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Estimate of an Hypoelliptic Heat-Kernel outside the Cut-Locus in Semi-Group Theory

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ABSTRACT

We give a proof in semi-group theory based on the Malliavin Calculus of Bismut type in semi-group theory and Wentzel-Freidlin estimates in semi-group of our result giving an expansion of an hypoelliptic heat-kernel outside the cut-locus where Bismut's non-degeneray condition plays a preominent role.

Keywords: Subriemannian Geometry; Heat-Kernels

1. Introduction

Let us consider some vector fields $X_i, i = 1,\cdots,m$ on $\mathbb{R}^d$ with bounded derivatives at each order. We consider the generator

$$L = 1/2\sum X_i^2 \tag{1}$$

It generates a Markov semi-group P_t acting on bounded continuous *f* functions on $\mathbb{R}^d$. The natural question is to know if the semi-group has an heat-kernel:

$$P_t[f](x) = \int_{\mathbb{R}^d} p_t(x,y) f(y)\mathrm{d}y \tag{2}$$

Let us suppose that the strong Hoermander hypothesis is checked: in such a case Hoermander ([1]) proved the existence of a smooth heat kernel. Malliavin [2] proved again this theorem by using a probabilistic representation of it. A lot of tools of stochastic analysis were translated recently by Léandre in semi-group theory. We refer to the review papers [3]. In particular [4] proved again the existence of the heat kernel by using the Malliavin Calculus of Bismut type in semi-group theory.

Let us recall what is strong Hoermander hypothesis.

Let

$$E_1 = \{X_1,\cdots,X_m\} \tag{3}$$

$$E_{l+1} = \bigcup_{Y\in E_l, i=1,\cdots,m}[Y, X_i] \tag{4}$$

Strong Hoermander hypothesis in x is the following: there exits an *l* such that

$$\inf_{|\xi|=1}\sum_{E_l}(Y(x),\xi)^2 \geq C > 0 \tag{5}$$

Under Hoermander hypothesis in *x*, $p_t(x,y)$ exists and is smooth in *y*.

Let *h* be a path from [0,1] into $\mathbb{R}^m$ with finite energy

$$\|h\|^2 = \int_0^1 \sum_{i=1}^m \left|\left(h_s^i\right)'\right|^2 \mathrm{d}s < \infty \tag{6}$$

The Hilbert space of h such that (6) is satisfied is denoted by $\mathbb{H}$.

We consider the horizontal curve $x_s(h)$ starting from x:

$$\mathrm{d}x_s(h) = \sum X_i(x_s(h))\mathrm{d}h_s^i \tag{7}$$

We consider the control distance $\mathrm{d}(x,y)$

$$\mathrm{d}^2(x,y) = \inf_{x_0(x)=x, x_1(h)=y} \|h\|^2 \tag{8}$$

By standard result of semi-riemannian geometry ([5], [6]), if the Hoermander hypothesis is checked in all x, $(x,y)\to \mathrm{d}(x,y)$ is finite continuous.

Bismut in his seminal book [7] has introduced the notion of cut-locus associated to the sub-riemannian distance $\mathrm{d}(\cdot,\cdot)$. We will recall in the first part what is the cut-locus in sub-riemannian geometry.

Bismut in his seminal book [3] pointed out the relationship between the Malliavin Calculus, Wentzel-Freidlin estimates and short time asymptotics of heat-kernels. This relationship was fully performed by Léandre in [8,9]. In particular, by using probabilistic technics we proved:

Theorem 1. (*Léandre* [9]). *If* x *and* y *are not in the cut-locus of the sub-riemannian distance, we have when* $t\to 0$

$$p_t(x,y) \sim C(x,y)t^{-\mathrm{d}/2}\exp\left[-\mathrm{d}^2(x,y)/2t\right] \tag{9}$$

where $C(x,y) > 0$.

In the proof we used a mixture between large deviation estimates, the Malliavin Calculus and the Bismutian procedure. Several authors laters ([10,11]) have presented other probabilistic proofs of (9). See [12] in a special case. We refer to [13] for an analytic proof of this result.

Remark. The complement of the cut-locus is an open-subset of $\mathbb{R}^d \times \mathbb{R}^d$: estimate (9) is uniform on any compact set of the complement of the cut-locus.

For readers interested by short time asymptotics of heat-kernels by using probabilistic methods, we refer to the review papers [14-16] and to the book of Baudoin [17]. We refer to the books of Davies [18] and of Varopoulos-Coulhon-Saloff-Coste [19] for analytical methods and to the review of Jerison-Sanchez [20] and Kupka [6].

The object of this paper is to translate in semi-group theory the proof of Theorem 1 of Takanobu-Watanabe [11], by using the tools of stochastic analysis for estimate of heat kernels we have translated in semi-group theory in [21,22] and [23] for Varadhan type estimates.

2. The Cut Locus Associated to a Sub-Riemannian Distance

The material of this part is taken on [7]. But we refer to [11] for a nice introduction to it.

We consider the map $h \to x_1(h)$ starting from x. This map is a Frechet smooth function from $\mathbb{H}$ into $\mathbb{R}^d$. We consider $U_t = D_x x_t(h)$. It satisfied the linear equation starting from I :

$$\mathrm{d}U_t = \sum_i D_x X_i(x_t(h))U_t \mathrm{d}h_t^i \tag{10}$$

We get

$$Dx_1(h)\cdot k = \sum_i U_1 \int_0^1 U_s^{-1} X_i(x_s(h))\mathrm{d}k_s^i \tag{11}$$

The Gram matrix associate to the map $h \to x_1(h)$ is

$$\begin{aligned}&\int_0^1 \langle U_1 U_s^{-1} X_i(x_s(h))\cdot,\rangle^2 \mathrm{d}s\\ &= \langle Dx_1(h), Dx_1(h)\rangle\end{aligned} \tag{12}$$

Bismut introduced the question to know if $h \to x_1(h)$ is a submersion. It is fullfilled if and only if the Gram matrix $\langle Dx_1(h), Dx_1(h)\rangle$ is invertible.

By standard result on Carnot-Caratheodory distance $\mathrm{d}^2(x,y) = \|h\|^2$ for some $h \in \mathbb{H}$ such that $x_0(h) = x, x_1(h) = y$.

Let be $K_{x,y}$ the set of *h* such that $x_0(h) = x, x_1(h) = y$. The main remark of Bismut [7] is the following: if $h \in K_{x,y}$ and $\langle Dx_1(h), Dx_1(h)\rangle$ is invertible, then $K_{x,y}$ is in a neighborhood of *h* a submanifold of $\mathbb{H}$ by using the implicit function theorem.

We recall the following definition:

Definition 2. (*Bismut* [7]) *We say that* (x,y) *are not in the cut-locus of the cut-locus of the sub-riemannian distance* $\mathrm{d}(\cdot,\cdot)$ *if the following* 3 *conditions are checked:*

1) $\mathrm{d}^2(x,y), = \|h_{x,y}\|^2$ for only one element of $K_{x,y}$.

2) The Gram matrix $\langle Dx_1(h_{x,y}), Dx_1(h_{x,y})\rangle$ is invertible.

3) $d^2(x,y)$ is a non-degenerated minimum of the energy function $h \to \|h\|^2$ on $K_{x,y}$.

Condition 3) has a meaning because $K_{x,y}$ is a manifold on a neighborhood of $h_{x,y}$.

As traditional in sub-riemannian geometry, we consider the Hamiltonian $H(x,p)$. It is the function from $\mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}^+$

$$(x,p) \to 1/2 \sum \langle X_i(x), p\rangle^2 \tag{13}$$

When there is an Hamiltonian, people introduced classically the Hamilton-Jacobi equation associated. In sub-riemannian geometry, this was introduced by Gaveau [24]. A bicharacteristic is the solution of the ordinary differential equation on $\mathbb{R}^d \times \mathbb{R}^d$ starting from (x,p):

$$\begin{aligned}x_t'(p) &= D_p H(x_t(p), p_t(p))\\ p_t'(p) &= -D_x H(x_t(p), p_t(p))\end{aligned} \tag{14}$$

We put

$$h_t^i(p) = \langle X_i(x_t(p)), p_t(p)\rangle \tag{15}$$

We recall some classical result on sub-riemannian geometry (See [11], p. 204):

$$\begin{aligned}H(x_t(p), p_t(p)) &= H(x,p)\\ &= 1/2\|h(p)\|^2\end{aligned} \tag{16}$$

$$p_t(p) = ({}^t U_t(h(p)))^{-1} p \tag{17}$$

$$h(p) = Dx_1(h(p))^* p_1(p) \tag{18}$$

Let us recall one of the main result of [7]. If (x,y) does not belong to the cut locus of $d(\cdot,\cdot)$, then $x_t(h_{x,y}) = x_t(p_{x,y})$ for a convenient bicharectiristic.

By using result of [11] pp. 206-207, we can compute the Hessian of the energy in $h_{x,y}$ in $K_{x,y}$. It is equal to

$$\begin{aligned}I''(k,l) &= \langle k,l\rangle_{\mathbb{H}}\\ &\quad -\langle p_1(p_{x,y}), D^2 x_1(h_{x,y})k,l\rangle\end{aligned} \tag{19}$$

We can compute $D^2 x_1(h)(k,l)$. It is given by

$$
\begin{aligned}
&D^2x_1(j)(k,l)\\
&=U_1(h)\Big\{\sum_i\int_0^1 U_s(h)^{-1}D_x^2X_i(x_s(h))Dx_s(h)(k)\\
&\qquad\otimes Dx_s(h)(l)\,\mathrm{d}h_s^i\\
&\quad+\sum_I\int_0^1 U_s(h)^{-1}D_xX_i(x_s(h))\\
&\qquad\cdot\big(Dx_s(h)(k)\,\mathrm{d}l_s^i+Dx_s(h)(l)\,\mathrm{d}k_s^i\big)\Big\}\\
&=A_1(k,l)+A_2(k,l)
\end{aligned}
\tag{20}
$$

3. Scheme of the Proof of Theorem 1

We translate in semi-group the proof of [9] in the way presented in [12].

See [22] for similar considerations for logarithmic estimates of the heat-kernel.

We consider $t=\epsilon^2$ classically and introduce the operator

$$L_\epsilon=1/2\,\epsilon^2\sum X_i^2 \tag{21}$$

Classically

$$\exp[L_\epsilon]=\exp[tL] \tag{22}$$

We consider the unique curve of minimum enegy $h_{x,y}$ sucht $x_1(h_{x,y})=y$ and we introduce the operator

$$L_\epsilon(h_{x,y})=L_\epsilon+\sum \mathrm{d}/\mathrm{d}sh^i_{x,y,s}X_i \tag{23}$$

This generates a time inhomogeneous semi-group. According the Girsanov formula in semi-group theory of Léandre [4], we introduce the vector field on $\mathbb{R}^{\mathrm{d}}\times\mathbb{R}$:

$$\tilde{X}_i(\epsilon)=\left(\epsilon X_i,-1/\epsilon\,\mathrm{d}/\mathrm{d}sh^i_{x,y,s}u\right) \tag{24}$$

and the generator written in Itô form

$$
\begin{aligned}
\tilde{L}_\epsilon(h_{x,y})\tilde{f}&=\sum_i \mathrm{d}/\mathrm{d}sh^i_{x,y,s}\left\langle X_i,\tilde{D}\tilde{f}\right\rangle\\
&\quad+1/2\,\epsilon^2\sum_{i>0}\left\langle DX_iX_i,\tilde{D}\tilde{f}\right\rangle\\
&\quad+1/2\sum_i\left\langle \tilde{X}(\epsilon),\tilde{D}^2\tilde{f},\tilde{X}_i(\epsilon)\right\rangle
\end{aligned}
\tag{25}
$$

According [21], p. 207, we have:

$$\exp[L_\varepsilon][f](x)=\exp\left[\tilde{L}_\in(h_{x,y})\right][uf](x,1) \tag{26}$$

We consider the generator

$$\overline{L}(h_{x,y})=\sum \mathrm{d}/\mathrm{d}sh^i_{x,y,s}X_i+1/2\sum\tilde{X}_i^2(\epsilon) \tag{27}$$

It differs from $\tilde{L}_\epsilon(h_{x,y})$ by $-1/2\sum\left|h^i_{x,y,s}\right|^2uD_u$. This last vector field commute with $\tilde{L}_\epsilon(h_{x,y})$. We deduce that

$$
\begin{aligned}
&\exp\left[\tilde{L}_\epsilon(h_{x,y})\right][uf](x,1)\\
&=\exp\left[-\mathrm{d}^2(x,y)/2t\right]\exp\left[\overline{L}_\epsilon(h_{x,y})\right][uf](x,1)
\end{aligned}
\tag{28}
$$

We consider the vector fields

$$\overline{Y}_i(\epsilon)=\left(\epsilon X_i,-\mathrm{d}/\mathrm{d}sh_t^i\right) \tag{29}$$

and the generator

$$\overline{Q}_\epsilon(h_{x,y})=\sum \mathrm{d}/\mathrm{d}sh^i_{x,y,s}X_i+1/2\sum\overline{Y}_i^2(\epsilon) \tag{30}$$

We have clearly that

$$
\begin{aligned}
&\exp\left[\overline{L}_\epsilon(h_{x,y})\right][uf](x,1)\\
&=\exp\left[\overline{Q}_\epsilon(h_{x,y})\right]\left[\exp[u/\epsilon]f\right](x,0)
\end{aligned}
\tag{31}
$$

Let us consider the flow Φ_s associated to the ordinary differential Equation (7) $x_s(h_{x,y})$. Let us introduce the vector fields

$$Y_i(\epsilon)=\left(\epsilon\Phi_s^{*-1}X_i,-\mathrm{d}/\mathrm{d}sh^i_{x,y,s}\right) \tag{32}$$

and the time-dependent generator

$$Q_\epsilon(h_{x,y})=1/2\sum Y_i^2(\epsilon) \tag{33}$$

We have the main formula

$$
\begin{aligned}
&\exp\left[\overline{Q}_\epsilon(h_{x,y})\right]\left[\exp[u/\epsilon]f\right](x,0)\\
&=\exp\left[Q_\epsilon(h_{x,y})\right]\left[\exp[u/\epsilon]f_1\right](x,0)
\end{aligned}
\tag{34}
$$

where f_1 is the map which to z associate $f(\Phi_1(z))$. Since $\Phi_1(x)=y$, we have only to estimate the density in x of the measure which to f associates

$$\exp\left[Q_\epsilon(h_{x,y})\right]\left[\exp[u/\epsilon]f\right](x,0) \tag{35}$$

We can suppose without any restriction that $x=0$.

We perform the dilation $y\to y/\epsilon$.

This means that we have to consider the vector fields

$$Z_i(\epsilon)=\left(\Phi_s^{*-1}X_i(\epsilon\,\cdot),-\mathrm{d}/\mathrm{d}sh^i_{x,y,s}\right) \tag{36}$$

and the generator

$$R_\epsilon(h_{x,y})=1/2\sum Z_i^2(\epsilon) \tag{37}$$

We consider the density $r_\varepsilon(\cdot)$ ot the measure which to the test function f associates

$$\exp\left[R_\epsilon(h_{x,y})\right]\left[\exp[u/\epsilon]f\right](0,0) \tag{38}$$

The main result of [21] is the following: for some $C(x,y)>0$

$$\left\{C(x,y)\exp\left[-\mathrm{d}^2(x,y)/2t\right]/t^{\mathrm{d}/2}\right\}r_\epsilon(0)=p_t(x,y) \tag{39}$$

The main difference with [21] is in treatment of the term $\exp[u/\epsilon]$. We refer to [9,10,12] for the treatment of that expression by using stochastic analysis.

In Part 2, $Dx_s(h_{x,y})\cdot k$ and $D^2x_s(h_{x,y})k\cdot k$ satisfy a system of stochastic differential equations in cascade with associated vector fields $Y_i(1), Y_i(2)$. We denote (x', u_1, u_2) the generic element of $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$. We consider the vector fields

$$\bar{Z}_i(\epsilon) = \left(\Phi_s^{*-1}X_i(\epsilon\cdot), Y_i(1), Y_i(2)\right) \quad (40)$$

and the generator

$$\bar{R}(h_{x,y}) = 1/2\sum \bar{Z}_i^2(\epsilon) \quad (41)$$

From (14), (15), (18), the density $r_\epsilon(0)$ is equal to the density $\bar{r}(0)$ in 0 of the measure which to f associates

$$\exp\left[\bar{R}(h_{x,y})\right] \cdot\left[\exp\left[\left\langle \frac{\Phi_1(\epsilon x') - y - \epsilon u_1 - \epsilon^2\frac{1}{2}u_2}{\epsilon^2}, p_1(x,y)\right\rangle\right] \cdot\exp\left[\left\langle\frac{1}{2}u_2, p_1(x,y)\right\rangle\right] f\right](0,0,0) \quad (42)$$

where $p_s(x,y)$ is associated to $h_s(x,y)$ by the procedure of the Part 2. Theorem 1 will follow from Theorem 6.

We consider (x', u_1, u_2, v) the generic element of $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}$ and

$$\bar{Z}_0(\epsilon) = \left(0,0,0,\epsilon^2(x')^2\right) \quad (43)$$

and the generator

$$\tilde{R}_\epsilon(h_{x,y}) = 1/2\sum_{i>0}\bar{Z}_i^2(\epsilon) + \bar{Z}_0(\epsilon) \quad (44)$$

The following lemma is proved in the appendix and was originally proved by stochastic analysis in [12].

Lemma 3. *For any positive* p, *there exists a* ρ *such that*

$$\exp\left[\tilde{R}(h_{x,y})\right] \cdot\left[\left(\exp\left[\left\langle \frac{\Phi_1(\epsilon x') - y - \epsilon u_1 - \epsilon^2\frac{1}{2}u_2}{\epsilon^2}, p_1(x,y)\right\rangle\right] - 1\right)^p \cdot 1_{|\epsilon x'|\le\rho}1_{|v|\le\rho}\right](0,0,0,0,0) \to 0 \quad (45)$$

when $\epsilon\to 0$

The next lemma is due to Bismut [7] and is proved without using stochastic analysis in the appendix:

Lemma 4. *Let* $\rho > 0$ *be very small. There exists a* $p > 1$ *such that*

$$\exp\left[\tilde{R}_\epsilon(h_{x,y})\right] \left[\left[\exp p\langle u_2, p_1(x,y)\rangle/2\right]1_{|x'|\le\rho}1_{|v|\le\rho}\right](0,0,0,0) < \infty \quad (46)$$

The remaining part of the scheme of the proof is to apply the Malliavin Calculus of Bismut type depending of a parameter of [21], Part 3 to the the semi-group $\exp\left[\tilde{R}_\epsilon(h_{x,y})\right]$. We will apply an improvement of Theorem 1 of [21]. We consider $(x', u_1, u_2, v, U, V) \in \mathbb{R}_d = \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{G}_d\times\mathbb{M}_d$ where $\mathbb{G}_d$ is the set on invertible matrices on $\mathbb{R}^d$ and $\mathbb{M}_d$ the set of symmetric matrices on $\mathbb{R}^d$ (V is called the Malliavin matrix). We consider if $i > 0$ the vector fields on $\mathbb{E}_d$

$$V_i(\epsilon) = \left(\bar{Z}_i(\epsilon), 0, \epsilon\Phi_s^{*-1}X_i(\epsilon\cdot)U, 0\right) \quad (47)$$

and

$$V_0(\epsilon) = \left(0,0,0,\sum_{i>0}\left\langle U^{-1}\Phi_s^{*-1}X_i(\epsilon\cdot),\cdot\right\rangle^2\right) \quad (48)$$

Let be the generator

$$L_{tot,\epsilon}(h_{x,y}) = 1/2\sum_{i>0}V_i^2(\epsilon) + \bar{Z}_0(\epsilon) + V_0(\epsilon) \quad (49)$$

It generates a time inhomogeneous semi-group. We have

Lemma 5. *For all positive* p, *the uniform Malliavin condition is checked*:

$$\sup_{\epsilon<1}\exp\left[L_{tot,\epsilon}(h_{x,y})\right]\left[V^{-p}\right](0,0,0,I,0) < \infty \quad (50)$$

Theorem 1 is a consequence of the next theorem, (which is an extension of Theorem 1 of [21]) and of (39):

Theorem 6. *When* $\epsilon\to 0$, $r_\epsilon(0)\to r_0(0)$ *where* $r_0(\cdot)$ *is the density of the measure which to f associates*

$$\exp\left[\tilde{R}_0(h_{x,y})\right]\left[\exp\left[\langle p_1(x,y), u_2\rangle\right] f\right](0,0,0,0) \quad (51)$$

First of all, we recall the Wentzel-Freidlin estimates translated in semi-group theory by Léandre [22,23,25]:

Theorem 7. (*Wentzel-Freidlin*) *Let* Y_i *some time dependent vector fields with bounded derivatives at each order on* $\mathbb{R}^{d_1}$, $i = 0,\cdots,m$. *We consider the control distance* $\mathrm{d}^Y(x_1, y_1)$ *as in* (8) *and the diffusion semi-group* $\exp\left[1/2\in^2\sum_{i>0}Y_i^2 + \in^2 Y_0\right]$. *We suppose that the control distance is continuous. Then for any open subset* O

$$\overline{\lim}_{\epsilon\to 0} 2\epsilon^2\mathrm{Log}\left(\exp\left[1/2\epsilon^2\sum Y_i^2 + \in Y_0\right]\left[1_O\right](x_1)\right) \le -\inf_{y_1'\in O}\mathrm{d}_Y^2(x_1, y_1') \quad (52)$$

Proof of Theorem 6. Let χ be a smooth function from $\mathbb{R}$ into $[0,1]$ equals to 1 and 0 and equals to 0 if $|v| > \rho$. By Wentzel-Freidlin estimates, we can find an $\eta > 0$ such that if $p > 1$.

$$\exp\left[\tilde{R}_{x,y}\right]\left[\exp\left[p< p_1(x,y),\Phi_1(\epsilon x')-y-\epsilon u_1>/\epsilon^2\right]\chi(x')(1-\chi)(v)\right](0,0,0,0)\le\exp\left[-\eta\epsilon^{-2}\right] \quad (53)$$

By the integration by part of the Malliavin Calculus and the Technical Lemma 5, we have if α is a multi-index

$$\left|\exp\left[\tilde{R}_{x,y}\right]\left[\exp\left[\left\langle p_1(x,y),\Phi_1(\epsilon x')-y-\epsilon u_1\right\rangle/\epsilon^2\right]\chi(x')(1-\chi)(v)D^{(\alpha)}f\right](0,0,0,0)\right|\le\|f\|_\infty\exp\left[-\eta/\epsilon^2\right] \quad (54)$$

Therefore we have only to estimate the density in 0 of the measure which to f associate

$$\exp\left[\tilde{R}_\epsilon(h_{x,y})\right]\left[\exp\left[\left\langle p_1(x,y),\Phi_1(\epsilon x')-y-\epsilon u_1-\epsilon^2/2u_2\right\rangle/\epsilon^2\right]\exp\left[\left\langle p_1(x,y),u_2\right\rangle/2\right]\chi(x')\chi(v)f\right][0,0,0,0] \quad (55)$$

By using Lemma 3, Lemma 4, Lemma 5 the density of this measure tends to $r_0(0)$ by using the Malliavin Calculus of Bismut type which depends of a parameter of [21]. □

4. Proof of the Technical Lemmas

Proof of Lemma 3. Let us first show that

$$\exp\left[\tilde{R}_\epsilon(h_{x,y})\right]\left[\left|\Phi_1(\epsilon x')-y-\epsilon u_1-\epsilon^2/2u_2\right|^2/\epsilon^4\right]\to 0 \quad (56)$$

(We will omitt to write later the obvious initial condition which appear in various semi-group later). We introduce a polynomial F of degre less or equal to 2 in $\Phi_1(\epsilon x')$ and in u_1,u_2. Let us compute the Taylor expansion of $\exp\left[\tilde{R}_\epsilon(h_{x,y})\right][F]$. We use Lemma 1 of [21]. If the degree of F in $\Phi_1(\epsilon x')$ is 2, the two first terms of the Taylor expansion are 0 and the term of order 2 is

$$\exp\left[\tilde{R}_0(h_{x,y})\right]\left[D^2F(0,u_1,u_2)u_1\cdot u_1\right] \quad (57)$$

where we take partial derivatives in the first component. If the polynomial F is of degree 1 in $\Phi_1(\epsilon x')$, the term of order 1 is

$$\exp\left[\tilde{R}_0(h_{x,y})\right]\left[DF(0,u_1,u_2)u_1\right] \quad (58)$$

and the term of order two is

$$\exp\left[\tilde{R}_0(h_{x,y})\right]\left[DF(0,u_1,u_2)u_2\right] \quad (59)$$

Lemma 3 will arise from the translation in semi-group theory of Lemma 3.4 of [12].

For all p there exists a ρ such that

$$\sup_{\epsilon<1}\exp\left[\tilde{R}_\epsilon(h_{x,y})\right]\left[\exp\left[p\left\langle\Phi_1(\epsilon x')-\epsilon u_1-\epsilon^2/2u_2, p_1(x,y)\right\rangle\right];1_{|v|\le\rho}\right]<\infty \quad (60)$$

The proof follows slightly the line of Lemma 3.4 of [12]. We don't write the convenient enlarged semi-groups when we enlarge the space. We follow the notation of [12], η being replaced by ϵ and V_α being replaced by X_i.

We introduce the new coordinate

$$\eta_s^\epsilon=1/\epsilon\left(\Phi_s(\epsilon x')-x_s(h_{x,y})\right) \quad (61)$$

We use the Itô formula in semi-group theory of [25]. This leads to introduce extra coordinates in the vector fields:

1) $X_i\left(\Phi_s(\epsilon x')\right)$.

2) $\partial X_i(s)\eta_s^\epsilon\,\mathrm{d}/\mathrm{d}s h_{x,y,s}^i$

$$=\int_0^1 D\left(x_s(h_{x,y}+u\left(\Phi_s(\epsilon x')-x_s(h_{x,y})\right)\mathrm{d}u\eta_s^\epsilon\mathrm{d}/\mathrm{d}s h_{x,y,s}^i\right.$$

We introduce the new variable Ξ_s^ε which is associated to the extra component vector fields

3) $\sum\partial X_i(s)\Xi_s^\epsilon\,\mathrm{d}/\mathrm{d}s h_{x,y,s}^i$.

We use another time the Itô formula in semi-group theory of [25] (11). This leads to introduce the vector field associated to another variable $\eta_s^{1,\epsilon}$.

4) $\left(\Xi_s^\epsilon\right)^{-1}X_i\left(\Phi_s(\epsilon x')\right)$

We introduce an extra variable $\eta_s^{3,\epsilon}$ associated to another component in the drift which is $\left(\eta_s^\epsilon\right)^2$.

We get for another enlarged semi-group $\exp\left[\hat{R}_\epsilon^1(h_{x,y})\right]$ an extension of formula 3.44 of [12], but with $\int_0^1\left|\eta_s^\epsilon\right|^2\mathrm{d}s$ instead of $\sup\left|\eta_s^\epsilon\right|^2$.

Lemma 8. *For all* ρ, *there exists* $p_0>0$ *such that*

$$\sup_{\epsilon<1}\exp\left[\hat{R}_\epsilon^1(h_{x,y})\right]\left[\exp\left[p_0\eta_1^{3,\epsilon}\right]1_{|v|\le\rho}\right]<\infty \quad (62)$$

We postpone later the proof of this lemma which is an analog of the quasi-continuity lemma of [25].

Next we consider another enlarged semi-group to look the couple η_s^ϵ and η_s together. We use the Itô formula in semi-group theory of [25] (11), [22,23]. We introduce

1) $\partial^2X_i^\epsilon(s)$

$$=2\int_0^1\mathrm{d}u\int_0^u D^2X_i\left(x_s(h_{x,y})+v\left(\Phi_s(\epsilon x')-x_s(h_{x,y})\right)\right)\mathrm{d}v.$$

By introducing a cascade of vector fields, we can translate in semi-group theory (3.45) of [12]. We introduce a variable $\eta_s^{4,\varepsilon}$ associate to the new component in the drift $\left|\eta_s^\epsilon - \eta_s\right|^2$ and we can state an analog of Lemma 8 for a convenient enlarged semi-group $\exp\left[\hat{R}_\epsilon^2\left(h_{x,y}\right)\right]$.

For every p, there exists a small ρ such that

$$\sup_{\epsilon\le 1}\exp\left[\hat{R}_\epsilon^2\left(h_{x,y}\right)\right]\left[\exp\left[p\eta_1^{4,\epsilon}\right]1_{|v|\le\rho}\right]<\infty \qquad (63)$$

which is the analog of (3.46) in [12] where we have replaced $\sup_{s\le 1}\left|\eta_s^\epsilon-\eta_s\right|^2$ by $\int_0^1\left|\eta_s^\epsilon-\eta_s\right|^2\mathrm{d}s$.

Let be $\theta_s^\epsilon=\eta_s^\epsilon-\eta_s\epsilon$ and θ associated to the extra-component vector fields:

1) $\partial V_i\left(x_s\left(h_{x,y}\right)\right)\eta_s$ for the diffusion part.

2) $\sum\partial X_i\left(x_s\left(h_{x,y}\right)\eta_s\mathrm{d}/\mathrm{d}s h_{s,x,y}^i\right.$

$+1/2\sum\partial^2X_i\left(x_s\left(h_{x,y}\eta_s\right)\otimes\eta_s\right.$ for the drift part.

We use another time the Itô formula in semi-group theory of [25] (11) for a convenient enlarged semi-group established to study together θ_s^ϵ and θ_s. This allow to study $\theta_s^\epsilon-\theta$ and we conclude exactly as in pages 29, 30 of [12] with a small improvement of Lemma 8 to study (3.46), (3.47) of [12]. □

Proof of Lemma 4. We assemble the semi-group $\exp\left[\tilde{R}_\epsilon\left(h_{x,y}\right)\right]$ and the semi-group $\exp\left[\tilde{R}_0\left(h_{x,y}\right)\right]$ together in a total semi-group $\exp\left[\tilde{R}_\epsilon^{tot}\left(h_{x,y}\right)\right]$. We have some variables x'_ϵ,u_1,u_2 and v. We have

$$D\Phi(0)\cdot x'_0=u_1 \qquad (64)$$

Let ρ_1 be small and ρ be very small. We use the exponential inequality in semi-group theory of Lemma 8. For a small ρ and a small ρ_1, we have (we omitt to write the obvious initial values in the considered semi-groups)

$$\begin{aligned}&\exp\left[\tilde{R}_\epsilon^{tot}h_{x,y}\right]\left[\left\langle u_2,p_1(x,y)\right\rangle>\eta^{-2};\left|x'_\epsilon\right|<\rho;|v|<\rho\right]\\&\le\exp\left[\tilde{R}_\epsilon^{tot}h_{x,y}\right]\\&\left[\left\langle u_2,p_1(x,y)\right\rangle>\eta^{-2};C\eta\left|D\Phi(0)x'_\epsilon-u_1\right|>\rho_1;|v|<\rho\right]\\&+\exp\left[\tilde{R}_\epsilon^{tot}h_{x,y}\right]\left[\left\langle u_2,p_1(x,y)\right\rangle>\eta^{-2};\eta\left|u_1\right|<\rho_1\right]\\&=A_1+A_2\end{aligned} \qquad (65)$$

We choose a small ρ_1 and a very small ρ. The exponential inequalities of the proof of Lemma 8 show

$$A_1\le\exp\left[-C\eta^{-2}\right] \qquad (66)$$

It remains to estimate A_2. We scale the vector fields $Y_1(1)$ by $\eta Y_i(1)$ and $Y_i(2)$ by $\eta Y_i(2)$. We get a generator $\bar{R}_\eta\left(h_{x,y}\right)$ and a new Markov semi-group $\exp\left[u\bar{R}_\eta\left(h_{x,y}\right)\right]$. By a scaling argument, we recognize in A_1

$$\exp\left[\bar{R}_\eta\left(h_{x,y}\right)\right]\left[\left\langle u_2,p_1(x,y)\right\rangle>1,\left|u_1\right|\le\rho_1\right](0,0,0) \qquad (67)$$

By a simple improvement of the large deviation estimates of Theorem 7, we get

$$\begin{aligned}&\overline{\lim}_{\eta\to 0}\operatorname{Log}\exp\left[\bar{R}_\eta\left(h_{x,y}\right)\right]\\&\cdot\left[\left\langle u_2,p_1(x,y)\right\rangle>1,\left|u_1\right|\le\rho_1\right](0,0,0)\\&=-\inf_{\left|\left\langle Dx_1\left(h_{x,y}\right),k\right\rangle\right|\le\rho_1;\left\langle p_1(x,y),D^2x_1\left(h_{x,y}\right)\cdot k\cdot k\right\rangle>1}\|k\|^2\end{aligned} \qquad (68)$$

We chose a small ρ_1 and we use (20) and the fact (x,y) don't belong to the cut-locus in part 2. We deduce that if ρ is very small, that there exists a $C>1$ such that

$$\begin{aligned}&\exp\left[\tilde{R}_\epsilon\left(h_{x,y}\right)\right]\left[\left\langle u_2,p_1(x,y)\right\rangle/2>\eta^{-2},|x'|<\rho;|v|<\rho\right]\\&\le\exp\left[-C\eta^{-2}\right]\end{aligned} \qquad (69)$$

□

Remark. This result is traditionnally hold by using the theory of Fredholm determinant.

Proof of Lemma 5. We assemble together the semi-group $L_{tot,\epsilon}\left(h_{x,y}\right)$ and $L_{tot,0}\left(h_{x,y}\right)$ in a global generator $\bar{L}_{tot,\epsilon}\left(h_{x,y}\right)$ We get therefore a total semi-group $\exp\left[u\bar{L}_{tot,\epsilon}\left(h_{x,y}\right)\right]$. We get the Malliavin matrix V_ε and V_0. But V_0 is nothing else that $\left\langle U_1^{-1}Dx_1\left(h_{x,y}\right),U_1^{-1}Dx_1\left(h_{x,y}\right)\right\rangle$ which is invertible because (x,y) don't belong to the cut-locus of the sub-riemannian geometry.

Moreover, by omitting to write the obvious starting conditions, we get for a small η:

$$\exp\left[\bar{L}_{tot,\epsilon}\left(h_{x,y}\right)\right]\left[\left|V_\epsilon V_0^{-1}\right|>\eta\right]\le C\epsilon^p \qquad (70)$$

for all p. Therefore for a small η:

$$\begin{aligned}&\exp\left[\bar{L}_{tot,\epsilon}\left(h_{x,y}\right)\right]\left[V_\epsilon^{-p}\right]\le A_1+A_2\\&=\exp\left[\bar{L}_{tot,\epsilon}\left(h_{x,y}\right)\right]\left[V_\epsilon^{-p};1_{\left|V_\epsilon V_0^{-1}\right|>\eta}\right]\\&+\exp\left[\bar{L}_{tot,\epsilon}\left(h_{x,y}\right)\right]\left[\left(V_\epsilon-V_0+V_0\right)^{-p};1_{\left|V_\epsilon V_0^{-1}\right|>\eta}\right]\end{aligned} \qquad (71)$$

Since V_0 is constant invertible, A_2 is bounded independent of p if η is small enough. By the results of [22,23], there exist $n(p)$ such that:

$$\exp\left[\bar{L}_{tot,\epsilon}\left(h_{x,y}\right)\right]\left[V_\epsilon^{-p}\right]\le C\epsilon^{-n(p)} \qquad (72)$$

By Hoelder inequality, we deduce that A_1 is bounded independent of p. □

Proof of Lemma 8. This follows clearly the line of the quasi-continuity lemma for Wentzel-Freidlin estimates in semi-group theory of [25]. We sketch the proof.

We recall the elementary Kolmogorov lemma of the theory of stochastic processes ([26,27]).

Let $s \to X_s$ be a family of random variables parametrized by $s \in [0,1]$ with values in $\mathbb{R}^d$ equals to 0 or 1 in $s = 0$ such that

$$E\left[\left|X_{s'} - X_s\right|^p\right] \le C(p)(s'-s)^{\alpha p} \quad (73)$$

for $s' > s$. There exists a continuous version of $s \to X_s$ and the L^p norm of $(X_1)^* = \sup_{s\le 1}|X_s|$ can be estimated only in terms of the data (73).

Let us recall that $R_\eta^1(h_{x,y})$ is a time dependent generator. For $s' > s$ there is a time inhomogeneous semi-group $\exp\left[\left(R_\epsilon^1\right)_s^{s'}(h_{x,y})\right]$. By the Burkholder-Davies-Gundy inequality in semi-group theory of [16] (19), we have

$$\exp\left[\left(R_\epsilon^1\right)_0^s\right]\left[\exp\left[\left(R_\epsilon^1\right)_s^{s'}\right]\left[\left|\eta_s^{1,\epsilon} - \eta_{s'}^{1,\epsilon}\right|^p\right]\right] \le C(p)t^{\alpha p} \quad (74)$$

There we can define a continuous stochastic process with probability measure $\mathrm{d}P$ associated to $\eta_s^{1,\epsilon}$.

We use the Paul Levy martingale exponential in semi-group theory of [25] (33), (46). We get

$$\begin{aligned} &E_P\left[\left|\exp\left[\left\langle A, \eta_{s'}^{1,\epsilon}\right\rangle\right] - \exp\left[\left\langle A, \eta_{s'}^{1,\epsilon}\right\rangle\right]\right|^p\right] \\ &\le C(p)(s'-s)^{\alpha p} \exp\left[C|A|^2\right] \end{aligned} \quad (75)$$

By the Kolmogorov lemma, we get

$$E_P\left[\left(\exp\left\langle A, \eta_1^{1,\epsilon}\right\rangle\right)^*\right] \le C \exp\left[C|A|^2\right] \quad (76)$$

By standard computations, we deduce that

$$P\left[\left(\eta_1^{1,\epsilon}\right)^* > C\right] \le K' \exp\left[-KC^2\right] \quad (77)$$

But $\left(\Xi_s^\epsilon\right)$ is bounded, and by the same type of argument we deduce that

$$P\left[\left(\eta_1^{\epsilon}\right)^* > C\right] \le K' \exp\left[-KC^2\right] \quad (78)$$

But

$$\eta_1^{3,\epsilon} = \int_0^1 \left|\eta_s^\epsilon\right|^2 \mathrm{d}s \quad (79)$$

such that

$$\sup_{\epsilon \le 1} \exp\left[\hat{R}_\epsilon^1(h_{x,y})\right]\left[\eta_1^{3,\epsilon} > C\right] \le K' \exp\left[-KC^2\right] \quad (80)$$

□

5. Conclusion

We have translated in semi-group theory some classical result of stochastic analysis for subelliptic heat-kernels where Bismutian non degeneracy condition [7] plays a preeminent role.

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