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General Markowitz Optimization Problems

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ABSTRACT

We solve two Markowitz optimization problems for the one-step financial model with a finite number of assets. In our results, the classical (inefficient) constraints are replaced by coherent measures of risk that are continuous from below. The methodology of proof requires optimization techniques based on functional analysis methods. We solve explicitly both problems in the important case of Tail Value at Risk.

Keywords: Markowitz Optimization Problems; Coherent Risk Measure; Tail Value at Risk

1. Introduction

We consider optimal investment for the one-step financial model with a finite number of assets. The classical Markowitz optimization problems are looking for portfolios that either maximize the expected return for a given variance threshold, or minimize the variance for a given expected return. However, using variance as a measure of risk has a serious drawback: high profits are penalized in the same way as high losses. Instead, in what follows we shall use coherent measures of risk (cf. [1]), that provide a much better quantification of risk.

In our set-up, the space of financial positions is a vector space with vector ordering $(E,+,\cdot,\leq)$. Besides the origin 0 in *E*, we distinguish a (strictly positive) reference cash stream denoted by 1. In the space of linear price systems E', *i.e.*, the algebraic dual of E, we fix a total subspace $E^{\times}$ (*i.e.*, if $f(x)=0$ for all $f\in E^{\times}$, then $x=0$) and consider the weak*-topology on $E^{\times}$ associated to the dual pair $(E,E^{\times})$.

A coherent measure of risk (see [2]) is a real valued mapping ρ defined on E which is subadditive: $\rho(x+y)\leq\rho(x)+\rho(y)$ for $x,y\in E$, positive homogeneous: $\rho(tx)=t\rho(x)$ for $x\in E, t\geq 0$, monotonic: $\rho(x)\leq 0$ if $x\geq 0$, and translation invariant: $\rho(x+t\times 1)=\rho(x)-t$ for $x\in E$ and any real t.

The following property will be needed in our results. A coherent measure of risk ρ is called continuous from below (cf. [1,3]) if $\rho(x_n)\searrow -1$ for any sequence $(x_n)_{n\geq 1}$ in E satisfying $0\leq x_n\nearrow 1$. Note that, if E has a strong order unit, continuity from below is equivalent to the more familiar condition: $\rho(x_n)\searrow\rho(x)$ provided $0\leq x_n\nearrow x$ (see [3,4]).

2. Main Results

Our first result formulates and solves in our set-up the first Markowitz problem, *i.e.*, the so-called "agent-independent optimization problem": find portfolios that maximize the expected return for a given (measure of) risk. Particular cases have been considered in [5-8].

Theorem 1. *Let* E *be an ordered locally convex vector space, and* $E^{\times}$ *a total Banach subspace of* E'. *Let* $x_1,\cdots,x_n\in E, c>0$ *and* $g\in E^{\times}$ *be fixed; if* ρ *is a coherent measure of risk continuous from below, then the following optimization problem*:

$$\begin{cases}\max t_1 g(-x_1)+\cdots+t_n g(-x_n) \text{ with } t_1,\cdots,t_n\in\mathbb{R}\\ \text{and subject to } 0\leq\rho(t_1x_1+\cdots+t_nx_n)\leq c,\end{cases}\tag{1}$$

admits optimal solutions.

Proof. According to the structure theorem for coherent measures of risk (see e.g. [3], Theorem 2.1), ρ admits the following representation

$$\rho(x)=\sup_{f\in\mathcal{A}} f(-x)\tag{2}$$

for some weak*-closed convex set $\mathcal{A}\subset E^{\times}$, in which all $f\in\mathcal{A}$ are positive (*i.e.*, $f(x)\geq 0$ for $0\leq x\in E$) and normalized (*i.e.*, $f(1)=1$). Note that continuity from below of ρ implies continuity in the order convergence topology of all $f\in\mathcal{A}$ in formula (2), see [3].

Let us define

$$G:=\overline{\{(f(-x_1),\cdots,f(-x_n)), f\in\mathcal{A}\}},$$

where the bar denotes closure. By the continuity from below of ρ and the Krein-Šmulian theorem (see e.g. [9]), the set $\mathcal{A}$ is weak*-compact, hence G is compact. Therefore, using (2), the definition of G, continuity from below of ρ and James' theorem (see [9]), we obtain for any $t_1,\cdots,t_n \in \mathbb{R}$:

$$\rho\left(t_1x_1+\cdots+t_nx_n\right)=\max_{(s_1,\cdots,s_n)\in G}\left(t_1s_1+\cdots+t_ns_n\right).$$

In particular the sup in (2) is achieved, and for any $t_1,\cdots,t_n \in \mathbb{R}$ one has

$$\begin{aligned}&\rho\left(t_1x_1+\cdots+t_nx_n\right)\le c \text{ iff}\\ &t_1s_1+\cdots+t_ns_n\le c \text{ for any } \left(s_1,\cdots,s_n\right)\in G,\end{aligned} \tag{3}$$

where the threshold $c>0$ is given in formula (1).

As $\rho\left(t_1x_1+\cdots+t_nx_n\right)\ge 0$ for $t_1,\cdots,t_n \in \mathbb{R}$, from (2) it follows that, for some $f_0 \in \mathcal{A}$, one has $f_0\left(t_1x_1+\cdots+t_nx_n\right)\ge 0$ for $t_1,\cdots,t_n \in \mathbb{R}$. Take $t_1=\pm 1, t_2=\cdots=t_n=0$ in the latter and, using linearity, obtain $f_0\left(\pm x_1\right)\ge 0$, *i.e.*, $f_0\left(x_1\right)=0$. Similarly obtain $f_0\left(x_2\right)=\cdots=f_0\left(x_n\right)=0$. This means $0 \in \operatorname{int} G$, hence the following is well defined:

$$\lambda^*=\inf\left\{\lambda>0:\lambda\left(g\left(-x_1\right),\cdots,g\left(-x_n\right)\right)\in G\right\}. \tag{4}$$

Then the max value in (1) equals c/λ^* and is achieved at every $\left(t_1,\cdots,t_n\right)\in \mathbb{R}^n/\{0\}$ satisfying:

$$\begin{aligned}&t_1s_1+\cdots+t_ns_n\ge \lambda^*\left(t_1g\left(-x_1\right)+\cdots+t_ng\left(-x_n\right)\right)\\ &\text{for any } \left(s_1,\cdots,s_n\right)\in G.\end{aligned} \tag{5}$$

Indeed, if $\lambda\left(g\left(-x_1\right),\cdots,g\left(-x_n\right)\right)\in G$ for some $\lambda>0$, take $\left(t_1,\cdots,t_n\right)\in \mathbb{R}^n/\{0\}$ satisfying

$$\begin{aligned}&t_1g\left(-x_1\right)+\cdots+t_ng\left(-x_n\right)\le\left(t_1s_1+\cdots+t_ns_n\right)/\lambda\\ &\text{for any } \left(s_1,\cdots,s_n\right)\in G.\end{aligned}$$

Condition (3) and definition (4) imply that $\max\, t_1g\left(-x_1\right)+\cdots+t_n\, g\left(-x_n\right)=c/\lambda^*$ and the max is achieved at every $\left(t_1,\cdots,t_n\right)\in \mathbb{R}^n/\{0\}$ satisfying condition (5). □

Problem (1) is for investing a sum of money in securities; it is possible that the investor already possesses a capital with terminal value y, in which case minimizing the risk leads to the second Markowitz optimization problem, or "single-agent optimization problem". Alternatively, one can seek the minimum price which allows us to sell a payment order, and then compile a hedging portfolio of assets such that the risk of the entire operation will be negative or zero. Our second result formulates and solves the second Markowitz problem in our set-up.

Theorem 2. *Let* E *be an ordered locally convex vector space, and* $E^\times$ *a total Banach subspace of* E'. *Let* $y,x_1,\cdots,x_n\in E$ *be fixed; if* ρ *is a coherent measure of risk continuous from below, then the following optimization problem*

$$\begin{aligned}&\min\rho\left(y+t_1x_1+\cdots+t_nx_n\right) \text{ with } t_1,\cdots,t_n\in\mathbb{R}\\ &\text{and subject to } \rho\left(t_1x_1+\cdots+t_nx_n\right)\ge 0,\end{aligned} \tag{6}$$

admits optimal solutions.

Proof. Let us denote

$$\tilde{G}:=\overline{\left\{\left(f\left(-x_1\right),\cdots,f\left(-x_n\right),f\left(-y\right)\right),\ f\in\mathcal{A}\right\}}.$$

Using a similar argument as in the proof of Theorem 1, we obtain for any $t_1,\cdots,t_n\in\mathbb{R}$:

$$\begin{aligned}&\rho\left(y+t_1x_1+\cdots+t_nx_n\right)\\ &=\max_{(s_1,\cdots,s_n,s_{n+1})\in\tilde{G}}\left(t_1s_1+\cdots+t_ns_n+s_{n+1}\right).\end{aligned}$$

In particular, for all $u>0$ and any $t_1,\cdots,t_n\in\mathbb{R}$ one has

$$\begin{aligned}&\rho\left(y+t_1x_1+\cdots+t_nx_n\right)\ge u \text{ iff}\\ &t_1s_1+\cdots+t_ns_n+s_{n+1}\ge u\\ &\text{for some } \left(s_1,\cdots,s_n,s_{n+1}\right)\in\tilde{G}.\end{aligned} \tag{7}$$

As $\rho\left(t_1x_1+\cdots+t_nx_n\right)\ge 0$ for $t_1,\cdots,t_n\in\mathbb{R}$, using a similar argument as in the proof of Theorem 1, we obtain that the following is well defined

$$u^*:=\inf\left\{u>0:\left(0,\cdots,0,u\right)\in\tilde{G}\right\}.$$

Then the min value in (6) is given by u^* and is achieved at every $\left(t_1,\cdots,t_n\right)\in\mathbb{R}^n$ satisfying:

$$\begin{aligned}&t_1s_1+\cdots+t_ns_n+s_{n+1}\ge u^*\\ &\text{for some } \left(s_1,\cdots,s_n,s_{n+1}\right)\in\tilde{G}.\end{aligned} \tag{8}$$

Indeed, take $t_1,\cdots,t_n\in\mathbb{R}$ satisfying $t_1s_1+\cdots+t_ns_n+s_{n+1}\ge u$ for some $\left(s_1,\cdots,s_n,s_{n+1}\right)\in\tilde{G}$. Condition (7) and definition (8) imply that $\min\rho\left(y+t_1x_1+\cdots+t_nx_n\right)=u^*$ and the min is achieved at every $\left(t_1,\cdots,t_n\right)\in\mathbb{R}^n$ satisfying condition (9). □

3. Applications

Examples. 1) We can solve explicitly problems (1) and (6) in the important case of Tail VaR (short for Tail Value at Risk). More precisely, consider $0<\alpha\le 1, g\in E^\times$ and define the Tail VaR of order α as the coherent measure of risk with the representation (2) in which $A=\left\{f\in E^\times:\alpha f\le g\right\}$, cf. [1,3,10]. One can easily check that Tail VaR is continuous from below. More, Tail VaR is one of the best coherent risk measures, because is the smallest law invariant coherent risk measure that dominates the *Value of Risk* (cf. [3,11]). In the context of Theorem 1, we have that

$$\begin{cases}\max t_1g\left(-x_1\right)+\cdots+t_ng\left(-x_n\right) \text{ with } t_1,\cdots,t_n\in\mathbb{R}\\ \text{and subject to } 0\le \text{Tail Var}\left(t_1x_1+\cdots+t_nx_n\right)\le c,\end{cases}$$

has the optimal solution equal to αc. Indeed, one can easily check that $\lambda^*=\alpha^{-1}$ and any positive constant multiple of $\left(1,1,\cdots,1\right)$ is an optimal solution of (1). In the context of Theorem 2, we have that

$$\begin{cases} \min \ \text{Tail Var}\left(y + t_1 x_1 + \cdots + t_n x_n\right) \text{ with } t_1, \cdots, t_n \in \mathbb{R} \\ \text{and subject to Tail Var}\left(t_1 x_1 + \cdots + t_n x_n\right) \geq 0, \end{cases}$$

has the optimal solution equal to Tail VaR(y). Indeed, one can check that $u^* =$ Tail VaR(y) because $(0, 0, \cdots, 0)$ is an optimal solution of (6). This situation occurs in problems (1) and (6) for complete models, such as Black-Scholes and Cox-Ross-Rubinstein.

2) Recall that a coherent measure of risk identifies unacceptable positions, *i.e*. with strictly positive risk $\rho(x)$. A good measure of the latter are the so-called relevant measures of risk: given $g \in E^{\times}$, a coherent measure of risk ρ is called *g*-relevant (cf. [1,3,10]) if $x \geq 0$ and $g(x) > 0$ imply $\rho(-x) > 0$.

Let us consider $E = L^{\infty}(\Omega, F, P)$; we have $E' = E^{\times} = ba(\Omega, F, P)$, the Banach space of bounded finitely additive measures on F and absolutely continuous with respect to *P*. In this case, all functionals $f \in A$ (given by formula (2) above) describing a coherent measure of risk continuous from below and g-relevant are genuine (*i.e*., σ-additive) probability measures equivalent to g. The particular case $g = P$, *i.e*., *g* represents integration with respect to P, has been treated in [2], Theorem 3.4, and the associated optimization problems (1) and (6) have been completely solved in [8,12] (see also [4]).

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