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On Continuous Limiting Behaviour for the $q(n)$-Binomial Distribution with $q(n)\to 1$ as $n\to\infty$

Malvina Vamvakari
Department of Informatics and Telematics, Harokopio University of Athens, Athens, Greece
Email: mvamv@hua.gr

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ABSTRACT

Recently, Kyriakoussis and Vamvakari [1] have established a *q*-analogue of the Stirling type for *q*-constant which have lead them to the proof of the pointwise convergence of the *q*-binomial distribution to a Stieltjes-Wigert continuous distribution. In the present article, assuming $q(n)$ a sequence of *n* with $q(n)\to 1$ as $n\to\infty$, the study of the affect of this assumption to the $q(n)$-analogue of the Stirling type and to the asymptotic behaviour of the $q(n)$-Binomial distribution is presented. Specifically, a $q(n)$ analogue of the Stirling type is provided which leads to the proof of deformed Gaussian limiting behaviour for the $q(n)$-Binomial distribution. Further, figures using the program MAPLE are presented, indicating the accuracy of the established distribution convergence even for moderate values of *n*.

Keywords: Stirling Formula; $q(n)$-Factorial Number of Order *n*; Saddle Point Method; $q(n)$-Binomial Distribution; Pointwise Convergence; Gauss Distribution

1. Introduction and Preliminaries

In last years, many authors have studied q-analogues of the binomial distribution (see among others [2-4]). Specifically, Kemp and Kemp [3] defined a q-analogue of the binomial distribution with probability function in the form

$$f_X(x)=P(X=x)=\binom{n}{x}_q q^{\binom{x}{2}}\theta^x\prod_{j=1}^{n}\left(1+\theta q^{j-1}\right)^{-1}, \quad x=0,1,\cdots,n, \tag{1}$$

where $\theta>0, 0<q<1,$ by replacing the loglinear relationship for the Bernoulli probabilities in Poissonian random sampling with loglinear odds relationship. Also, Kemp [4] defined (1) as a steady state distribution of birth-abort-death process.

Futhermore, Charalambides [2] considering a sequence of independent Bernoulli trials and assuming that the odds of success at the *i*th trial given by

$$\pi_i=\theta q^{i-1}, i=1,2,\cdots,0<q<1,0<\theta<\infty,$$

is a geometrically decreasing sequence with rate *q*, derived that the probability function of the number X of successes up to *n*-trail is the *q*-analogue of the binomial distribution with p.f. given by Equation (1).

For q constant, the *q*-binomial distribution has finite mean and variance when $n\to\infty$ .Thus, the asymptotic normality in the sense of the DeMoivre-Laplace classical limit theorem did not conclude, as in the case of ordinary hypergeometric series discrete distributions. Also, asymptotic methods—central or/and local limit theorems—are not applied as in Bender [5], Canfield [6], Flajolet and Soria [7], Odlyzko [8] *et al*.

Recently, Kyriakoussis and Vamvakari [1], for *q* constant, established a limit theorem for the *q*-binomial distribution by a pointwise convergence in a *q*-analogue sense of the DeMoivre-Laplace classical limit theorem. Specifically, the pointwise convergence of the *q*-binomial distribution to a Stieltjes-Wigert continuous distribution was proved. In detail, transferred from the random variable X of the *q*-binomial distribution (1) to the equal-distributed deformed random variable $Y=[X]_{1/q}$, then, for $n\to\infty$, the *q*-binomial distribution was approximated by a deformed standardized continuous Stieltjes-Wigert distribution as follows

$$f_X(x) \cong \frac{q^{1/8}\left(\log q^{-1}\right)^{1/2}}{(2\pi)^{1/2}} \cdot \left(q^{-3/2}(1-q)^{1/2}\frac{[x]_{1/q}-\mu_q}{\sigma_q}+q^{-1}\right)^{1/2} \cdot \exp\left(\frac{\log^2\left(q^{-3/2}(1-q)^{1/2}\frac{[x]_{1/q}-\mu_q}{\sigma_q}+q^{-1}\right)}{2\log q}\right), x \geq 0, \quad (2)$$

where $\theta = \theta_n, n = 0,1,2,\cdots$ such that $\theta_n = q^{-\alpha n}$ with $0 < a < 1$ constant and μ_q and σ_q^2 the mean value and variance of the random variable Y, respectively. To obtain the above pointwise convergence (2), a q-analogue of the well known Stirling formula for the n factorial $(n!)$ has been provided.

In statistical mechanics and in computer science such as in probabilistic and approximation algorithms, applications of the q-binomial distribution involve sequences of independent Bernoulli trials where in the geometrically decreasing odds of success at the i th trial, the rate q is considered to be a sequence of n with $q = q(n) \to 1$ as $n \to \infty$. In this work, under this consideration, a question arises. How this assumption affects the continuous limiting behaviour of this q-binomial distribution?

The answer to this question is given in this manuscript by establishing a deformed Gaussian limiting behaviour for the $q(n)$-Binomial distribution is proved. The proofs are concentrated on the study of the sequence $q(n)$ and the parameters of the considered distribution as sequences of n. Further, figures using the program MAPLE are presented, indicating the accuracy of the established distribution convergence even for moderate values of n.

2. Main Results

2.1. An Asymptotic Expansion of the *q*(*n*)-Factorial Number of Order *n* with $q(n) \to 1$ as $n \to \infty$

To initiate our study we need to derive an asymptotic expansion for $n \to \infty$ of the q-factorial number of order n

$$[n]_q! = [1]_q[2]_q\cdots[n]_q = \prod_{k=1}^{n}\frac{1-q^k}{(1-q)^n} = \frac{(q;q)_n}{(1-q)^n}, \quad (3)$$

where $q = q(n)$ with $q(n) \to 1$ as $n \to \infty$ and $[t]_q = \frac{1-q^t}{1-q}$, the q-number t.

The derived estimate for the q-factorial numbers of order n, is based on the analysis of the q-Exponential function

$$E_q((1-q)x) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(1-q)^n}{(q;q)_n}x^n = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{[n]_q!}x^n = \prod_{j=1}^{\infty}\left(1+(1-q)xq^{j-1}\right) \quad (4)$$

which is the ordinary generating function (g.f.) of the numbers $\frac{q^{\binom{n}{2}}}{[n]_q!}, n = 0,1,2,\cdots$.

Rewriting $E_q((1-q)x)$ as follows

$$E_q(x) = \exp(g(x)), \quad (5)$$

where

$$g(x) = \sum_{j=1}^{\infty}\log\left(1+((1-q)x)q^{j-1}\right), \quad (6)$$

because of the large dominant singularities of the generating function $E_q(x)$, a well suited method for analyzing this is the *saddle point* method.

Using an approach of the saddle point method inspired from [9-12] and [1], the following theorem gives an asymptotic for the $q(n)$-factorial number of order *n*.

Theorem 1. *The q-factorial numbers of order* n, $[n]_q!$, *where*

A) $q = q(n)$ with $q(n) \to 1$ as $n \to \infty$ and $q(n)^n = \Omega(1)$

or

B) $q = q(n)$ with $q(n)^n = o(1)$

have the following asymptotic expansion for $n \to \infty$

$$[n]_q! = (2\pi)^{1/2} q^{\binom{n}{2}} \exp(-g(r))\left[rg'(r)+r^2g''(r)\right]^{1/2} r^n \cdot\left[1+\sum_{k'=1}^{N}S_{k'}(r)\left(q^n(1-q)\right)^{k'}+O\left(\left(q^n(1-q)\right)^{\frac{N+1}{2}}\right)\right]^{-1} \quad (7)$$

where N *is a positive integer*, r *is the real solution of the equation*

$$rg'(r) = n$$

and

$$S_{k'}(r)=\frac{r^{k'}}{(2k')!}\cdot\sum_{j=1}^{2k'}B_{2k',j}\left(\alpha_1(r),\cdots,\alpha_{2k'}(r)\right)\frac{\Gamma\left(j+k'+1/2\right)}{\pi^{1/2}},$$

$$\alpha_{k'}(r)=\frac{\left[2^{-1}\left(1+\frac{rg''(r)}{g'(r)}\right)\right]^{-k'/2-1}\mathrm{i}^{k'+2}}{g'(r)(k'+1)(k'+2)r^{k'/2+1}}\cdot\sum_{\nu=1}^{k'+2}S(k'+2,\nu)r^{\nu}g^{(\nu)}(r) \tag{8}$$

with $B_{k',j}(\psi_1,\cdots,\psi_{k'})$ *the partial Bell polynomials,* $S(k',j)$ *the Stirling numbers of the second kind and* $i^2=-1$.

Proof. We shall study the asymptotic behaviour of the q-factorial numbers of order n, $[n]_q!$, by expressing them via Cauchy's integral formula that gives the coefficients of a power series:

$$\frac{q^{\binom{n}{2}}}{[n]_q!}=\frac{1}{2\pi i}\int_{|x|=r}\frac{\exp(g(x))}{x^{n+1}}\mathrm{d}x \tag{9}$$

where the contour of integration is taken to be a circle of radius r. This integral will be estimated with the saddle point method. The saddle point is defined by the equation $xg'(x)=n+1$. It turns out that it is convenient to switch to polar coordinates, setting $x=r\mathrm{e}^{i\theta}$. Then the original integral becomes

$$\frac{q^{\binom{n}{2}}}{[n]_q!}=\frac{\exp(g(r))}{r^n2\pi}\cdot\int_{-\pi}^{\pi}\exp\left[g\left(r\mathrm{e}^{i\theta}\right)-g(r)-in\theta\right]\mathrm{d}\theta. \tag{10}$$

In accordance with the saddle point method principles, we choose the radius r to be the solution of $rg'(r)=n$. Setting $G(\theta)=g\left(r\mathrm{e}^{i\theta}\right)-g(r)-in\theta$ with a Maclaurin series expansion about $\theta=0$ we have

$$G(\theta)=-\phi^2+\phi^2\sum_{k'=1}^{\infty}\alpha_{k'}(r)\frac{(\psi\phi)^{k'}}{k'!} \tag{11}$$

where

$$\phi=\left[\frac{1}{2}\left(rg'(r)+r^2g''(r)\right)\right]^{1/2}\theta,\psi=\left[g'(r)\right]^{-\frac{1}{2}} \tag{12}$$

and

$$\alpha_{k'}(r)=\frac{\left[2^{-1}\left(1+\frac{rg''(r)}{g'(r)}\right)\right]^{-k'/2-1}g^{(k'+2)}\left(r\mathrm{e}^{i\theta}\right)\Big|_{\theta=0}}{g'(r)(k'+1)(k'+2)r^{k'/2+1}}, \tag{13}$$

where

$$g^{(k'+2)}\left(r\mathrm{e}^{i\theta}\right)\Big|_{\theta=0}=\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{k'+2}g\left(r\mathrm{e}^{i\theta}\right)\Big|_{\theta=0}.$$

The absence of a linear term in θ indicates a saddle point. The function $\left|\mathrm{e}^{G(\theta)}\right|$ is unimodal with its peak at $\theta=0$.

An estimation of the q-factorial numbers of order $n,[n]_q!$ with q defined by conditions (A) or (B) should naturally proceed by isolating separately small portions of the contour (corresponding to x near the real axis) as follows.

A) For $q=q(n)$ with $q(n)^n=\Omega(1)$ we set

$$I_1=\frac{1}{\sqrt{2\pi}}\int_{-\delta}^{\delta}\exp\left[G(\theta)\right]\mathrm{d}\theta,\qquad I_2=\frac{1}{\sqrt{2\pi}}\int_{\delta}^{2\pi-\delta}\exp\left[G(\theta)\right]\mathrm{d}\theta, \tag{14}$$

and choose δ such that the following conditions are true (see [12]):

C1) $n\delta^2\to\infty$, that is $\delta\gg n^{-1/2}$

C2) $n\delta^3\to 0$, that is $\delta\ll n^{-1/3}$,

where "$\ll$" means "much smaller than". A suitable choice for δ is $n^{-3/8}$.

As $\left|\mathrm{e}^{G(\theta)}\right|$ decreases in $[\delta,\pi]$,

$$\left|\mathrm{e}^{G(\theta)}\right|\le\left|\mathrm{e}^{G(\delta)}\right|,\delta\le\theta\le 2\pi-\delta. \tag{15}$$

We will show in the sequel that from C1) and C2) it follows that $\mathrm{e}^{G(\delta)}$ is exponentially small, being dominated by a term of the form $\mathrm{e}^{-O\left(n^{1/4}\right)}$.

Indeed we have

$$G(\delta)\sim-\frac{1}{2}\left(rg'(r)+r^2g''(r)\right)\delta^2$$

or

$$G(\delta)\sim-\frac{1}{2}\left(rg'(r)+r^2g''(r)\right)n^{-3/4}. \tag{16}$$

But

$$rg'(r)+r^2g''(r)\sim\frac{1}{\log q^{-1}}\left(q^{-n}-1\right)$$

or

$$rg'(r)+r^2g''(r)\sim\frac{(1-q)}{\log q^{-1}}\left(q^{n-1}+q^{n-2}+\cdots+1\right). \tag{17}$$

For $q=q(n)$ with $q(n)^n=\Omega(1)$ we get

$$G(\delta)=O\left(-\frac{1}{2}n^{1/4}\right). \tag{18}$$

From which we find that

$$|I_2|=O\left(\mathrm{e}^{G(\delta)}\right)=O\left(\mathrm{e}^{-\frac{1}{2}n^{1/4}}\right). \tag{19}$$

Thus, by C1), δ has been taken large enough so that the central integral I_1 "captures" most of the contribution, while the remainder integral I_2 is exponentially small by (19).

We now turn to the precise evaluation of the central integral I_1. We have

$$I_1 = \frac{1}{\left[rg'(r)+r^2g''(r)\right]^{1/2}}\frac{1}{\sqrt{\pi}} \cdot \int_{-\epsilon}^{\epsilon} \exp\left[-\phi^2+\phi^2\sum_{k'=1}^{\infty}\alpha_{k'}(r)\frac{(\psi\phi)^{k'}}{k'!}\right]\mathrm{d}\phi \tag{20}$$

where

$$\epsilon = \left[\frac{1}{2}\left(rg'(r)+r^2g''(r)\right)\right]^{1/2}\delta. \tag{21}$$

Note that $\epsilon\to\infty$ as $n\to\infty$, since

$$\epsilon = n^{-3/8}\left[\frac{1}{2}\left(rg'(r)+r^2g''(r)\right)\right]^{1/2} = n^{1/8}\left[\frac{1}{2}\left(1+\frac{rg''(r)}{g'(r)}\right)\right]^{1/2} > Cn^{1/8},$$

where C a positive constant.

B) For $q=q(n)$ with $q(n)^n = o(1)$ we set

$$I_3 = \frac{1}{\sqrt{2\pi}}\int_{-\delta}^{\delta} r^{-n}\exp\left[G(\theta)\right]\mathrm{d}\theta, \quad I_4 = \frac{1}{\sqrt{2\pi}}\int_{\delta}^{2\pi-\delta} r^{-n}\exp\left[G(\theta)\right]\mathrm{d}\theta, \tag{22}$$

and choose δ such that the conditions C1) and C2) are true. We suitably select $\delta = n^{-3/8}$.

As $\left|\mathrm{e}^{G(\theta)}\right|$ decreases in $[\delta,\pi]$,

$$\left|e^{G(\theta)}\right| \le \left|\mathrm{e}^{G(\delta)}\right|, \delta\le\theta\le 2\pi-\delta. \tag{23}$$

We will now show that $\mathrm{e}^{G(\delta)}$ is dominated by a term of the form $O(1)$. Indeed, form C1), C2), 16) and 17) it follows that

$$\exp\left(G(\delta)\right) \sim \exp\left(-\frac{1}{2\log q^{-1}}\left(1-q^n\right)n^{-3/4}\right). \tag{24}$$

From which we get

$$\left|I_4\right| = O\left(r^{-n}\mathrm{e}^{G(\delta)}\right) = O\left(q^{n^2}\mathrm{e}^{-\frac{1}{2}\frac{1}{\log q^{-1}}\left(1-q^n\right)n^{-3/4}}\right). \tag{25}$$

Thus, for $q=q(n)$ with $q(n)^n=o(1)$ the integral I_4 is negligibly small. We now turn to the precise evaluation of the central integral I_3. Since

$$\begin{aligned} I_3 &= \frac{1}{\sqrt{2\pi}}\int_{-\delta}^{\delta} r^{-n}\exp\left[G(\theta)\right]\mathrm{d}\theta \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} r^{-n}\exp\left[G(\theta)\right]\mathrm{d}\theta \\ &\quad -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\delta} r^{-n}\exp\left[G(\theta)\right]\mathrm{d}\theta \\ &\quad -\frac{1}{\sqrt{2\pi}}\int_{\delta}^{\infty} r^{-n}\exp\left[G(\theta)\right]\mathrm{d}\theta \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} r^{-n}\exp\left[G(\theta)\right]\mathrm{d}\theta \\ &\quad -O\left(2q^{n^2}\mathrm{e}^{-\frac{1}{2}\frac{1}{\log q^{-1}}\left(1-q^n\right)n^{-3/4}}\right) \end{aligned}$$

we have

$$I_3 = \frac{r^{-n}}{\left[rg'(r)+r^2g''(r)\right]^{1/2}}\frac{1}{\sqrt{\pi}} \cdot \int_{-\infty}^{\infty}\exp\left[-\phi^2+\phi^2\sum_{k=1}^{\infty}\alpha_k(r)\frac{(\psi\phi)^k}{k!}\right]\mathrm{d}\phi. \tag{26}$$

We now unifiable proceed our proof for both conditions A) and B) and working analogously as in Kyriakoussis and Vamvakari [1] we get our final estimation (7). ◊

In the previous theorem due to saddle point method principles, we have chosen the radius r of the derived asymptotic expansion (7) to be the solution of $rg'(r)=n$. By solving this saddle point equation we get that

$$r = q^{-1}[n]_{1/q}$$

and

$$g'(r)+r^2g''(r) \cong \frac{1-q}{\log q^{-1}}[n]_{1/q}\,q^{n-1}.$$

So, by substituting these to our estimation (7) the following corollary is proved.

Corollary 1. *The q-factorial numbers of order* $n,[n]_q!$, *where*

A) $q=q(n)$ with $q(n)\to 1$ as $n\to\infty$ and $q(n)^n=\Omega(1)$

or

B) $q=q(n)$ with $q(n)^n=o(1)$

have the following asymptotic expansion for $n\to\infty$

$$[n]_q! = \frac{\left(2\pi(1-q)\right)^{1/2}}{\left(q\log q^{-1}\right)^{1/2}} \cdot \frac{q^{\binom{n}{2}}q^{n/2}[n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty}\left(1+\left(q^{-n}-1\right)q^{j-1}\right)}\left(1+O\left(q^n(1-q)\right)\right). \tag{27}$$

2.2. Deformed Gaussian Limiting Behaviour for the $q(n)$-Binomial Distributions with $q(n)\to 1$ as $n\to\infty$

Transferred from the random variable X of the q-binomial distribution (1) to the equal-distributed deformed random variable $Y=[X]_{1/q}$, the mean value and variance of the random variable Y, say μ_q and σ_q^2 respectively, are given by the next relations

$$\mu_q=[n]_q\frac{\theta}{1+\theta q^{n-1}} \tag{28}$$

and

$$\sigma_q^2=\frac{\theta^2[n]_q[n-1]_q}{q\left(1+\theta q^{n-1}\right)\left(1+\theta q^{n-2}\right)}+\frac{\theta[n]_q}{\left(1+\theta q^{n-1}\right)}-\frac{\theta^2[n]_q^2}{\left(1+\theta q^{n-1}\right)^2} \tag{29}$$

(see Kyriakoussis and Vamvakari [1]).

Using the standardized r.v.

$$Z=\frac{[X]_{1/q}-\mu_q}{\sigma_q}$$

with μ_q and σ_q given in (28) and (29), the q-analogue Stirling asymptotic formula (27) and inspired by [1], the following theorem explores the continuous limiting behaviour of the $q(n)$-binomial distribution with $q(n)\to 1$ as $n\to\infty$.

Theorem 2. *Let the p.f. of the q-binomial distribution be of the form*

$$f_X(x)=\binom{n}{x}_q q^{\binom{x}{2}}\theta^x\prod_{j=1}^{n}\left(1+\theta q^{j-1}\right)^{-1},x=0,1,\cdots,n,$$

where $\theta=\theta_n,n=0,1,2,\cdots$ *such that* $\theta_n\to\infty,$ as $n\to\infty$. *Then, for*

A) $q=q(n)$ with $q(n)^n=\Omega(1)$

or

B) $q=q(n)$ with $q(n)\to 1$ as $n\to\infty$ and $q(n)^n=o(1)$ and $\theta_n=q^{-\alpha n}$ with $0<a<1$ constant

the $q(n)$*-binomial distribution is approximated, for* $n\to\infty,$ *by a deformed standardized Gauss distribution as follows*

$$f_X(x)\cong\frac{\left(\log q^{-1}\right)^{1/2}}{(2\pi)^{1/2}}\cdot\exp\left(-\frac{1}{2}\left(\frac{\sigma_q}{\mu_q\left(\log q^{-1}\right)^{1/2}}\frac{[x]_{1/q}-\mu_q}{\sigma_q}\right)^2\right),x\geq 0. \tag{30}$$

Proof. Using the q-analogue of Stirling type (27), for $q=q(n)$ with $q(n)\to 1$ and $q(n)^n=\Omega(1)$ or $q(n)^n=o(1)$, the q-binomial distribution (1), is approximated by

$$f_X(x)\cong\frac{\left(q\log q^{-1}\right)^{1/2}}{\left(2\pi(1-q)\right)^{1/2}}\frac{\theta_n^x}{(1-q)^x}\cdot\frac{\prod_{j=1}^{\infty}\left(1+\left(q^{-x}-1\right)q^{j-1}\right)}{\prod_{j=1}^{\infty}\left(1+\theta_n q^{j-1}\right)q^{x/2}[x]_{1/q}^{x+1/2}}. \tag{31}$$

Let the random variable $[X]_{\frac{1}{q}}=\frac{1-q^{-X}}{1-q^{-1}}$ and the q-standardized r.v. $Z=\frac{[X]_{1/q}-\mu_q}{\sigma_q}$ with μ_q and σ_q given by (28) and (29) respectively, then all the following listed estimations are easily derived

$$[x]_{1/q}\cong\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right), \tag{32}$$

$$q^{-x}\cong\left(q^{-1}-1\right)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)+1, \tag{33}$$

$$x\cong\frac{1}{\log q^{-1}}\log\left(\left(q^{-1}-1\right)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)+1\right), \tag{34}$$

$$\left([x]_{1/q}\right)^x\cong\mu_q^x\cdot\exp\left(\frac{1}{\log q^{-1}}\log\left(\left(q^{-1}-1\right)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)+1\right)\cdot\log\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)\right) \tag{35}$$

Also, the estimation of the next product

$$\prod_{j=1}^{\infty}\left(1+q\left(q^{-x}-1\right)q^{j-1}\right)=\prod_{j=1}^{\infty}\left(1+q\left(q^{-1}-1\right)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)q^{j-1}\right)=\exp\left(\sum_{j=1}^{\infty}\log\left(1+(1-q)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)q^{j-1}\right)\right), \tag{36}$$

is derived by applying the Euler-Maclaurin summation formula (see Odlyzko [8], p. 1090) in the sum of the above Equation (36) as follows

$$\sum_{j=1}^{\infty}\log\left(1+(1-q)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)q^{j-1}\right)$$
$$=\frac{1}{2\log q^{-1}}\log^2\left((1-q)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)\right)$$
$$+Li_2\left(\frac{(1-q)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)}{(1-q)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)+1}\right) \quad (37)$$
$$+\frac{1}{2}\log\left(1+(1-q)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)\right)$$
$$+\frac{\beta_2\log q}{2}\frac{(1-q)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)}{1+(1-q)\mu_q\left(\frac{\sigma_q}{\mu_q}z+1\right)}+O(\log q),$$

where Li_2 the dilogarithmic function and β_2 the Bernoulli number of order 2.

Moreover, working similarly for the sum appearing in the product

$$\prod_{j=1}^{\infty}\left(1+\theta_n q^{j-1}\right)=\exp\left(\sum_{j=1}^{\infty}\log\left(1+\theta_n q^{j-1}\right)\right) \quad (38)$$

the next estimation is obtained

$$\sum_{j=1}^{\infty}\log\left(1+\theta_n q^{j-1}\right)=\frac{1}{2\log q^{-1}}\log^2(\theta_n)$$
$$+Li_2\left(\frac{\theta_n}{\theta_n+1}\right)+\frac{1}{2}\log(1+\theta_n) \quad (39)$$
$$+\frac{\beta_2\log q}{2}\frac{\theta_n}{1+\theta_n}+O\left(\theta_n^{-1}\right).$$

Applying all the previous the estimations (32)-(39) to the approximation (31), carrying out all the necessary manipulations and for $\theta_n\to\infty$, by both conditions A) and B), we derive our final asymptotic (30). ◊

Remark 2. A realization of the sequence $q(n), n=0,1,2,\cdots$ considered in the above theorem 1A) is

$$q(n)=1-\frac{\beta}{n}, 0<\beta\le 1$$

with

$$q(n)^n=\exp(-\beta).$$

Remark 3. Possible realizations of the sequence $q(n), n=0,1,2,\cdots$ considered in the above theorem 2B) are among others the next two ones

$$q(n)=1-\frac{1}{\ln(n)} \text{ or } q(n)=1-\frac{1}{n^c}, 0<c<1.$$

Corollary 2 Let the random variable X with p.f. that of the $q(n)$-binomial distribution as in Theorem 2. Then for $n\to\infty$ the following approximation holds

$$P(a\le X\le b)\cong\frac{1}{2}Erf(u_{b+1})-\frac{1}{2}Erf(u_a), \quad 0\le a\le b, \quad (40)$$

where

$$u_a=\frac{\left\{[a-1/2]_{1/q}-\mu_q\right\}/\sigma_q}{\mu_q\left(2\log q^{-1}\right)^{1/2}} \quad (41)$$

with $Erf(t)=\frac{2}{\sqrt{\Pi}}\int_0^t\exp\left(-x^2\right)\mathrm{d}x$ the Gauss error function.

Proof. Using the approximation (2) and the classical continuity correction we have that

$$P[a\le X\le b]=\sum_{x\in[a,b]}P(X=x)$$
$$\cong\int_{a-\frac{1}{2}}^{b+\frac{1}{2}}\frac{\left(\log q^{-1}\right)^{1/2}}{(2\pi)^{1/2}} \quad (42)$$
$$\cdot\exp\left(-\frac{1}{2}\left(\frac{\sigma_q}{\mu_q\left(\log q^{-1}\right)^{1/2}}\frac{[x]_{1/q}-\mu_q}{\sigma_q}\right)^2\right)\mathrm{d}x.$$

Setting

$$z=\frac{[x]_{1/q}-\mu_q}{\sigma_q}$$

the approximation (42) becomes

$$P[a\le X\le b]=\sum_{x\in[a,b]}P(X=x)$$
$$\cong\frac{\sigma_q\mu_q^{-1}}{\left(\log q^{-1}\right)^{1/2}(2\pi)^{1/2}} \quad (43)$$
$$\cdot\int_{\frac{[a-1/2]_{1/q}-\mu_q}{\sigma_q}}^{\frac{[b+1/2]_{1/q}-\mu_q}{\sigma_q}}\exp\left(-\frac{1}{2}\left(\frac{\sigma_q z}{\mu_q\left(\log q^{-1}\right)^{1/2}}\right)^2\right)\mathrm{d}z.$$

Carrying out all the necessary manipulations, we get the final approximation (40). ◊

3. Figures Using Maple

In this section, we present a computer realization of approximation (30), by providing figures using the computer program MAPLE and the q-series package developed by F. Garvan [13] which indicate good convergence even for moderate values of *n*. Analytically, for the random variable *X*, we give the **Figures 1** and **2** realizing Theorem 2(A), by demonstrating with *diamond blue points* the *exact probability*

$$f_X(x) = P(X = x) = \text{Prob}\left(x - \frac{1}{2} \le X \le x + \frac{1}{2}\right), \quad (44)$$

$$x = 0, 1, 2, \cdots, n,$$

and with *diamond green points* the continuous *probability approximation*

$$b_n^q(x) = \text{Prob}\left(x - \frac{1}{2} \le X \le x + \frac{1}{2}\right)$$

$$\cong \frac{1}{2} Erf(u_{x+1}) - \frac{1}{2} Erf(u_x), \quad (45)$$

$$0 \le x \le n$$

with u_x and u_{x+1} given by Equation (41), for

$$q = q(n) = 1 - \frac{1}{n},$$

$$\theta = \theta_n = (\exp(1) - 1) / (\exp(1) - 2\exp(1) q^n + q^n)$$

and $n = 50, 100$.

Note that similar good convergence even for moderate values of n have been implemented for Theorem 2B).

The procedure in MAPLE which realizes the exact probability (44) and its approximation (45) for given n, q and *theta* for both Theorem 2A) and 2B), is available under request.

4. Concluding Remarks

In this article, a deformed Gaussian limiting behaviour for the $q(n)$-Binomial distribution has been established. The proofs have been concentrated on the study of the sequence $q(n)$ and the parameters of the considered distributions as sequences of *n*. Further, figures using the program MAPLE have been presented, indicating the accuracy of the established distribution convergence even for moderate values of *n*.

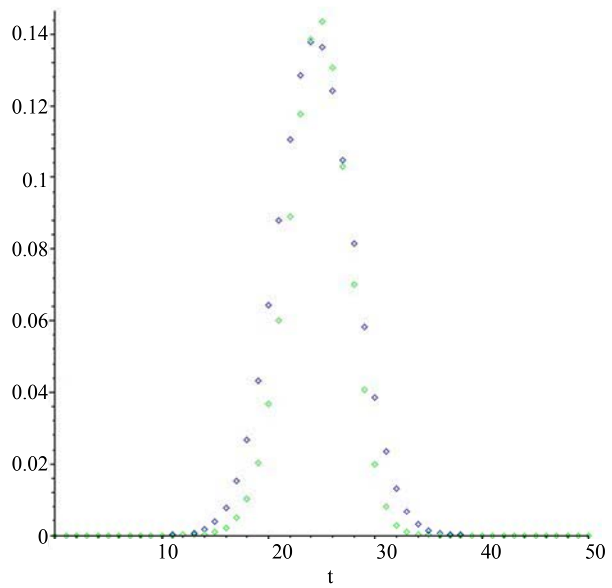

Figure 1. Sketch of exact probability (44) by blue diamond points and probability approximation (45) by green diamond points, for *n* = 50.

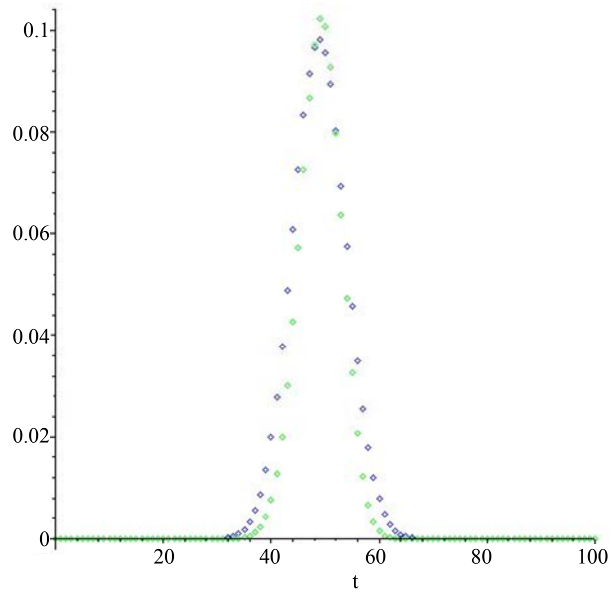

Figure 2. Sketch of exact probability (44) by blue diamond points and probability approximation (45) by green diamond points, for *n* = 100.

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REFERENCES

[1] A. Kyriakoussis and M. G. Vamvakari, "On a *q*-Analogue of the Stirling Formula and a Continuous Limiting Behaviour of the *q*-Binomial Distribution-Numerical Calculations," *Methodology and Computing in Applied Probability*, 2011, pp. 1-27.

[2] Ch. A. Charalambides, "Discrete *q*-Distributions on Bernoulli Trials with a Geometrically Varying Success Probability," *The Journal of Statistical Planning and Inference*, Vol. 140, No. 9, 2010, pp. 2355-2383.

[3] A. W. Kemp and C. D. Kemp, "Welson's Dice Data Revisited," *The American Statistician*, Vol. 45, No. 3, 1991, pp. 216-222.

[4] A. W. Kemp, "Steady-State Markov Chain Models for Certain *q*-Confluent Hypergeometric Distributions," *Journal of Statistical Planning and Inference*, Vol. 135, No. 1, 2005, pp. 107-120. doi:10.1016/j.jspi.2005.02.009

[5] E. A. Bender, "Central and Local Limit Theorem Applied to Asymptotic Enumeration," *Journal of Combinatorial Theory Series A*, Vol. 15, No. 1, 1973, pp. 91-111.

doi:10.1016/0097-3165(73)90038-1

[6] E. R. Canfield, "Central and Local Limit Theorems for the Coefficients of Polynomials of Binomial Type," *Journal of Combinatorial Theory*, *Series A*, Vol. 23, No. 3, 1977, pp. 275-290. doi:10.1016/0097-3165(77)90019-X

[7] P. Flajolet and M. Soria, "Gaussian Limiting Distributions for the Number of Components in Combinatorial Structures," *Journal of Combinatorial Theory Series A*, Vol. 53, No. 2, 1990, pp. 165-182. doi:10.1016/0097-3165(90)90056-3

[8] A. M. Odlyzko, "Handbook of Combinatorics," In: R. L. Graham, M. Grötschel and L. Lovász, Eds., *Asymptotic Enumeration Methods*, Elsevier Science Publishers, Amsterdam, 1995, pp. 1063-1229.

[9] L. M. Kirousis, Y. C. Stamatiou and M. Vamvakari, "Upper Bounds and Asymptotics for the *q*-Binomial Coefficients," *Studies in Applied Mathematics*, Vol. 107, No. 1, 2001, pp. 43-62. doi:10.1111/1467-9590.1071177

[10] A. Kyriakoussis and M. Vamvakari, "On Asymptotics for the Signless Noncentral *q*-Stirling Numbers of the First Kind," *Studies in Applied Mathematics*, Vol. 117, No. 3, 2006, pp. 191-213. doi:10.1111/j.1467-9590.2006.00352.x

[11] Ch. A. Charalambides and A. Kyriakoussis, "An Asymptotic Formula for the Exponential Polynomials and a Central Limit Theorem for Their Coefficients," *Discrete Mathematics*, Vol. 54, No. 3, 1985, pp. 259-270. doi:10.1016/0012-365X(85)90110-4

[12] P. Flajolet and R. Sedgewick, "Analytic Combinatorics, Chapter V: Application of Rational and Meromorphic Asymptotics, Chapter VIII: Saddle Point Asymptotics," Cambridge University Press, Cambridge, 2009.

[13] F. Garvan, "The Maple QSERIES Package," 2010. http:www.math.ufl.edu/ frank/qmaple/qmaple.html