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TE, TM Fields in Toroidal Electromagnetism

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ABSTRACT

We analyze the behaviour of TE, TM electromagnetic fields in a toroidal space through Maxwell and wave equations. Their solutions are discussed in a space endowed with a refractive index making separable the wave equations.

Keywords: TE, TM Fields; Toroidal Space; Wave Equation; Laplace Equation

1. Introduction

Toroidal functions have been known for a long time and an important bibliography is given in [1,2] with some applications to diffraction of acoustic and electromagntic plane waves by a torus. These functions have played a major role in the analysis of plasma confinement in a torus configuration [3], where further references can be found] and, more recently, important works have been devoted to their application in particular situations [4-6].

We are interested here in TE, TM electromagnetic fields in a toroidal medium which becomes of importance for the so-called transformation media: a tool used to tackle invisibility problems [7]. We start with a presentation of the main properties of the toroidal coordinates ξ, θ, ϕ. Then, we discuss the Maxwell equations satisfied by the components E_ϕ, H_ξ, H_θ of TE modes (it is easy to transpose these results to TM modes H_ϕ, E_ξ, E_θ) and, we finally get the wave equation satisfied by E_ϕ. Approximate solutions of this equation are obtained when the index of refraction in the toroidal space makes separable the wave equation.

2. Toroidal Coordinates

2.1. Geometric Parameters

In terms of the Cartesian coordinates x, y, z, the toroidal coordinates ξ, θ, ϕ are defined by the relations [8-10]

$$\begin{aligned} x &= a\sin\xi\cos\phi/(\cosh\xi-\cos\theta) \\ y &= a\sin\xi\sin\phi/(\cosh\xi-\cos\theta) \\ z &= a\sin\theta/(\cosh\xi-\cos\theta) \end{aligned} \tag{1}$$

in which $\xi \ge 0, -\pi < \theta < \pi, 0 < \phi \le 2\pi, a > 0$, with the inverse relations

$$\begin{aligned} &\tan\phi = y/x, \xi = \ln(d_1/d_2), \\ &\cos\theta = -\left(4a^2 - d_1^2 - d_2^2\right)/2d_1 d_2 \end{aligned} \tag{2}$$

$$\begin{aligned} &d_1^2 = (\rho+a)^2 + z^2, d_2^2 = (\rho-a)^2 + z^2, \\ &\rho^2 = x^2 + y^2 \end{aligned} \tag{2a}$$

The surfaces of constant θ correspond to spheres with different radii

$$x^2 + y^2 + (z - a\cot\theta)^2 = a^2/\sin^2\theta \tag{3a}$$

the surfaces of constant ξ are non intersecting tori of different radii (anchor rings)

$$z^2 + \left(\sqrt{\left(x^2+y^2\right)} - a\coth\xi\right)^2 = a^2/\sinh^2\xi \tag{3b}$$

the normal derivative to these surfaces is

$$\begin{aligned} &\text{On } \theta : \partial_n = a^{-1}(\cosh\xi - \cos\theta)\partial_\theta, \\ &\text{on } \xi : \partial_n = a^{-1}(\cosh\xi - \cos\theta)\partial_\xi \end{aligned} \tag{4}$$

Now, from the toroidal metric

$$\begin{aligned} &ds^2 = g\left(d\xi^2 + d\theta^2 + \sin h^2\xi d\phi^2\right), \\ &g = a^2(\cosh\xi - \cos\theta)^{-2} \end{aligned} \tag{5}$$

we get the scale functions

$$\begin{aligned} &h_\xi = h_\theta = a/(\cos h\xi - \cos\theta) && \text{(a)} \\ &h_\phi = a\sin h\xi/(\cos h\xi - \cos\theta) && \text{(b)} \end{aligned} \tag{6}$$

so that the volume element is

$$dV = h_\xi^3 \sin h\xi d\xi d\theta d\phi \tag{7a}$$

and the surface elements

$$\begin{aligned} &\text{on } \theta : d\sigma = h_\xi^2 \sin h\xi d\xi d\phi \\ &\text{on } \xi : d\sigma = h_\xi^2 \sin h\xi d\theta d\phi \end{aligned} \tag{7b}$$

Remark: the metric (5) may be written with $(i,j) = 1, 2, 3$ and summation on the repeated indices

$ds^2 = g_{ij}dx^i dx^j$, $x^1 = \xi, x^2 = \theta, x^3 = \phi$, and, g denoting the matrix with components g_{ij}, it comes g = diag. $a^2\underline{h}^2$ (1, 1, $\sinh^2\xi$) where $\underline{h} = (\cos h\xi - \cos\theta)^{-1}$.

2.2. Differential Operators

Using the general definition of the differential operators in orthogonal coordinates [10,11], we get
gradient:

$$\Delta\psi = \left(\partial_\xi\psi/h_\xi,\quad \partial_\theta\psi/h_\theta,\quad \partial_\theta\psi/h_\phi\right) \tag{8a}$$

divergence:

$$\begin{aligned}\Delta F &= 1/h_\xi\, h_\theta h_\phi \\ &\cdot\left[\partial_\xi\left(h_\theta h_\phi F_\xi\right) + \partial_\theta\left(h_\phi h_\xi F_\theta\right) + \partial_\theta\left(h_\xi h_\phi F_\theta\right)\right]\end{aligned} \tag{8b}$$

curl:

$$\begin{aligned}(\nabla\wedge F)_\xi &= 1/h_\theta h_\phi\left[\partial_\theta\left(h_\phi F_\phi\right) - \partial_\theta\left(h_\theta F_\theta\right)\right] \\ (\nabla\wedge F)_\theta &= 1/h_\phi h_\xi\left[\partial_\phi\left(h_\xi F_\xi\right) - \partial_\xi\left(h_\phi F_\phi\right)\right] \\ (\nabla\wedge F)_\phi &= 1/h_\xi h_\theta\left[\partial_\xi\left(h_\theta F_\theta\right) - \partial_\theta\left(h_\xi F_\xi\right)\right]\end{aligned} \tag{8c}$$

and, for the laplacian of a scalar field

$$\begin{aligned}\Delta\psi &= 1/h_\xi h_\theta\, h_\phi\left[\partial_\xi\left(h_\theta h_\phi/h_\xi\right)\partial_\xi\psi\right. \\ &\left.+\partial_\theta\left(h_\phi h_\xi/h_\theta\right)\partial_\theta\psi + \partial_\phi\left(h_\xi h_\theta/h_\phi\right)\partial_\phi\psi\right]\end{aligned} \tag{9}$$

3. TE Field

The components of the TE field $E_\phi(\xi,\theta), H_\xi(\xi,\theta), H_\theta(\xi,\theta)$ are not function of the ϕ coordinate and, using (8c), it is easy to get, in a homogeneous isotropic medium with permittivity ε and permeability μ the Maxwell equations they satisfy

$$\begin{aligned}-(i\varepsilon\omega/c)H_\xi &= \left(1/h_\theta h_\phi\right)\partial_\theta\left(h_\phi E_\phi\right) & \text{(a)} \\ -(i\varepsilon\omega/c)H_\theta &= -\left(1/h_\xi h_\phi\right)\partial_\xi\left(h_\phi E_\phi\right) & \text{(b)} \\ (i\mu\omega/c)E_\phi &= \left(1/h_\theta h_\xi\right)\left[\partial_\xi\left(h_\theta H_\theta\right) - \partial_\theta\left(h_\xi H_\xi\right)\right] & \text{(c)}\end{aligned} \tag{10}$$

Substituting (10(a) and (b)) into (10(c)) gives the wave equation satisfied by $E_\phi\left(n^2 = \varepsilon\mu\right)$

$$\begin{aligned}\left(\omega^2 n^2/c^2\right)E_\phi &= -\left(1/h_\xi\, h_\theta\right)\partial_\xi\left[\left(h_\theta/h_\phi h_\xi\right)\partial_\xi\left(h_\phi E_\phi\right)\right] \\ &-\left(1/h_\xi\, h_\theta\right)\partial_\theta\left[\left(h_\xi/h_\phi h_\theta\right)\partial_\theta\left(h_\phi E_\phi\right)\right]\end{aligned} \tag{11}$$

Multiplying (11) by $h_\phi\partial_\xi{}^2$ and using (6(a)), we get with

$$\psi(\xi,\theta) = h_\phi E_\phi(\xi,\theta) \tag{11a}$$

$$\begin{aligned}&h_\phi\partial_\xi\left(1/h_\phi\partial_\xi\psi\right) + h_\phi\partial_\theta\left(1/h_\phi\partial_\theta\psi\right) \\ &+\left(\omega^2 n^2 h_\xi{}^2/c^2\right)\psi = 0\end{aligned} \tag{12}$$

and, since $h_\phi\partial_\xi\left(1/h_\phi\partial_\xi\psi\right) = -1/h_\phi\,\partial_\xi\left(h_\phi\partial_\xi\psi\right)$, $h_\phi\partial_\theta\left(1/h_\phi\partial_\theta\psi\right) = -1/h_\phi\,\partial_\theta\left(h_\phi\partial_\theta\psi\right)$, this equation becomes

$$\begin{aligned}&1/h_\phi\,\partial_\xi\left(h_\phi\partial_\xi\psi\right) + 1/h_\phi\,\partial_\theta\left(h_\phi\partial_\theta\psi\right) \\ &-\left(\omega^2 n^2 h_\xi{}^2/c^2\right)\psi = 0\end{aligned} \tag{13}$$

To get a more tractable wave equation, we introduce [8,10] fhe variables ς,τ and the function $\chi(\varsigma,\tau)$

$$s = \cos h\xi, \tau = \cos\theta,$$

$$\omega(\xi,\theta) = (s-\tau)^{1/2}\chi(s,\tau) \tag{14}$$

and we write (13) A + B + C = 0.

Then, the expression (6(b)) of h_ϕ is written with the parameter a deleted

$$h_\phi = \sin h\xi(s-\tau)^{-1} \tag{15a}$$

and ∂_ξ is changed into $\sin h\xi\partial_\varsigma$ implying

$$\begin{aligned}1/h_\phi\,\partial_\xi &= (s-\tau)\partial_s \\ h_\phi\partial_\xi &= \left(s^2-1\right)(s-\tau)^{-1}\partial_s\end{aligned} \tag{15b}$$

so that

$$h_\phi\partial_\xi\psi = \left(s^2-1\right)(s-\tau)^{-1}\partial_s\left[(s-\tau)^{1/2}\chi\right] \tag{16}$$

Then, taking into account (14), (15b), (16), the first term $A = 1/h_\phi\partial_\xi\left(h_\phi\partial_\xi\psi\right)$ of (13) becomes

$$A = (s-\tau)\partial_s\left[\left\{\left(s^2-1\right)/(s-\tau)\right\}\partial_s\left\{\left[(s-\tau)^{1/2}\chi\right]\right\}\right] \tag{17}$$

and a simple calculation gives since the $\partial_\varsigma\chi$ terms cancel

$$\begin{aligned}A/(s-\tau) &= (s-\tau)^{-1/2}\partial_s\left[\left(s^2-1\right)\partial_s\chi\right] \\ &+1/2\left\{\partial_s\left[\left(s^2-1\right)/(s-\tau)^{3/2}\right]\right\}\chi\end{aligned} \tag{17a}$$

Similarly with

$$\begin{aligned}h_\phi\partial_\theta &= -\left(1-\tau^2\right)^{1/2}\left(s^2-1\right)^{1/2}(s-\tau)^{-1}\partial_\tau \\ 1/h_\phi\partial_\theta &= -\left(1-\tau^2\right)^{1/2}\left(s^2-1\right)^{-1/2}(s-\tau)\partial_\tau\end{aligned} \tag{18}$$

the second term $B = 1/h_\phi\partial_\theta\left(h_\phi\partial_\theta\psi\right)$ of (13) becomes

$$\begin{aligned}B &= \left(1-\tau^2\right)^{1/2} \\ &\cdot\left[(s-\tau)\partial_\tau\left\{\left(1-\tau^2\right)^{1/2}(s-\tau)^{-1}\partial_\tau\left[(s-\tau)^{1/2}\chi\right]\right\}\right]\end{aligned} \tag{19}$$

and, a simple calculation gives since the terms $\partial_\tau\chi$ cancel

$$\begin{aligned}B/\left(1-\tau^2\right)^{1/2}(s-\tau) &= (s-\tau)^{-1/2}\partial_\tau\left[\left(1-\tau^2\right)^{1/2}\partial_\tau\chi\right] \\ &-1/2\,\partial_\tau\left\{\left(1-\tau^2\right)^{1/2}(s-\tau)^{-3/2}\right\}\chi\end{aligned} \tag{19a}$$

Taking into account (17a) and (19a), we get

$$A+B=(s-\tau)^{1/2}\,\partial_S\left[\left(s^2-1\right)\partial_S\chi\right]$$
$$+(s-\tau)^{1/2}\left(1-\tau^2\right)^{1/2}\partial_\tau\left[\left(1-\tau^2\right)^{1/2}\partial_\tau\chi\right]+D \tag{20}$$

$$D=1/2(s-\tau)\left\{\partial_s\left[\left(s^2-1\right)/(s-\tau)^{3/2}\right]\right\}\chi$$
$$-1/2(s-\tau)\left(1-\tau^2\right)^{1/2}\left\{\partial_\tau\left(1-\tau^2\right)^{1/2}(s-\tau)^{-3/2}\right\}\chi \tag{20a}$$

which reduces as easy to prove to the simple expression

$$D=\left(s^2-\tau\right)^{1/2}\chi/4 \tag{21}$$

Now, using (14) and the expression $a(s-\tau)^{-1}$ of h_ξ, the last term $C=-\left(\omega^2n^2h_\xi/c^2\right)\psi$ of (13) becomes

$$C=-\omega^2n^2a^2c^{-2}(s-\tau)^{-3/2}\chi \tag{22}$$

So, according to (20) (20a), (21), (22), the equation (13) $A+B+C=0$ takes the simple form justifying the choice of the function (14) made in [8,10] where calculations for A, B are also made

$$\partial_s\left[\left(s^2-1\right)\partial_\varsigma\chi\right]+\partial_\theta{}^2\chi+\chi/4$$
$$-\left(\omega^2n^2a^2/c^2(s-\tau)^2\right)\chi=0 \tag{23}$$

Deleting the last term in Equation (23) reduces the wave equation to the Laplace equation with the variables s,τ separated so that looking for $\chi(s,\tau)$ in the form where m is an integer

$$\chi(s,\tau)=\exp(im\arccos\tau)\varpi_0(s) \tag{24}$$

gives since $\partial_\tau\chi=im\left(1-\tau^2\right)^{1/2}\chi$

$$\partial_s\left[\left(s^2-1\right)\partial_s\right]\varpi_0-\left(m^2-1/4\right)\varpi_0=0 \tag{25}$$

with as elementary solutions [8,10] of this Laplace equation, the half order Legendre functions $P_{m-1/2}(s),Q_{m-1/2}(s)$ of first and second kind.

Now, the wave E_q (23) for TE fields may be generalized to a medium with a constant permitivity while the permeability, consequently the refractive index, depends on s and τ. So, to make E_q (23) separable, we assume that the refractive index n is with n_0 constant

$$n(s,\tau)=n_0(s-\tau) \tag{26}$$

Then, the last term of (23) becomes $b^2\chi$ where $b^2=\omega^2a^2n_0{}^2/c^2$ and with

$$\chi(s,\tau)=\exp(im\arccos\tau)\varpi(s) \tag{27}$$

in which m is no more assumed to be an integer, the wave equation satisfied by $\varpi(s)$ is

$$\partial_S\left[\left(s^2-1\right)\partial_S\right]\varpi(s)-\left(m^2-1/4\right)\varpi(s)$$
$$-b^2\varpi(s)=0 \tag{28}$$

and with $m^2+b^2=-k^2$ we get

$$\partial_s\left[\left(1-s^2\right)\right]\partial_\varsigma\varpi(s)-\left(k^2+1/4\right)\varpi(s)=0 \tag{29}$$

whose solutions are the conical (Mehler) harmonics [10, 12] $P_{-1/2+ik}(s)$ and $Q_{-1/2+ik}(s)$.

Then, according to (11a), (14), (27), the component $E_\phi(\xi,\theta)$ of the TE field is

$$E_\phi(\xi,\theta)=\psi(\xi,\theta)/h_\phi=(s-\tau)^{1/2}\chi(s,\tau)/h_\phi$$
$$=(s-\tau)^{1/2}\exp(im\theta)\varpi(s)/h_\phi \tag{30}$$

in which $h_\phi=\sin h\xi(s-\tau)^{-1},s=\cos h\xi,\tau=\cos\theta$ and $\varpi(s)$ solution of (23). Substituting (30) into (10(a) and (b)) gives the magnetic components $H_\xi(\xi,\theta),H_\theta(\xi,\theta)$ of the TE field.

TM field: These results are easy to transpose to TM fields: just change E_ϕ, H_ξ, H_θ, and ε. $\mu(s,\tau)$ into H_ϕ, E_ξ, E_θ and $-\mu$. $-\varepsilon(s,\tau)$ with a constant permeability while the permittivity is defined so that the refractive index has still the expression (26).

4. Discussion

This work is an illustration of the importance of the function (14) in toroidal coordinates either as in [8,10] to make manageable the 3D Laplace equation which separates into three equations. or as here to get a similar result with the 2D Helmholtz equation. It is remarkable that for TE, TM propagation in a toroidal medium this choice of a particular form for the ϕ component of these fields works so efficiently that we get an exact equation whatever is the refractive index. In some sense, the function (14) is consubstantial to toroidal and bispherical coordinates.

The results obtained here are only illutrative because to get analytical solutions of the exact Equation (23), satisfied by $\chi(s,\tau)$, we had to impose rather drastic conditions on the refractive index with a constant permittivity (TE) or constant permeability (TM) , in order to get a separable equation. In practice, to solve (23), requires numerical calculations with algorithms to tackle 2D partial differential equations which is not a real difficulty in particular for propagation in isotropic homogeneous media where the refractive index is constant.

The toroidal geometry has made a comeback in two different domains: first in the string theory of elementary particles [13,14] and second in cosmology. The quest of cosmic origin [15] has suggested [16] that the Universe could have the shape of a drough nut that is of a 3D torus.

Clearly, the present work on electromagnetic wave propagation in a toroidal space could be of some interest to toroidal cosmology [17-19].

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