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A Modified Homotopy Analysis Method for Solving Boundary Layer Equations

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ABSTRACT

A new modification of the Homotopy Analysis Method (HAM) is presented for highly nonlinear ODEs on a semi-infinite domain. The main advantage of the modified HAM is that the number of terms in the series solution can be greatly reduced; meanwhile the accuracy of the solution can be well retained. In this way, much less CPU is needed. Two typical examples are used to illustrate the efficiency of the proposed approach.

Keywords: Homotopy Analysis Method; Boundary Layer Equations; Orthonormal Functions

1. Introduction

In 1992, Liao [1] proposed the Homotopy Analysis Method (HAM) to solve nonlinear differential equations analytically. Since then, HAM has been used to investigate a variety of mathematical and physical problems [2]. As is well known, HAM has the advantage of independence on small physical parameters and adjusting the convergence region and convergence rate of the series solution over perturbation method. However, for some type of auxiliary operator (*i.e*. base functions), it is usually time-consuming to get high-order approximation, and the number of terms appearing in high order approximation is very huge.

To improve the efficiency of HAM, many scholars have proposed different techniques. Yabushita [3] suggested an optimal HAM approach by minimizing the residual of governing equations. Marinca [4], Niu [5], Liao [6] developed this kind of approach. Lin [7] suggested an iterative technique, in which the initial guess is continuously replaced by intermediate approximation to proceed the computation. Recently, we use orthonormal polynomials/functions to approximate the right-hand side of the high-order deformation equations to prohibit the rapid growth of the terms appearing in approximate solutions. For differential equations defined on a finite interval, trigonometric functions (or polynomial functions) are usually selected to express solutions. In this case, orthonormal trigonometric functions (or Chebyshev polynomials) are used to approximate right-hand side of high-order deformation equations. In this paper, we generalize this kind of approach for nonlinear problems defined on semi-infinite intervals. The main idea is that orthonormal functions derived from Schmidt-Gram process are used to approximate the right-hand side of high-order deformation equations during computation. For different types of problems, the derived orthonormal functions are different, which are closely related to the solution expression.

In the following section, the modified HAM (MHAM) is presented for boundary layer problems. In Section 3, examples are given to demonstrate it. Conclusions and some discussions are given in the last section.

2. Analysis of the Method

For convenience, a brief description of the standard HAM will be present first. Then the proposed truncation technique will be followed.

Without loss of generality, consider the differential equation

$$N\left[u(t)\right]=0, \quad (1)$$

where N is a nonlinear operator, t denotes the independent variable, $u(t)$ is an unknown function. Suppose $u(t)$ could be expressed by a set of functions

$$\left\{\phi_k(t)\middle| k\geq 0\right\} \quad (2)$$

such that

$$u(t)=\sum_{k=0}^{+\infty} a_k\phi_k(t) \quad (3)$$

is uniformly valid, where a_k is a coefficient. In HAM, the *zeroth-order deformation equation* is constructed as

$$(1-q)L\left[\phi(t;q)-u_0(t)\right]=qc_0H(t)N\left[\phi(t;q)\right], \quad (4)$$

where

$$\phi(t;q)=\sum_{m=0}^{+\infty}u_m(t)q^m, \quad (5)$$

L is an auxiliary linear operator, and $H(t)$ the auxiliary function. Applying the *homotopy-derivative* [8]

$$D_m(\square)=\frac{1}{m!}\frac{\partial^m\square}{\partial q^m}\bigg|_{q=0} \quad (6)$$

to both sides of Equation (4), we get the corresponding *m*th-order deformation equation

$$L\left[u_m(t)-\chi_m u_{m-1}(t)\right]=c_0 H(t)R_{m-1}, m\ge 1, \quad (7)$$

where

$$R_{m-1}=D_{m-1}\left\{N\left[\phi(t;q)\right]\right\}, \quad (8)$$

and

$$\chi_m=\begin{cases}0, m\le 1,\\ 1, m>1.\end{cases} \quad (9)$$

Note that $u_1,u_2,\cdots$ can be obtained by solving linear Equation (7) one after the other. The *m*th-order approximation of $u(t)$ is given by

$$U_m=\sum_{k=0}^{m}u_k \quad (10)$$

To measure the accuracy of U_m, the squared residual error for Equation (1) is defined as

$$\delta_m=\int_A\left[N(U_m)\right]^2\mathrm{d}t,$$

where A is the domain.

If a successful homotopy analysis solution is obtained, the difficulty to get better approximation is that with the growth of order, the number of terms in higher-order approximation will grow rapidly, resulting in an enormous amount of computing time. To address this problem, we propose a truncation technique. The basic idea is that the right-hand side of Equation (7) is approximated by a set of orthonormal functions.

Suppose that R_{m-1} can be expressed by a finite linear combination of linearly independent functions $\varphi_k(t)$, $k=1,2,\cdots$. Note that $\{\varphi_k\}$ may be slightly different from $\{\phi_k\}$. Define a proper inner product in the linear space spanned by $\varphi_1,\varphi_2,\cdots,\varphi_N$ as

$$(f,g)=\int_A\rho(t)f(t)g(t)\mathrm{d}t, \quad (11)$$

where $\rho(t)$ is a weight function. In the framework of HAM, two typical kinds of base functions are usually used for boundary layer problems.

Case 1: suppose that R_{m-1} can be expressed by finite linear combination of linearly independent functions

$$\left\{t^n\mathrm{e}^{-m\lambda t}\,\middle|\,n\ge 0, m\ge 1, \lambda>0\right\} \quad (12)$$

Note that (12) is dependent on three parameters λ, m and n. To implement the orthonormalization, an order is given to (12) as follows (called triangular order):

$$t^n\mathrm{e}^{-m\lambda t}=\varphi_{n+1+\frac{(m+n-1)(m+n)}{2}}.$$

The inner product is defined as

$$(f,g)=\int_0^{+\infty}f(t)g(t)\mathrm{d}t.$$

Case 2: suppose that R_{m-1} can be expressed by finite linear combination of linearly independent functions

$$\left\{t^{-(n+\alpha)}\,\middle|\,n\ge 0, \alpha\ge 1\right\}. \quad (13)$$

The inner product is defined as

$$(f,g)=\int_1^{+\infty}f(t)g(t)\mathrm{d}t. \quad (14)$$

Applying the Schmidt-Gram process to the first N functions $\varphi_1,\varphi_2,\cdots,\varphi_N$, we obtain N orthonormal functions $e_1,e_2,\cdots,e_N$. Every time when R_{m-1} is got, we approximate it by $e_1,e_2,\cdots,e_N$, to ensure that the number of terms in the right-hand side of Equation (7) will be no more than N. That is to say, we replace R_{m-1} with its approximation

$$\tilde{R}_{m-1}=\sum_{i=1}^{N}(R_m,e_i)e_i \quad (15)$$

to proceed the computation in HAM.

3. Numerical Experiment

To illustrate the efficiency of the truncation technique, two typical examples are considered. The codes are written in Maple 13 on a PC with an Intel Core 2 Quad 2.66 GHz CPU. The variable *Digits* in the experiments is to control the number of digits when calculating with software floating point numbers in Maple.

3.1. Example 1

Let us consider the Blasius Equation (9)

$$f'''(\eta)+\frac{1}{2}f(\eta)f'(\eta)=0, \quad (16)$$

subject to the boundary conditions

$$f(0)=f'(0)=0,\ f'(+\infty)=1, \quad (17)$$

where the prime denotes the derivative with respect to η. Following Liao [9], we seek the solution $f(\eta)$ in the form

$$f(\eta)=a_{0,0}+\eta+\sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty}a_{m,n}\eta^n\mathrm{e}^{-m\lambda\eta},$$

where $\lambda>0$ is the spatial-scale parameter, and $a_{m,n}$ is a coefficient. The auxiliary linear operator is chosen as

$$L\left[\phi(\eta;q)\right]=\frac{\partial^3\phi(\eta;q)}{\partial\eta^3}+\lambda\frac{\partial^2\phi(\eta;q)}{\partial\eta^2}.$$

The initial guess is

$$f_0(\eta)=\eta+\frac{1-e^{-\lambda\eta}}{\lambda}.$$

The zeroth-order deformation equation is constructed as in (4), and the m th-order deformation equation as in (7) with homogeneous boundary conditions

$$f_m(0)=f_m'(0)=f_m'(+\infty)=0,\ m\ge 1,$$

where

$$R_{m-1}=f_{m-1}'''(\eta)+\frac{1}{2}\sum_{k=0}^{m-1}f_k(\eta)\,f_{m-1-k}''(\eta).$$

For this example, the first kind orthonormal functions are used to approximate R_{m-1} every time when R_{m-1} is got. Then $\tilde{R}_{m-1}$ is used to compute f_m instead of R_{m-1}. In the experiment, we set $\lambda=4$, $N=36$, $Digits=100$, and $H=1$. From **Tables 1** and **2**, we can see that though we use approximate R_{m-1} (*i.e.* $\tilde{R}_{m-1}$) to do the computation, the residual error δ_m and $f''(0)$ given by MHAM are almost the same as that given by standard HAM. From **Table 3**, we can see that the number of terms in high-order approximation given by MHAM is 39, while that given by HAM grows exponentially. Moreover, it shows that MHAM needs less than ninth the CPU time used by HAM to get 50th-order approximate solution. The curve of residual error and CPU time is plotted in **Figure 1**. From it we can see that the truncation technique greatly improves the efficiency of HAM.

3.2. Example 2

Consider a set of two coupled nonlinear differential equations (see Kuiken [10] for details)

$$f'''(\eta)+\theta(\eta)-f'^2(\eta)=0 \tag{18}$$

$$\theta''(\eta)=3\sigma f'(\eta)\theta(\eta) \tag{19}$$

with the boundary conditions

$$f(0)=f'(0)=0,\ \ \theta(0)=1,\ \ f'(+\infty)=\theta(+\infty)=0,$$

where σ is the Prandtl number.

Under the transformation

$$\xi=1+\lambda\eta,\ \ F(\xi)=f'(\eta),$$

Equations (18) and (19) become

$$\lambda^2F''(\xi)+S(\xi)-F^2(\xi)=0, \tag{20}$$

$$\lambda^2S''(\xi)=3\sigma F(\xi)S(\xi), \tag{21}$$

with the boundary conditions

Table 1. Comparison of $f''(0)$ given by MHAM and HAM in Example 1.

Order m	MHAM	HAM
10	0.3277556	0.3277556
20	0.3318513	0.3318513
30	0.3320404	0.3320403
40	0.3320557	0.3320555
50	0.3320573	0.3320571

Table 2. Comparison of δ_m given by MHAM and HAM in Example 1.

Order m	MHAM	HAM
10	1.9500×10^{-1}	1.9500×10^{-1}
20	3.8793×10^{-3}	3.8793×10^{-3}
30	1.1616×10^{-4}	1.1616×10^{-4}
40	3.9192×10^{-6}	3.9192×10^{-6}
50	1.4013×10^{-7}	1.4008×10^{-7}

Table 3. Comparison of CPU time (seconds) and number of terms appearing in *m*th order approximation given by MHAM and HAM in Example 1.

Order m	MHAM		HAM	
	Terms	Time (sec)	Terms	Time (sec)
10	39	49.530	123	0.791
20	39	103.837	443	9.52
30	39	158.952	963	72.832
40	39	215.133	1683	537.956
50	39	271.667	2603	2488.035

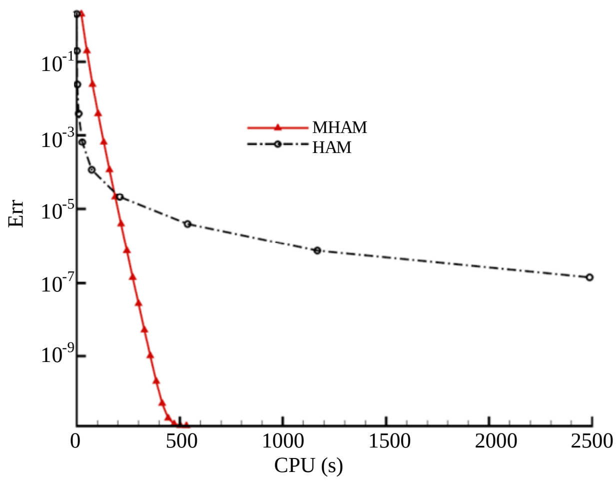

Figure 1. Residual error versus CPU time in Example 1. Solid line: MHAM; Dash-dotted line: HAM.

$$F(1)=0,\ S(1)=1,\ F(+\infty)=S(+\infty)=0.$$

We seek the solution $F(\xi)$ and $S(\xi)$ in the form

$$F(\xi)=\sum_{n=2}^{+\infty}a_n\xi^{-n},\quad S(\xi)=\sum_{n=4}^{+\infty}b_n\xi^{-n},$$

where a_n, b_n are coefficients.

Following Liao [2], the auxiliary linear operators are chosen as

$$L_F=\left(\frac{\xi}{3}\right)\frac{\partial^2}{\partial\xi^2}+\frac{\partial}{\partial\xi},\quad L_S=\left(\frac{\xi}{5}\right)\frac{\partial^2}{\partial\xi^2}+\frac{\partial}{\partial\xi}$$

and the initial guess of $F(\xi)$ and $S(\xi)$ as

$$F_0(\xi)=\gamma\left(\xi^{-2}-\xi^{-3}\right),S_0(\xi)=\xi^{-4}.$$

Then the high-order deformation equations become

$$L_F\left[F_n(\xi)-\chi_nF_{n-1}(\xi)\right]=c_FH_F(\xi)R_{n-1}^F,$$

$$L_S\left[S_n(\xi)-\chi_nS_{n-1}(\xi)\right]=c_SH_S(\xi)R_{n-1}^S,$$

subject to the homogeneous boundary conditions

$$F_n(1)=S_n(1)=F_n(+\infty)=S_n(+\infty)=0,$$

and

$$R_{n-1}^F=\lambda^2F''_{n-1}(\xi)+S_{n-1}(\xi)-\sum_{j=0}^{n-1}F_j(\xi)F_{n-1-j}(\xi),$$

$$R_{n-1}^S=\lambda^2S''_{n-1}(\xi)-3\sigma\sum_{j=0}^{n-1}F_j(\xi)S_{n-1-j}(\xi).$$

We find that R_{n-1}^F and R_{n-1}^S can be expressed in the form of finite combination of functions

$$\left\{\xi^{-n}\middle|n\ge 4\right\}, \tag{22}$$

and

$$\left\{\xi^{-n}\middle|n\ge 6\right\}, \tag{23}$$

respectively. Applying the Schmidt-Gram process to the first N functions in (22) and (23), respectively, we obtain two set of N orthonormal functions, denoted by e^F and e^S. We use e^F to approximate R_{n-1}^F every time it is got, and e^S to approximate R_{n-1}^S. Then $\tilde{R}_{n-1}^F$ and $\tilde{R}_{n-1}^S$ are used to proceed the computation.

In the experiment, we set $N=15$, $\sigma=1$, $\lambda=1/3$, $c_F=c_S=-1/2$, $H_F=H_S=1$, and $\text{Digits}=100$. For different order approximation given by the two approaches, the quantity $f''(0)$ is showed in **Table 4**, and the residual error for Equation (20) is compared in **Table 5**. We can see that although we use $\tilde{R}_{n-1}^F$ to proceed the computation instead of R_{n-1}^F, the accuracy of the approximate solution is well retained. From **Table 6**, we can see that the number of terms in the high-order approximation given by MHAM was kept within 16, while that given by HAM grows with the order m. Moreover, it shows that MHAM needs less CPU time than the standard HAM to get high-order approximate solution. The curve of residual error and CPU time is plotted in **Figure 2**. From it we can see that the truncation technique is more powerful to get higher-order approximation than the standard HAM.

Table 4. Comparison of $f''(0)$ given by MHAM and HAM in Example 2.

Order *m*	MHAM	HAM
20	0.6971702145	0.6971702149
40	0.6932675852	0.6932675846
60	0.6932116278	0.6932116273
80	0.6932116054	0.6932116060
100	0.6932116316	0.6932116326

Table 5. Comparison of δ_m given by MHAM and HAM in Example 2.

Order *m*	MHAM	HAM
20	1.4233×10^{-5}	1.4233×10^{-5}
40	4.3708×10^{-9}	4.3708×10^{-9}
60	8.9385×10^{-12}	1.1883×10^{-12}
80	6.0625×10^{-13}	5.9355×10^{-13}
100	9.7190×10^{-14}	9.5370×10^{-14}

Table 6. Comparison of CPU time (seconds) and number of terms appearing in F_m given by MHAM and HAM in Example 2.

Order *m*	MHAM		HAM	
	Terms	Time (sec)	Terms	Time (sec)
20	16	11.606	42	0.967
40	16	25.428	82	6.177
60	16	40.607	122	24.694
80	16	57.112	162	77.267
100	16	74.958	202	207.590

4. Conclusion and Discussions

In this paper, an efficient modification of HAM is proposed for solving boundary layer problems. Using the derived orthonormal functions, the right-hand sides of high-order deformation equations are approximated to reduce the rapid growth of terms in high-order approximate solution. Two typical examples show that the new approach can greatly reduce the terms in the approximate solution; meanwhile the accuracy can be largely retained. The new approach needs less time to get high-order approximation than the standard HAM. However, one unsolved problem

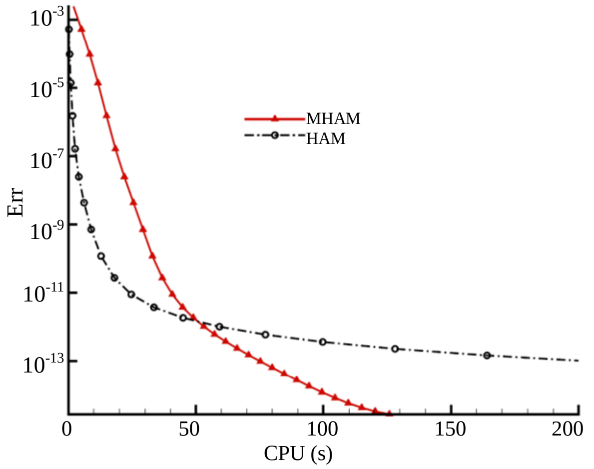

Figure 2. Residual error versus CPU time in Example 2. Solid line: MHAM; Dash-dotted line: HAM.

of this approach is that there is so far no estimation theory on how many orthonormal functions should be used to approximate R_{n-1} when accuracy is prior given. We will try to generalize this truncation technique to solve PDEs in the next step.

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