_*\right|\le 1\right),\ j\in J$$

and according to the proof of Theorem 2.2, we have

$$C_\eta\left[W(z),\bar{D}\right]\le M_9,\quad L_{p_0,2}\left[|W_{\bar{z}}|+|W_z|,\tilde{D}\right]\le M_{10}. \tag{2.26}$$

From the above estimates, it immediately follows that (2.22) holds.

Next, we prove the uniqueness of solutions of Problem B_1 and Problem B_2 for the complex Equation (1.1). For this, we need to add the following condition: For any continuous functions $w_1(z),w_2(z)$ on $\bar{D}$ and $U(z)\in L_{p_0,2}(\bar{D})$, there is

$$F(z,w_1,U)-F(z,w_2,U) = A(z,w_1,w_2,U)(w_1-w_2), \tag{2.27}$$

where $A(z,w_1,w_2,U)\in L_{p_0,2}(\bar{D})$. When (1.1) is linear, (3.27) obviously holds.

Theorem 2.4. If Condition C and (2.27) hold, then the solution of Problem B_1 (or Problem B_2) for (1.1) is unique.

Proof. Let $w_1(z),w_2(z)$ be two solutions of Problem B_1 for (1.1). By Condition C and (2.27), we see that $w(z)=w_1(z)-w_2(z)$ is a solution of the following boundary value problem

$$w_{\bar{z}}-\tilde{Q}w_z=\tilde{A}w,\ z\in D, \tag{2.28}$$

$$\operatorname{Re}\left[\overline{\lambda(z)}w(z)\right]=h(z),\ z\in\Gamma, \tag{2.29}$$

$$\operatorname{Im}\left[\overline{\lambda(a_j)}w(a_j)\right]=0,\ j\in J, \tag{2.30}$$

where

$$\tilde{Q} = \begin{cases} \left[F(z,w_1,w_{1z}) - F(z,w_1,w_{2z})\right] / (w_1 - w_2)_z \\ \text{for } w_{1z} \neq w_{2z}, \\ 0 \text{ for } w_{1z} = w_{2z}, z \in D, \end{cases}$$

$$\tilde{A} = \begin{cases} \left[F(z,w_1,w_{2z}) - F(z,w_2,w_{2z})\right] / (w_1 - w_2) \\ \text{for } w_1(z) \neq w_2(z), \\ 0 \text{ for } w_1(z) = w_2(z), z \in D, \end{cases}$$

and $|\tilde{Q}| \le q_0 < 1, z \in D, L_{p_0,2}(\tilde{A}, \bar{D}) < \infty$. According to the representation (2.1), we have

$$w(z) = \Phi[\zeta(z)] e^{\phi(z)}, \tag{2.31}$$

where $\phi(z), \zeta(z), \Phi(\zeta)$ are as stated in Theorem 2.1. It can be seen that the analytic function $\Phi(z)$ satisfies the boundary conditions of Problem B_0:

$$\text{Re}\left[\overline{\Lambda(\zeta)}\Phi(\zeta)\right] = H(\zeta), \zeta \in L = \zeta(\Gamma), \tag{2.32}$$

$$\text{Im}\left[\overline{\Lambda(a'_j)}\Phi(a'_j)\right] = 0, j \in J, \tag{2.33}$$

where $\Lambda(\zeta), H(\zeta)\ (\zeta \in L), a'_j(j \in J)$ are as stated in (2.11)-(2.13). In accordance with Theorem 2.2, it can be derived that $\Phi(\zeta) = 0, \zeta \in G = \zeta(D)$. Hence, $w(z) = \Phi[\zeta(z)] e^{\phi(z)} = 0$, *i.e.* $w_1(z) = w_2(z), z \in D$.

3. The Continuity Method of Solving Boundary Value Problem

Next, we discuss the modified Riemann-Hilbert boundary value problems (Problem B_1 and Problem B_2) for the nonlinear elliptic complex Equation (1.1) in the (N+1)-connected unbounded domain D as stated in Section 1, here we use the Newton imbedding method of another form and give an error estimate, which is better than that as stated before. In the following, we only deal with Problem B_1, because by using the same method, Problem B_2 can be discussed.

Theorem 3.1. Suppose that the nonlinear elliptic Equation (1.1) satisfies Condition C and (1.6), (1.10), on $\bar{D}$. Then Problem B_1 for (1.1) has a solution $w(z) \in W^1_{p_0,2}(D)$.

Proof We introduce the nonlinear elliptic complex equation with the parameter $t \in [0,1]$:

$$w_{\bar{z}} - tF(z,w,w_z) = A(z), \tag{3.1}$$

where $A(z)$ is any measurable function in D and $A(z) \in L_{p_0,2}(\bar{D}), 2 < p_0 \le p$. When $t = 0$, it is not difficult to see that there exists a unique solution $w(z)$ of Problem B_1 for the complex Equation (3.1), which possesses the form

$$w(z) = \Phi(z) + \psi(z),$$
$$\psi(z) = \hat{T}A = -\frac{1}{\pi}\iint_D \frac{A(1/\zeta)}{\bar{\zeta}^2 \zeta(1-\zeta z)} d\sigma_\zeta, \tag{3.2}$$

where $\Phi(z)$ is an analytic function in D and satisfies the boundary conditions

$$\text{Re}\left[\overline{\lambda(z)}\Phi(z)\right] = r(z) - \text{Re}\left[\overline{\lambda(z)}\psi(z)\right] + h(z),\ z \in \Gamma, \tag{3.3}$$

$$\text{Im}\left[\overline{\lambda(a_j)}\Phi(a_j)\right] = b_j - \text{Im}\left[\overline{\lambda(a_j)}\psi(a_j)\right],\ j \in J. \tag{3.4}$$

From Theorem Theorem 2.2, We see that

$$w(z) \in B = C_\beta(\bar{D}) \cap W^1_{p_0,2}(D),$$
$$\beta = \alpha\left[\min(\alpha, 1 - 2/p_0)\right]^2.$$

Suppose that when $t = t_0\ (0 \le t_0 < 1)$, Problem B_1 for the complex Equation (1.18) has a unique solution, we shall prove that there exists a neighborhood of $t_0 : E = \{|t - t_0| \le \delta, 0 \le t \le 1, \delta > 0\}$, so that for every $t \in E$ and any function $A(z) \in L_{p_0,2}(\bar{D})$, Problem B_1 for (1.18) is solvable. In fact, the complex Equation (3.2) can be written in the form

$$w_{\bar{z}} - t_0 F(z,w,w_z) = (t - t_0) F(z,w,w_z) + A(z). \tag{3.5}$$

We arbitrarily select a function $w_0(z) \in B = C_\beta(\bar{D}) \cap W^1_{p_0,2}(D)$, in particular $w_0(z) = 0$ on $\bar{D}$. Let $w_0(z)$ be replaced into the position of $w(z)$ in the right hand side of (1.22). By Condition C, it is obvious that

$$B_0(z) = (t - t_0) F(z,w_0,w_{0z}) + A(z) \in L_{p_0}(\bar{D}).$$

Noting the (3.5) has a solution $w_1(z) \in B$. Applying the successive iteration, we can find out a sequence of functions: $w_n(z) \in B, n = 1,2,\cdots,$ which satisfy the complex equations

$$\begin{aligned} &w_{n+1\bar{z}} - t_0 F(z,w_{n+1},w_{n+1z}) \\ &= (t - t_0) F(z,w_n,w_{nz}) + A(z), n = 1,2,\cdots. \end{aligned} \tag{3.6}$$

The difference of the above equations for $n+1$ and n is as follows:

$$\begin{aligned} &(w_{n+1} - w_n)_{\bar{z}} \\ &-t_0\left[F(z,w_{n+1},w_{n+1z}) - F(z,w_n,w_{nz})\right] \\ &= (t - t_0)\left[F(z,w_n,w_{nz}) - F(z,w_{n-1},w_{n-1z})\right], \\ &n = 1,2,\cdots. \end{aligned} \tag{3.7}$$

From Condition C, on $\bar{D}$, it can be seen that

$$
\begin{aligned}
&F\left(z,w_{n+1},w_{n+1z}\right)-F\left(z,w_n,w_{nz}\right)\\
&=F\left(z,w_{n+1},w_{n+1z}\right)-F\left(z,w_{n+1},w_{nz}\right)\\
&+\left[F\left(z,w_{n+1},w_{nz}\right)-F\left(z,w_n,w_{nz}\right)\right]\\
&=\tilde{Q}_{n+1}(z)\left(w_{n+1}-w_n\right)_z+\tilde{A}_{n+1}(z)\left(w_{n+1}-w_n\right),\\
&\left|\tilde{Q}_{n+1}(z)\right|\le q_0<1,\ \tilde{A}_{n+1}(z)\in L_{p_0,2}\left(\bar{D}\right),\ n=1,2,\cdots,
\end{aligned}
$$

and

$$
\begin{aligned}
&L_{p_0,2}\left[F\left(z,w_n,w_{nz}\right)-F\left(z,w_{n-1},w_{n-1z}\right),\bar{D}\right]\\
&\le q_0L_{p_0,2}\left[\left(w_n-w_{n-1}\right)_z,\bar{D}\right]+k_0C\left[w_n-w_{n-1},\bar{D}\right]\\
&\le\left(q_0+k_0\right)C_\beta\left[w_n-w_{n-1},\bar{D}\right]\\
&+L_{p_0,2}\left[\left|\left(w_n-w_{n-1}\right)_{\bar{z}}\right|+\left|\left(w_n-w_{n-1}\right)_z\right|,\bar{D}\right]\\
&=\left(q_0+k_0\right)L_n.
\end{aligned}
$$

Moreover, $w_{n+1}(z)-w_n(z)$ satisfies the homogeneous boundary conditions

$$\operatorname{Re}\left[\overline{\lambda(z)}\left(w_{n+1}(z)-w_n(z)\right)\right]=h(z),\ z\in\Gamma, \quad (3.8)$$

$$\operatorname{Im}\left[\overline{\lambda\left(a_j\right)}\left(w_{n+1}\left(a_j\right)-w_n\left(a_j\right)\right)\right]=0,\ j\in J. \quad (3.9)$$

On the basis of Theorem 2.3, we have

$$
\begin{aligned}
L_{n+1}&=C_\beta\left[w_{n+1}-w_n,\bar{D}\right]\\
&+L_{p_0,2}\left[\left|\left(w_{n+1}-w_n\right)_z\right|+\left|\left(w_{n+1}-w_n\right)_z\right|,\bar{D}\right] \quad (3.10)\\
&\le M\left|t-t_0\right|\left(q_0+k_0\right)L_n,
\end{aligned}
$$

where $M=M_9+M_{10}$, M_9,M_{10} is as stated in (2.22). Provided $\delta(>0)$ is small enough, so that $\eta=\delta M\left(q_0+k_0\right)<1$, it can be obtained that

$$
\begin{aligned}
L_{n+1}&\le\eta L_n\le\eta^nL_1\\
&=\eta^n\left[C_\beta\left(w_1,\bar{D}\right)+L_{p_0,2}\left(\left|w_{1\bar{z}}\right|+\left|w_{1z}\right|,\bar{D}\right)\right]
\end{aligned} \quad (3.11)
$$

for every $t\in E$. Thus

$$
\begin{aligned}
&S\left(w_n-w_m\right)\\
&=C_\beta\left[w_n-w_m,\bar{D}\right]\\
&+L_{p_0,2}\left[\left|\left(w_n-w_m\right)_{\bar{z}}\right|+\left|\left(w_n-w_m\right)_z\right|,\bar{D}\right]\\
&\le L_n+L_{n-1}+\cdots+L_{m+1} \quad (3.12)\\
&\le\left(\eta^{n-1}+\eta^{n-2}+\cdots+\eta^m\right)L_1\\
&=\eta^m\left(1+\eta+\cdots+\eta^{n-m-1}\right)L_1\\
&\le\eta^{N+1}\frac{1-\eta^{n-m}}{1-\eta}L_1\le\frac{\eta^{N+1}}{1-\eta}L_1
\end{aligned}
$$

for $n\ge m>N$, where N is a positive integer. This shows that $S\left(w_n-w_m\right)\to0$ as $n,m\to\infty$. Following the completeness of the Banach space $B=C_\beta\left(\bar{D}\right)\cap W^1_{p_0,2}(D)$, there is a function $w_*(z)\in B$, such that when $n\to\infty$,

$$
\begin{aligned}
&S\left(w-w_*\right)=C_\beta\left[w_n-w_*,\bar{D}\right]\\
&+L_{p_0,2}\left[\left|\left(w_n-w_*\right)_{\bar{z}}\right|+\left|\left(w_n-w_*\right)_z\right|,\bar{D}\right]\to0.
\end{aligned}
$$

By Condition C and (1.6), (1.10), from the above formula it follows that $w_*(z)$ is a solution of Problem B_1 for (3.5), *i.e.* (3.1) for $t\in E$. It is easy to see that the positive constant δ is independent of $t_0\left(0\le t_0<1\right)$. Hence from Problem B_1 for the complex Equation (3.1) with $t=t_0=0$ is solvable, we can derive that when $t=\delta,2\delta,\cdots,[1/\delta]\delta,1$, Problem B_1 for (3.1) are solvable, especially Problem B_1 for (3.2) with $t=1$ and $A(z)=0$, namely Problem B_1 for (1.1) has a unique solution.

4. Error Estimates of Approximate Solutions for Boundary Value Problem

In this section, we shall introduce an error estimate of the above approximate solutions of the boundary value problem and can give the following error estimate of the approximate solutions.

Theorem 4.1 Let $w=w(z)$ be a solution of Problem B_1 for the complex Equation (1.1) satisfying Condition C and (1.6), (1.10) on $\bar{D}$, and $w_n^t=w_n(z,t)$ be its appro- ximation as stated in the proof of Theorem 2.2 with $A(z)=(1-t)F(z,0,0)$. Then we have the following error estimate

$$
\begin{aligned}
&S\left(w-w_n^t\right)=C_\beta\left(w-w_n^t,\bar{D}\right)\\
&+L_{p_0,2}\left[\left|\left(w-w_n^t\right)_{\bar{z}}\right|+\left|\left(w-w_n^t\right)_z\right|,\bar{D}\right] \quad (4.1)\\
&\le\gamma k\left[\frac{1-\gamma\left|t-t_0\right|^n}{1-\gamma\left|t-t_0\right|}(1-t)+\left(\gamma\left|t-t_0\right|\right)^n\left(1-t_0\right)\right],
\end{aligned}
$$

where $\gamma=M\left(q_0+k_0\right)$, $k=M\left(k_1+k_2+k_3\right)$, $M=M_9+M_{10},M_9,M_{10},q_0$ and $k_j\ (j=0,1,2,3)$ are constants in (1.3), (1.6) and (1.10).

Proof From (1.1) and (2.23) with $A(z)=(1-t)F(z,0,0)$, we have

$$
\begin{aligned}
&\left(w-w_{n+1}^t\right)_{\bar{z}}=F\left(z,w,w_z\right)-t_0F\left(z,w_{n+1}^t,w_{n+1z}^t\right)\\
&-\left(t-t_0\right)F\left(z,w_n^t,w_{nz}^t\right)-(1-t)F(z,0,0)\\
&=(1-t)\left[F\left(z,w,w_z\right)-F(z,0,0)\right]\\
&+t_0\left[F\left(z,w,w_z\right)-F\left(z,w_{n+1}^t,w_{n+1z}^t\right)\right]\\
&+\left(t-t_0\right)\times\left[F\left(z,w,w_z\right)-F\left(z,w_n^t,w_{nz}^t\right)\right] \quad (4.2)\\
&=t_0\left[\tilde{Q}\left(z,w,w_z,w_{n+1}^t\right)\left(w-w_{n+1}^t\right)_z\right.\\
&\qquad\left.+\tilde{A}\left(z,w,w_{n+1}^t,w_{n+1z}^t\right)\left(w-w_{n+1}^t\right)\right]\\
&+(1-t)\left[F\left(z,w,w_z\right)-F(z,0,0)\right]\\
&+\left(t-t_0\right)\left[F\left(z,w,w_z\right)-F\left(z,w_n^t,w_{nz}^t\right)\right].
\end{aligned}
$$

It is clear that $w-w_{n+1}^t$ satisfies the homogeneous boundary conditions

$$\operatorname{Re}\left[\overline{\lambda(z)}\left(w(z)-w_{n+1}^t(z)\right)\right]=h(z),\ z\in\Gamma, \quad (4.3)$$

$$\operatorname{Im}\left[\overline{\lambda(a_j)}\left(w(a_j)-w_{n+1}^t(a_j)\right)\right]=0,\ j\in J. \quad (4.4)$$

Noting that $\tilde{Q}=\tilde{Q}\left(z,w,w_z,w_{n+1}^t\right),\tilde{A}=\tilde{A}\left(z,w,w_{n+1}^t,w_{n+1z}^t\right)$ satisfy $\left|\tilde{Q}\right|\le q_0<1, L_{p_0,2}\left[\tilde{A},\bar{D}\right]\le k_0$, and

$$\begin{aligned}&L_{p_0,2}\left[F(z,w,w_z)-F(z,0,0),\bar{D}\right]\\&\le q_0L_{p_0,2}\left[w_z,\bar{D}\right]+k_0C\left[w,\bar{D}\right]\\&\le(q_0+k_0)\left[L_{p_0,2}\left(w_z,\bar{D}\right)+C\left(w,\bar{D}\right)\right]\\&\le(q_0+k_0)S(w),\end{aligned}$$

$$\begin{aligned}&L_{p_0,2}\left[F(z,w,w_z)-F\left(z,w_n^t,w_{nz}^t\right),\bar{D}\right]\\&\le q_0L_{p_0,2}\left[\left(w-w_n^t\right)_z,\bar{D}\right]+k_0C\left[w-w_n^t,\bar{D}\right]\\&\le(q_0+k_0)\left[L_{p_0,2}\left(\left(w-w_n^t\right)_z,\bar{D}\right)+C\left(w-w_n^t,\bar{D}\right)\right]\\&\le(q_0+k_0)S\left(w-w_n^t\right),\end{aligned}$$

and according to Theorem 2.2, it can be concluded

$$\begin{aligned}&S\left(w-w_{n+1}^t\right)\\&\le M\left[(1-t)(q_0+k_0)S(w)+|t-t_0|(q_0+k_0)S\left(w-w_n^t\right)\right]\\&=M(q_0+k_0)\left[(1-t)S(w)+|t-t_0|S\left(w-w_n^t\right)\right],\end{aligned} \quad (4.5)$$

where $M=M_3(q_0,p_0,k_0,\alpha,K,D)$, and

$$S(w)\le M(k_1+k_2+k_3)=k, \quad (4.6)$$

where k_1,k_2,k_3 are non-negative constants as stated in (1.3), (1.6) and (1.10). From (4.5) and (4.6), it follows

$$\begin{aligned}&S\left(w-w_{n+1}^t\right)\\&\le\gamma\left[(1-t)S(w)+|t-t_0|S\left(w-w_n^t\right)\right]\\&\le\gamma(1-t)S(w)\left(1+\gamma|t-t_0|+\gamma^2|t-t_0|^2+\cdots+\gamma^n|t-t_0|^n\right)\\&+\gamma^{n+1}|t-t_0|^{n+1}S\left(w-w_0^t\right)\\&\le\gamma(1-t)S(w)\frac{1-\left(\gamma|t-t_0|\right)^{n+1}}{1-\gamma|t-t_0|}\\&+\gamma^{n+1}|t-t_0|^{n+1}S\left(w-w_0^t\right),\end{aligned}$$

where $\gamma=M(q_0+k_0)$, and we choose that $w_0^t=w(z,t_0)$ is the solution of Problem B_1 for (2.22) with $t=t_0$ and $A(z)=(1-t_0)F(z,0,0)$. Due to $w-w_0^t$ is a solution of Problem B_1 for the complex equation

$$\begin{aligned}&\left(w-w_0^t\right)_{\bar{z}}\\&=F(z,w,w_z)-t_0F\left(z,w_0^t,w_{0z}^t\right)-(1-t_0)F(z,0,0)\\&=t_0\left[F(z,w,w_z)-F\left(z,w_0^t,w_{0z}^t\right)\right]\\&+(1-t_0)\left[F(z,w,w_z)-F(z,0,0)\right]\\&=t_0Q\left(w-w_0^t\right)_z+A\left(w-w_0^t\right)\\&+(1-t_0)\left[F(z,w,w_z)-F(z,0,0)\right],\end{aligned} \quad (4.7)$$

hence

$$\begin{aligned}&S\left(w-w_0^t\right)\\&\le M(1-t_0)\left[q_0L_{p_0,2}\left(w_z,\bar{D}\right)+k_0C\left(w,\bar{D}\right)\right]\\&\le M(q_0+k_0)(1-t_0)S(w)\le\gamma(1-t_0)k.\end{aligned} \quad (4.8)$$

Finally, we obtain

$$\begin{aligned}&S\left(w-w_{n+1}^t\right)\\&\le\gamma k\left[\frac{1-\gamma|t-t_0|^{n+1}}{1-\gamma|t-t_0|}(1-t)+\gamma k\gamma^{n+1}|t-t_0|^{n+1}(1-t_0)\right]\\&=\gamma k\left[\frac{1-\gamma|t-t_0|^{n+1}}{1-\gamma|t-t_0|}(1-t)+\left(\gamma|t-t_0|\right)^{n+1}(1-t_0)\right],\end{aligned} \quad (4.9)$$

this shows that (4.1) holds. If the positive constant δ is small enough, so that when $|t-t_0|\le\delta, \gamma|t-t_0|<1$, n is sufficiently large and t is close to 1, then the right hand side becomes small.

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