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Gnedenko-Raikov's Theorem Fails for Exchangeable Sequences

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ABSTRACT

We study the connection between the central limit theorem and law of large numbers for exchangeable sequences, and provide a counterexample to the Gnedenko-Raikov theorem for such sequences.

Keywords: Exchangeable Sequences; Central Limit Theorem; Weak Law of Large Numbers

The celebrated Gnedenko-Raikov theorem states that sums of independent, infinitesimal random variables are asymptotically normal if and only if the sum of squares, centered at truncated means, is relatively stable. The following variant for i.i.d. random variables has been recently proved in [1]:

Theorem 1. *Let* $\{\xi_n, n \geq 1\}$ *be i.i.d. random variables with mean zero, and* $\{b_n, n \geq 1\}$ *a sequence of positive reals increasing to* $+\infty$. *Then*

$$\frac{\xi_1 + \cdots + \xi_n}{b_n} \to N(0,1) \text{ in distribution}$$

if and only if

$$\frac{\xi_1^2 + \cdots + \xi_n^2}{b_n^2} \to 1 \text{ in probability.}$$

A classical extension of independence is exchangeability, and in this context we shall prove that the Gnedenko-Raikov theorem fails. First, let us recall the basic facts. A sequence of random variables $\{X_n, n \geq 1\}$ on the probability space (Ω, F, P) is said to be exchangeable if for each *n*,

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) = P\left(X_{\pi(1)} \leq x_1, \cdots, X_{\pi(n)} \leq x_n\right)$$

for any permutation π of $\{1, 2, \cdots, n\}$ and any $x_i \in \mathbb{R}, i = 1, \cdots, n$. Two trivial examples are i.i.d. random variables and totally determined random variables $\{X, X, \cdots\}$. Two nontrivial but simple examples are $\{X + r_n,\ n \geq 1\}$ and $\{Y \cdot r_n,\ n \geq 1\}$ where the r_n's are i.i.d. and independent of X or Y, respectively.

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By de Finetti's theorem, an infinite sequence of exchangeable random variables is conditionally i.i.d. given either the tail σ-field of $\{X_n, n \geq 1\}$ or the σ-field G of permutable events. Furthermore, there exists a regular conditional distribution P^{ω} for $\{X_n, n \geq 1\}$ given G such that for each $\omega \in \Omega$ the coordinate random variables $\{\xi_n \equiv \xi_n^{\omega}\}$, called mixands, of the probability space $(\mathbb{R}^{\infty}, B^{\infty}, P^{\omega})$ are i.i.d. Namely, for each natural number *n*, any Borel function $f: \mathbb{R}^n \to \mathbb{R}$, and any Borel set B on $\mathbb{R}$,

$$P\left(f(X_1, \cdots, X_n) \in B\right) = \int_{\Omega} P^{\omega}\left(f(\xi_1, \cdots, \xi_n) \in B\right) \mathrm{d}P. \quad (1)$$

The following central limit theorem for exchangeable sequences has been proved in [2]:

Theorem 2. *Let* $\{X_n, n \geq 1\}$ *be a sequence of exchangeable random variables. Then there exist constants* a_n, b_n *with* $b_n \to \infty$, *such that*

$$\frac{X_1 + \cdots + X_n - a_n}{b_n} \to N(0,1) \quad \text{in distribution if and only}$$

if there exists a positive sequence $\varepsilon_n \searrow 0$ such that

$$nP^{\omega}\left(|\xi_1| > \varepsilon_n b_n\right) \to 0 \text{ in probability.}$$

and either $b_n / \sqrt{n}$ is slowly varying with

$$\frac{v_n(\omega)}{b_n} \to 1, \quad \frac{a_n(\omega) - a_n}{b_n} \to 0 \text{ in probability}$$

or b_n / n is slowly varying with

$$\frac{v_n(\omega)}{b_n} \to 0 \text{ in probability,}$$

$$\frac{a_n(\omega)-a_n}{b_n}\to N(0,1) \text{ in distribution,}$$

where

$$a_n(\omega):=nE^{\omega}\left(\xi_1 1_{(|\xi_1|\le\varepsilon_n b_n)}\right)$$

and

$$v_n^2(\omega)=n\left[E^{\omega}\left(\xi_1^2 1_{(|\xi_1|\le\varepsilon_n b_n)}\right)-\left(E^{\omega}\left(\xi_1 1_{(|\xi_1|\le\varepsilon_n b_n)}\right)\right)^2\right].$$

In the above theorem, the case where $b_n/\sqrt{n}$ is slowly varying characterizes the situation when the classical central limit theorem holds for the mixands, whereas the case where b_n/n is slowly varying characterizes the situation when the law of large numbers holds for the mixands and those limits have a standard normal distribution. Recently, we "cleaned" the latest statement and proved in [3] the following variant of the law of large numbers for exchangeable sequences:

Theorem 3. *Let* $\{X_n, n\ge 1\}$ *be a sequence of exchangeable random variables and* $\{b_n, n\ge 1\}$ *a sequence of positive reals increasing to* $+\infty$, *that satisfy the following conditions*:

$$nP^{\omega}\left(|\xi_1|>b_n\right)\to 0 \text{ in probability}$$

and

$$\tilde{v}_n(\omega)b_n\to 0 \text{ in probability,}$$

where

$$\tilde{v}_n^2(\omega)=n\left[E^{\omega}\left(\xi_1^2 1_{(|\xi_1|\le b_n)}\right)-\left(E^{\omega}\left(\xi_1 1_{(|\xi_1|\le b_n)}\right)\right)^2\right].$$

Then

$$\frac{X_1+\cdots+X_n-\tilde{a}_n(\omega)}{b_n}\to 0 \text{ in probability,}$$

where

$$\tilde{a}_n(\omega)=nE^{\omega}\left(\xi_1 1_{(|\xi_1|\le b_n)}\right).$$

Unless the sequence $\{X_n, n\ge 1\}$ is i.i.d., the converse in the above theorem is not true; more is needed, see [4].

We are now ready to provide the counterexample mentioned in the introduction. It will rely on both Theorems 2 and 3, and some specific constants $\{b_n, n\ge 1\}$. More precisely, we have:

Theorem 4. *Let* $\{X_n, n\ge 1\}$ *be a sequence of exchangeable random variables and* $\{b_n, n\ge 1\}$ *a sequence of norming constants that satisfy the following condition:*

$$nP^{\omega}\left(|\xi_1|>\varepsilon_n b_n\right)\to 0 \text{ in probability,} \qquad (2)$$

where $\{\varepsilon_n, n\ge 1\}$ is the sequence appearing in Theorem 2.

1) Assume that the sequence $\{\varepsilon_n b_n, n\ge n_0\}$ is non-decreasing for some $n_0\ge 1$ and satisfies

$$\sum_{k=1}^{n}\frac{\varepsilon_k^2 b_k^2}{k^2}\le c\frac{\varepsilon_n^2 b_n^2}{n}$$

for all $n\ge 1$ and some constant $c>0$. Then

$$\frac{\left(X_1^2+\cdots+X_n^2\right)}{b_n^2}\to 0 \text{ in probability.}$$

2) If b_n/n and $\varepsilon_n/n^{\rho-1}$ are slowly varying for some $1/2<\rho<1$, then

$$X_1+\cdots+X_n b_n\to N(0,1) \text{ in distribution,}$$

and the Gnedenko-Raikov theorem fails in this case.

Proof of Theorem 4. 1) Under the assumptions on the sequence $\{\varepsilon_n b_n, n\ge n_0\}$ and according to [5], p. 680, we have that

$$nE^{\omega}\left(\xi_1^2 1_{(|\xi_1|\le\varepsilon_n b_n)}\right)\varepsilon_n^2 b_n^2\to 0 \text{ in probability.}$$

Also, cf. Section 2 in [5], we have that $\varepsilon_n b_n\to+\infty$ and $b_n\to+\infty$. These facts imply that

$$\frac{nE^{\omega}\left(\xi_1^2 1_{(|\xi_1|\le\varepsilon_n b_n)}\right)}{b_n^2}\to 0 \text{ in probability.} \qquad (3)$$

Taking into account the following identity (with the notations in Theorem 2):

$$v_n^2(\omega)=n\left[E^{\omega}\left(\xi_1^2 1_{(|\xi_1|\le\varepsilon_n b_n)}\right)-a_n^2(\omega)/n^2\right],$$

which gives

$$\frac{nE^{\omega}\left(\xi_1^2 1_{(|\xi_1|\le\varepsilon_n b_n)}\right)}{b_n^2}=\frac{a_n^2(\omega)}{nb_n^2}+\frac{v_n^2(\omega)}{b_n^2}$$

from formula (3) it follows that

$$\begin{aligned}&a_n^2(\omega)/nb_n^2\to 0 \text{ in probability,}\\ &\text{and } v_n/b_n\to 0 \text{ in probability.}\end{aligned} \qquad (4)$$

Now let $\varepsilon>0$ be given. By formula (1) and the triangle inequality we have

$$\begin{aligned}&P\left(\left|\left(X_1^2+\cdots+X_n^2\right)/b_n^2\right|>\varepsilon\right)\\ &=\int_{\Omega}P^{\omega}\left(\left|\left(\xi_1^2+\cdots+\xi_n^2\right)/b_n^2\right|>\varepsilon\right)\mathrm{d}P\\ &\le\int_{\Omega}P^{\omega}\left(\left|\frac{1}{b_n^2}\sum_{k=1}^{n}\xi_k^2 1_{(|\xi_k|\le\varepsilon_n b_n)}\right|>\frac{\varepsilon}{2}\right)\mathrm{d}P\\ &+\int_{\Omega}P^{\omega}\left(\left|\frac{1}{b_n^2}\sum_{k=1}^{n}\xi_k^2 1_{(|\xi_k|\le\varepsilon_n b_n)}\right|>\frac{\varepsilon}{2}\right)\mathrm{d}P.\end{aligned} \qquad (5)$$

Using (2), we estimate the first term in the right hand side of (5) as follows:

$$\begin{aligned} & P^{\omega}\left(\left|\frac{1}{b_n^2}\sum_{k=1}^{n}\xi_k^2 1_{(|\xi_k|\le\varepsilon_n b_n)}\right| > \frac{\varepsilon}{2}\right) \\ & \le P^{\omega}\left(\bigcup_{k=1}^{n}\left(|\xi_k| > \varepsilon_n b_n\right)\right) \qquad (6) \\ & \le nP^{\omega}\left(|\xi_k| > \varepsilon_n b_n\right) \to 0 \text{ in probability.} \end{aligned}$$

We then break down the second term in the right hand side of (5) as follows:

$$\begin{aligned} & P^{\omega}\left(\left|\frac{1}{b_n^2}\sum_{k=1}^{n}\xi_k^2 1_{(|\xi_k|\le\varepsilon_n b_n)}\right| > \frac{\varepsilon}{2}\right) \\ & \le P^{\omega}\left(\left|\frac{1}{b_n^2}\sum_{k=1}^{n}\xi_k^2 1_{(|\xi_k|\le\varepsilon_n b_n)} - \frac{1}{b_n^2}\sum_{k=1}^{n}E^{\omega}\left(\xi_k^2 1_{(|\xi_k|\le\varepsilon_n b_n)}\right)\right| > \frac{\varepsilon}{4}\right) \\ & + P^{\omega}\left(\left|\frac{1}{b_n^2}\sum_{k=1}^{n}E^{\omega}\left(\xi_k^2 1_{(|\xi_k|\le\varepsilon_n b_n)}\right) - \frac{n}{b_n^2}\left(E^{\omega}\xi_k 1_{(|\xi_k|\le\varepsilon_n b_n)}\right)^2\right| > \frac{\varepsilon}{8}\right) \\ & + P^{\omega}\left(\left|\frac{n}{b_n^2}\left(E^{\omega}\xi_k 1_{(|\xi_k|\le\varepsilon_n b_n)}\right)^2\right| > \frac{\varepsilon}{8}\right). \end{aligned} \qquad (7)$$

Using (4), we have

$$\begin{aligned} & P^{\omega}\left(\left|\frac{1}{b_n^2}\sum_{k=1}^{n}\xi_k^2 1_{(|\xi_k|\le\varepsilon_n b_n)} - \frac{1}{b_n^2}\sum_{k=1}^{n}E^{\omega}\left(\xi_k^2 1_{(|\xi_k|\le\varepsilon_n b_n)}\right)\right| > \frac{\varepsilon}{4}\right) \\ & \le \frac{16}{\varepsilon^2}\frac{1}{b_n^4} n\mathrm{Var}\,\xi_1^2 1_{(|\xi_1|\le\varepsilon_n b_n)} \le \frac{16}{\varepsilon^2}\frac{\varepsilon_n^2 v_n^2(\omega)}{b_n^2} \\ & \to 0 \text{ in probability.} \end{aligned} \qquad (8)$$

Also, cf. (4),

$$\begin{aligned} & P^{\omega}\left(\left|\frac{1}{b_n^2}\sum_{k=1}^{n}E^{\omega}\left(\xi_k^2 1_{|\xi_k|}\le\varepsilon_n b_n\right) - \frac{n}{b_n^2}\left(E^{\omega}\xi_k 1_{(|\xi_k|\le\varepsilon_n b_n)}\right)^2\right| > \frac{\varepsilon}{8}\right) \\ & = 1_{\left(v_n^2(\omega)/b_n^2 > \varepsilon/8\right)} \to 0 \text{ in probability,} \end{aligned} \qquad (9)$$

and, again cf. (4),

$$\begin{aligned} & P^{\omega}\left(\left|\frac{n}{b_n^2}\left(E^{\omega}\xi_k 1_{(|\xi_k|\le\varepsilon_n b_n)}\right)^2\right| > \frac{\varepsilon}{8}\right) \qquad (10) \\ & = 1_{\left(\left|a_n^2(\omega)/nb_n^2\right| > \varepsilon/8\right)} \to 0 \text{ in probability.} \end{aligned}$$

From (5)-(10) we deduce that $\left(X_1^2 + \cdots + X_n^2\right)/b_n^2 \to 0$ in probability.

Now, let us prove 2). If b_n/n is slowly varying, and using (4), Theorems 2 and 3 imply that $\left(X_1 + \cdots + X_n\right)/b_n \to N(0,1)$ in distribution. If, in addition, $\varepsilon_n/n^{\rho-1}$ is slowly varying for some $1/2 < \rho < 1$, then the hypotheses on the sequence $\{\varepsilon_n b_n, n \ge n_0\}$ in part 1) of Theorem 4 are satisfied cf. section 2 in [5], hence the Gnedenko-Raikov theorem fails in this case. □

Remark. It is worth noting that the Gnedenko-Raikov theorem is valid in the case where $b_n/\sqrt{n}$ is slowly varying in Theorem 2, as well as in both self-normalized central limit theorem [6] and self-normalized law of large numbers [7] for exchangeable sequences. This is why the counterexample in Theorem 4 above was rather hard to get.

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