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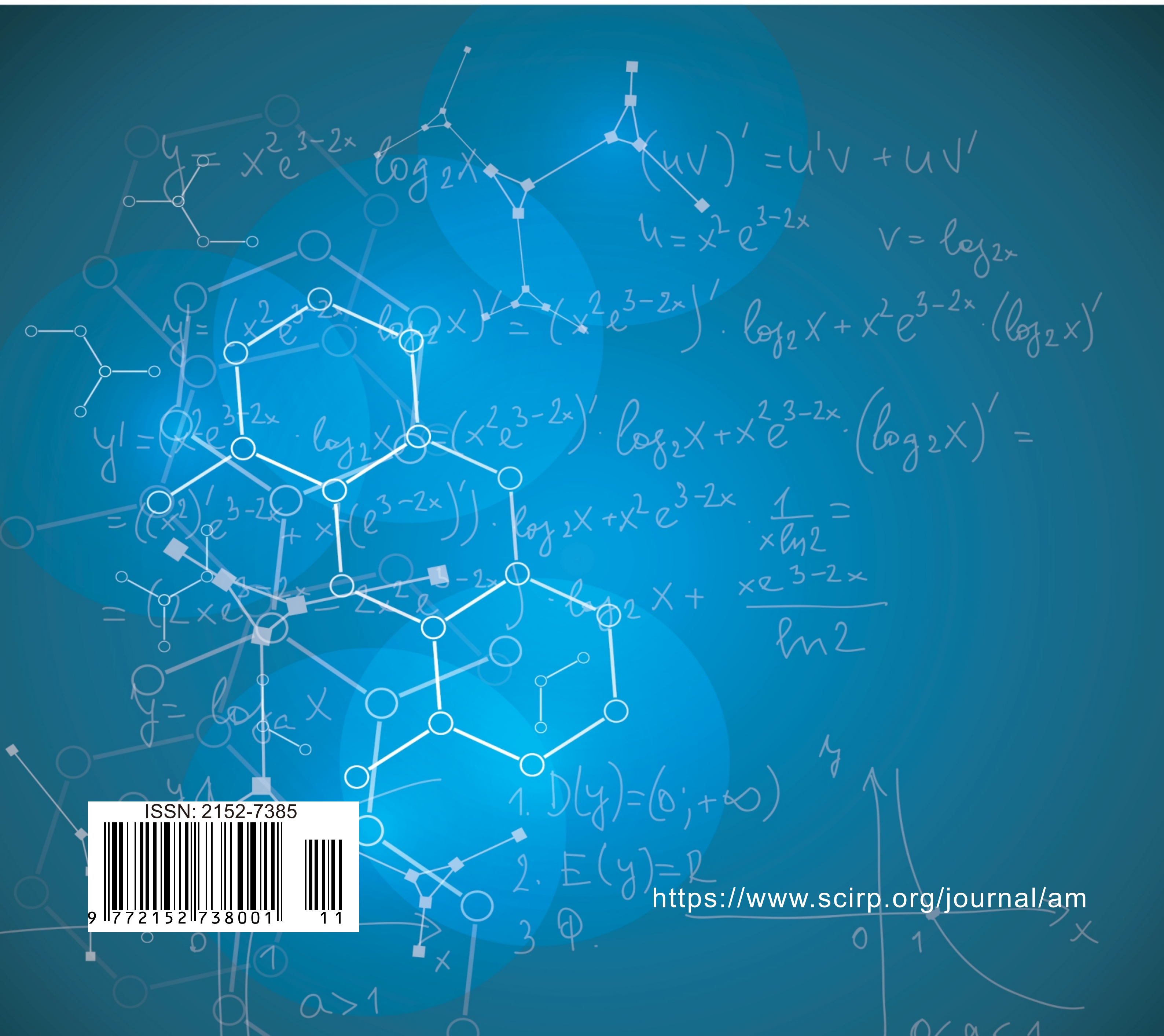

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An Alternative Way to Mapping Cone: The Algebraic Topology of the Pinched Tensor

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Abstract

In this research, we explore the properties and applications of the mapping cone and its variant, the pinched mapping cone. The mapping cone is a construction that arises naturally in algebraic topology and is used to study the homotopy type of spaces. It has several key properties, including its homotopy equivalence to the cofiber of a continuous map, and its ability to compute homotopy groups using the long exact sequence associated with the cofiber. We also provide an overview of the properties and applications of the mapping cone and the pinched mapping cone in algebraic topology. This work highlights the importance of these constructions in the study of homotopy theory and the calculation of homotopy groups. The study also points to the potential for further research in this area which includes the study of higher homotopy groups and the applications of these constructions to other areas of mathematics.

Keywords

Complex, Tensor Product, Pinched Tensor Product, Mapping Cone, Morphism

1. Introduction

The mapping cone is a construction used in algebraic topology and homological algebra to study maps between two topological spaces or algebraic objects. Let *S*, *T*, be two topological spaces and let $f: S \to T$ be a continuous map from *S* to *T*. The mapping cone associated with *f* is a new space that captures information about the failure of *f* to be a homotopy equivalence.

Intuitively, the mapping cone is constructed by taking a cylinder $S \times [0,1]$ and gluing one end to the image of *S* under *f* and the other end to a single point. The resulting space has a cone-like shape and encodes information about the twisting or winding of the map *f*.

In recent years, the mapping cone is used in algebraic topology to define the cone of a chain complex, which in turn is used to define the homology groups of a space. In homological algebra, the mapping cone is used to construct long exact sequences that relate the homology groups of two spaces connected by a map. Also it has been used for some discrete problems [1] and by bearing cone to estimate gazes in geological aspect of its applications [2].

Algebraic topology was significantly advanced by the work of German mathematician Tammo tom Dieck [3]. His research had a profound impact on homotopy theory, particularly in stable homotopy theory and its applications. In [4] Dieck's contributions encompassed work on the stable homotopy groups of spheres, surgery theory, and various other topics within algebraic topology. His research has greatly influenced the field, inspiring subsequent generations of mathematicians to explore these foundational ideas further.

Geometrically, the mapping cone of f can be visualized as the space obtained by gluing the cone over S to T via f. The chain complex $C(f)$ plays a pertinent role in algebraic topology, of note is its role in the study of spectral sequences and the long exact sequence of homology groups associated to a short exact sequence of chain complexes.

The pinched mapping cone of f has several important properties. First, it is homotopy equivalent to the suspension of the mapping cone $C(f)$. This follows from the fact that the suspension of $C(f)$ is obtained by attaching two cones to its base, and the pinched mapping cone identifies the basepoints of these cones.

Second, the homotopy groups of the pinched mapping cone can be computed using the long exact sequence in homotopy associated with the mapping cone. This sequence relates the homotopy groups of S, T, and $C(f)$, and can be used to compute the homotopy groups of Z.

Third, the pinched mapping cone is used in the proof of the Freudenthal suspension theorem, which relates the stable homotopy groups of spheres to the homotopy groups of suspensions of spheres. The pinched mapping cone is used to construct a sequence of maps that are used to show that the stable homotopy groups of spheres stabilize at a certain point.

For more details you may see in [5] [6] and [7].

2. Basic Setup

It is assumed in this paper that E, F, S, and T are complexes of R-modules, with R being an associative ring. We will see most of the definitions and results of this section in [8] [9] and [10].

Definition 2.1. A *chain complex* S of R-modules is a sequence of R-module homomorphisms,

$$S:\cdots \rightarrow S_{n+1} \xrightarrow{\partial_{n+1}^S} S_n \xrightarrow{\partial_n^S} S_{n-1} \rightarrow \cdots$$

such that $Im\partial_{n+1}^S \subseteq \ker \partial_n^S$ for all n. Equivalently $\partial_n^S \partial_{n+1}^S = 0$, for all n. The maps

∂_n are called the *differentials* of *S*.

Definition 2.2. Given a complex *S* of *R*-modules

$$S:\cdots \to S_{n+1} \xrightarrow{\partial_{n+1}^S} S_n \xrightarrow{\partial_n^S} S_{n-1} \to \cdots$$

We say that it is exact at S_n if $Im\partial_{n+1}^S = \ker \partial_n^S$. Moreover, we say *S* is an *exact sequence* if $Im\partial_{n+1}^S = \ker \partial_n^S$ for all *n*.

Definition 2.3. A *morphism*, or *a degree zero chain map*, $f: S \to T$ between complexes *S* and *T* is a family of *R*-module homomorphisms f_n, such that each square in the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & S_{n+1} & \xrightarrow{\partial_{n+1}^S} & S_n & \xrightarrow{\partial_n^S} & S_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & T_{n+1} & \xrightarrow{\partial_{n+1}^T} & T_n & \xrightarrow{\partial_n^T} & T_{n-1} & \longrightarrow & \cdots
\end{array}$$

commutes. In other words, for each *n* we have $f_{n-1}\partial_n^S = \partial_n^T f_n$.

Also, if $f_n: S_n \to T_n$ is an isomorphism for all *n*, then, *S* and *T* are said to be *isomorphic*, denoted by $\cong$.

Definition 2.4. Let *E* be a complex of *R*-modules. Then the *Shift* of *E*, ΣE is complex of *R*-modules defined by $(\Sigma E)_n = E_{n-1}$, and $\partial_n^{\Sigma E} = -\sigma_{n-2}\partial_{n-1}^E\sigma_{n-1}^{-1}$ for all *n*. Also, *the canonical map* $\sigma: E \to \Sigma E$, is obtained by shifting degrees of elements, specifically, if $e \in E$. Then, $|\sigma(e)| = |e| + 1$.

On the other hand, Lars Christensen and David Jorgensen unveiled in 2014 a variant of the tensor product of complexes in their paper [11] known as the pinched tensor product. We recall some essential definitions and recommendations.

Definition 2.5. Let *S* and *T* a complexes of *R*-modules. The tensor product $S \otimes_R T$ of *S* and *T* is specified by letting

$$(S \otimes_R T)_n = \coprod_{i\in\mathbb{Z}} S_i \otimes_R T_{n-i}.$$

The differential is defined by,

$$\partial_n^{S\otimes_R T}(s \otimes t) = \partial_i^S(s) \otimes t + (-1)^i s \otimes \partial_{n-i}^T(t).$$

For $s \in S_i$ and $t \in T_{n-i}$. The sign $(-1)^i$ ensures that $\partial_n^{S\otimes_R T}\partial_{n+1}^{S\otimes_R T} = 0$ for all *n*.

Definition 2.6. Let *S* and *T* be complexes of *R*-modules. We define the pinched tensor product $S \otimes_R^{\bowtie} T$ of *S* and *T* by:

$$\left(S \otimes_R^{\bowtie} T\right)_n = \begin{cases} \left(S_{\geq 0} \otimes_R T_{\geq 0}\right)_n & \text{for } n \geq 0 \\ \left(S_{\leq -1} \otimes_R (\Sigma T)_{\leq 0}\right)_n & \text{for } n \leq -1. \end{cases}$$

With differential $\partial^{S\otimes_R^{\bowtie} T}$ defined by,

$$\partial_n^{S\otimes_R^{\bowtie} T} = \begin{cases} \partial_n^{S_{\geq 0}\otimes_R T_{\geq 0}} & \text{for } n \geq 1 \\ \partial_0^S \otimes_R \left(\sigma\partial_0^T\right) & \text{for } n = 0 \\ \partial_n^{S_{\leq -1}\otimes_R (\Sigma T)_{\leq 0}} & \text{for } n \leq -1, \end{cases}$$

where σ denotes the *canonical map* $T \to \Sigma T$.

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For more details you may see [11] [12] and [13].

3. Mapping Cone

Definition 3.1. Let *A*, *B*, *S* and *T* be complexes, and let $f:S\to T$ and $g:A\to B$ be chain maps. Then for $s\otimes a\in S_i\otimes A_{n-i}$ we have $(f\otimes g)(s\otimes a)=(-1)^{|s||g|}f_i(s)\otimes g_{n-i}(a)$.

Definition 3.2. If $f:S\to T$ is a chain map, then its *mapping cone*, $\operatorname{cone}(f)$, is a complex of *R*-modules whose term of degree *n* is $\operatorname{cone}(f)_n=(\Sigma S)_n\oplus T_n$ and whose differentials $\partial_n:\operatorname{cone}(f)_n\to\operatorname{cone}(f)_{n-1}$ is given by

$$\partial_n^{\operatorname{cone}(f)}=\begin{bmatrix}\partial_n^{\Sigma S} & 0\\ f_{n-1}\sigma_{n-1}^{-1} & \partial_n^T\end{bmatrix}.$$

A straightforward computation shows that $\partial_{n-1}^{\operatorname{cone}(f)}\partial_n^{\operatorname{cone}(f)}=0$:

$$\begin{aligned}\partial_{n-1}^{\operatorname{cone}(f)}\partial_n^{\operatorname{cone}(f)}&=\begin{bmatrix}\partial_{n-1}^{\Sigma S} & 0\\ f_{n-2}\sigma_{n-2}^{-1} & \partial_{n-1}^T\end{bmatrix}\begin{bmatrix}\partial_n^{\Sigma S} & 0\\ f_{n-1}\sigma_{n-1}^{-1} & \partial_n^T\end{bmatrix}\\ &=\begin{bmatrix}\partial_{n-1}^{\Sigma S}\partial_n^{\Sigma S} & 0\\ f_{n-2}\sigma_{n-2}^{-1}\partial_n^{\Sigma S}+\partial_{n-1}^T f_{n-1}\sigma_{n-1}^{-1} & \partial_{n-1}^T\partial_n^T\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.\end{aligned}$$

Since we know that *f* is a morphism and $\partial_n^{\Sigma S}=-\sigma_{n-2}\partial_{n-1}^S\sigma_{n-1}^{-1}$.

Theorem 3.3. Let *A be an R-complex and* $f:S\to T$ *be a morphism of complexes of R-modules. Then,*

$$A\otimes_R\operatorname{cone}(f)\cong\operatorname{cone}(A\otimes_R f).$$

Proof. The proof is well-known. (Proposition 4.1.12 in [14].) □

4. Main Results

In the following section, we will present the main theorem applied in the special case of the pinched mapping cone and we provide the detailed proof of it.

Theorem 4.1. *Let A be an R-complex and* $f:S\to T$ *be a morphism of complexes of R-modules. Then, there exist a morphism from* $A\otimes_R^{\bowtie}\operatorname{cone}(f)$ *to* $\operatorname{cone}(A\otimes_R^{\bowtie}h)$. *Where,* $h_n=f_n$ *when* $n\geq -1$ *and* $h_n=-f_n$ *when* $n<-1$

Proof. We consider three cases: $n=0$, $n\geq 1$ and $n\leq -1$.

$n=0$:

In this case the diagram is

$$\begin{array}{ccc}(A\otimes_R^{\bowtie}\operatorname{cone}(f))_0 & \xrightarrow{\partial_0^{A\otimes_R^{\bowtie}\operatorname{cone}(f)}} & (A\otimes_R^{\bowtie}\operatorname{cone}(f))_{-1}\\ \downarrow{\scriptstyle\tau_0} & & \downarrow{\scriptstyle\tau_{-1}}\\ (\operatorname{cone}(A\otimes_R^{\bowtie}f))_0 & \xrightarrow{\partial_0^{\operatorname{cone}(A\otimes_R^{\bowtie}f)}} & (\operatorname{cone}(A\otimes_R^{\bowtie}f))_{-1}\end{array}$$

Define $\tau_0(a\otimes(s,t))=\left(\sigma_{-1}^{A\otimes_R^{\bowtie}S}\left(\partial_0^A(a)\otimes s\right),(a\otimes t)\right)$ and $\tau_{-1}\left(a\otimes\sigma_{-1}^{\operatorname{cone}(f)}(s,t)\right)=\left(\sigma_{-2}^{A\otimes_R^{\bowtie}S}(a\otimes t),\left(a\otimes\sigma_{-1}^T(t)\right)\right)$. Choose an element

$\left(a\otimes(s,t)\right)\in A_0\otimes_R\left((\Sigma S)_0\oplus T_0\right)$. Then,

$$\partial_0^{\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)}\tau_0\left(a\otimes(s,t)\right)$$

$$=\partial_0^{\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)}\left(\sigma_{-1}^{A\otimes_R^{\bowtie}S}\left(\partial_0^A(a)\otimes s\right),\left(a\otimes t\right)\right)$$

$$=\Big[\partial_0^{\Sigma\left(A\otimes^{\bowtie}S\right)}\left(\sigma_{-1}^{A\otimes_R^{\bowtie}S}\left(\partial_0^A(a)\otimes s\right)\right),$$

$$\left(A_{-1}\otimes_R(\Sigma f)_0\right)\left(\sigma_{-1}^{A\otimes_R^{\bowtie}S}\right)^{-1}\sigma_{-1}^{A\otimes_R^{\bowtie}S}\left(\partial_0^A(a)\otimes(s)\right)+\partial_0^{A\otimes^{\bowtie}T}\left(a\otimes t\right)\Big]$$

$$=\Big[-\sigma_{-2}^{A\otimes_R^{\bowtie}S}\partial_{-1}^{A\otimes^{\bowtie}S}\left(\sigma_{-1}^{A\otimes_R^{\bowtie}S}\right)^{-1}\left(\sigma_{-1}^{A\otimes_R^{\bowtie}S}\left(\partial_0^A(a)\otimes s\right)\right),$$

$$\left(A_{-1}\otimes_R(\Sigma f)_0\right)\left(\partial_0^A(a)\otimes(s)\right)+\partial_0^A(a)\otimes\sigma_{-1}^T\partial_0^T(t)\Big]$$

$$=\left[-\sigma_{-2}^{A\otimes_R^{\bowtie}S}\partial_{-1}^{A\otimes^{\bowtie}S}\left(\partial_0^A(a)\otimes s\right),\left(\partial_0^A(a)\otimes_R(\Sigma f)_0(s)\right)+\partial_0^A(a)\otimes\sigma_{-1}^T\partial_0^T(t)\right]$$

$$=\Big[-\sigma_{-2}^{A\otimes_R^{\bowtie}S}\left(\partial_{-1}^A\partial_0^A(a)\otimes s+(-1)\partial_0^A(a)\otimes\partial_{-1}^{\Sigma S}(s)\right),$$

$$\left(\partial_0^A(a)\otimes_R(\Sigma f)_0(s)\right)+\partial_0^A(a)\otimes\sigma_{-1}^T\partial_0^T(t)\Big]$$

$$=\left[\sigma_{-2}^{A\otimes_R^{\bowtie}S}\left(\partial_0^A(a)\otimes\partial_{-1}^{\Sigma S}(s)\right),\partial_0^A(a)\otimes_R(\Sigma f)_0(s)+\partial_0^A(a)\otimes\sigma_{-1}^T\partial_0^T(t)\right].$$

And

$$\tau_{-1}\partial_0^{A\otimes_R^{\bowtie}\mathrm{cone}(f)}\left(a\otimes(s,t)\right)$$

$$=\tau_{-1}\left[\partial_0^A(a)\otimes\partial_0^{\mathrm{Cone}(f)}(s,t)\right]$$

$$=\tau_{-1}\left[\partial_0^A(a)\otimes\left(\partial_{-1}^{\Sigma S}(s),\,f_{-1}\left(\sigma_{-1}^S\right)^{-1}(s)+\partial_0^T(t)\right)\right]$$

$$=\left[\sigma_{-2}^{A\otimes_R^{\bowtie}S}\left(\partial_0^A(a)\otimes\partial_{-1}^{\Sigma S}(s)\right),\partial_0^A(a)\otimes_R\sigma_{-1}^T\left(f_{-1}\left(\sigma_{-1}^S\right)^{-1}(s)+\partial_0^T(t)\right)\right].$$

Clear that $(\Sigma f)_0(s)=\sigma_{-1}^T f_{-1}\left(\sigma_{-1}^S\right)^{-1}(s)$ since the following diagram commutes

$$\begin{array}{ccc} S_{-1} & \xrightarrow{\quad f_{-1}\quad} & T_{-1} \\ \Big\downarrow{\scriptstyle \sigma_{-1}^S} & & \Big\downarrow{\scriptstyle \sigma_{-1}^T} \\ (\Sigma S)_0 & \xrightarrow{\quad (\Sigma f)_0\quad} & (\Sigma T)_{-1} \end{array}$$

Therefore, $\partial_0^{\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)}\tau_0=\tau_{-1}\partial_0^{A\otimes_R^{\bowtie}\mathrm{cone}(f)}$, which is what we wanted to show.

$n\geq 1$:

In this case the diagram is

$$\begin{array}{ccc} \left(A\otimes_R^{\bowtie}\mathrm{cone}(f)\right)_n & \xrightarrow{\quad \partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}(f)}\quad} & \left(A\otimes_R^{\bowtie}\mathrm{cone}(f)\right)_{n-1} \\ \Big\downarrow{\scriptstyle \tau_n} & & \Big\downarrow{\scriptstyle \tau_{n-1}} \\ \left(\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)\right)_n & \xrightarrow{\quad \partial_n^{\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)}\quad} & \left(\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)\right)_{n-1} \end{array}$$

Define $\tau_n\left(a\otimes(s,t)\right)=\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\left(a\otimes\sigma_i^{-1}(s)\right),(a\otimes t)\right)$. We have two subcases $i=0$ and $i\geq 1$

$i=0$:

Choose an element $\left(a\otimes(s,t)\right)\in A_n\otimes_R\left((\Sigma S)_0\oplus T_0\right)$. Then,

$$\begin{aligned}
&\partial_n^{\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)}\tau_n\left(a\otimes(s,t)\right)\\
&=\partial_n^{\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)}\left((a\otimes t)\right)\\
&=\left[\partial_n^{A\otimes^{\bowtie}T}(a\otimes t)\right]\\
&=\left[\partial_n^A(a)\otimes t+(-1)^n a\otimes\partial_0^T(t)\right].
\end{aligned}$$

And

$$\begin{aligned}
&\tau_{n-1}\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}(f)}\left(a\otimes(s,t)\right)\\
&=\tau_{n-1}\left[\partial_n^A(a)\otimes(s,t)+(-1)^n a\otimes\partial_0^{\mathrm{cone}(f)}(s,t)\right]\\
&=\tau_{n-1}\left[\partial_n^A(a)\otimes(s,t)+(-1)^n a\otimes\left(\partial_0^{\Sigma S}(s),f_{-1}\left(\sigma_{-1}^S\right)^{-1}(s)+\partial_0^T(t)\right)\right]\\
&=\left[\partial_n^A(a)\otimes t+(-1)^n a\otimes\partial_0^T(t)\right].
\end{aligned}$$

$i\geq 0$:

Choose an element $\left(a\otimes(s,t)\right)\in A_{n-i}\otimes_R\left((\Sigma S)_i\oplus T_i\right)$. Then,

$$\begin{aligned}
&\partial_n^{\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)}\tau_n\left(a\otimes(s,t)\right)\\
&=\partial_n^{\mathrm{cone}\left(A\otimes_R^{\bowtie}f\right)}\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\left(a\otimes\left(\sigma_i^S\right)^{-1}(s),(a\otimes t)\right)\right)\\
&=\left[\partial_n^{\Sigma\left(A\otimes^{\bowtie}S\right)}\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\left(a\otimes\left(\sigma_i^S\right)^{-1}(s)\right)\right),\right.\\
&\qquad\left.\left(A_{n-i}\otimes_R f_i\right)\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\right)^{n-1}\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\left(a\otimes\left(\sigma_i^S\right)^{-1}(s)\right)+\partial_n^{A\otimes^{\bowtie}T}(a\otimes t)\right]\\
&=\left[-\sigma_{n-2}^{A\otimes_R^{\bowtie}S}\partial_{n-1}^{A\otimes^{\bowtie}S}\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\right)^{-1}\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\left(a\otimes\left(\sigma_i^S\right)^{-1}(s)\right)\right),\right.\\
&\qquad\left.\left(A_{n-i}\otimes_R f_i\right)\left(a\otimes\left(\sigma_i^S\right)^{-1}(s)\right)+\partial_{n-i}^A(a)\otimes t+(-1)^{n-i}a\otimes\partial_i^T(t)\right]\\
&=\left[-\sigma_{n-2}^{A\otimes_R^{\bowtie}S}\partial_{n-1}^{A\otimes^{\bowtie}S}\left(a\otimes\left(\sigma_i^S\right)^{-1}(s)\right),\right.\\
&\qquad\left.\left(a\otimes_R f_i\left(\sigma_i^S\right)^{-1}(s)\right)+\partial_{n-i}^A(a)\otimes t+(-1)^{n-i}a\otimes\partial_i^T(t)\right]\\
&=\left[-\sigma_{n-2}^{A\otimes_R^{\bowtie}S}\left(\partial_{n-i}^A(a)\otimes\left(\sigma_i^S\right)^{-1}(s)+(-1)^{n-i}a\otimes\partial_{i-1}^S\left(\sigma_{i-1}^S\right)^{-1}(s)\right),\right.\\
&\qquad\left.\left(a\otimes_R f_i\left(\sigma_i^S\right)^{-1}(s)\right)+\partial_{n-i}^A(a)\otimes t+(-1)^{n-i}a\otimes\partial_i^T(t)\right].
\end{aligned}$$

And

$$\begin{aligned}
&\tau_{n-1}\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}(f)}\left(a\otimes(s,t)\right)\\
&=\tau_{n-1}\left[\partial_{n-i}^A(a)\otimes(s,t)+(-1)^{n-i}a\otimes\partial_i^{\mathrm{cone}(f)}(s,t)\right]
\end{aligned}$$

$$= \tau_{n-1}\left[\partial_{n-i}^{A}(a)\otimes(s,t)+(-1)^{n-i}a\otimes\left(\partial_i^{\Sigma S}(s),\, f_{i-1}\left(\sigma_{i-1}^S\right)^{-1}(s)+\partial_i^T(t)\right)\right]$$

$$=\left[-\sigma_{n-2}^{A\otimes_R^{\bowtie}S}\left(\partial_{n-i}^A(a)\otimes\left(\sigma_i^S\right)^{-1}(s)+(-1)^{n-i}a\otimes\left(\sigma_{i-1}^S\right)^{-1}\partial_i^{\Sigma S}(s)\right),\right.$$

$$\left. a\otimes f_{i-1}\left(\sigma_{i-1}^S\right)^{-1}(s)+\partial_{n-i}^A(a)\otimes t+(-1)^{n-i}a\otimes\partial_i^T(t)\right].$$

Therefore, $\partial_n^{\mathrm{cone}\left(A\otimes_R^{\bowtie} f\right)}\tau_n=\tau_{n-1}\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}(f)}$, which is what we wanted to show.

$n\le -1$:

In this case the diagram is

$$\begin{array}{ccc}
\left(A\otimes_R^{\bowtie}\mathrm{cone}(f)\right)_n & \xrightarrow{\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}(f)}} & \left(A\otimes_R^{\bowtie}\mathrm{cone}(f)\right)_{n-1} \\
\downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_{n-1}} \\
\left(\mathrm{cone}\left(A\otimes_R^{\bowtie} f\right)\right)_n & \xrightarrow{\partial_n^{\mathrm{cone}\left(A\otimes_R^{\bowtie} f\right)}} & \left(\mathrm{cone}\left(A\otimes_R^{\bowtie} f\right)\right)_{n-1}
\end{array}$$

Define $\tau_n\left(a\otimes\sigma_n^{\mathrm{cone}(f)}(s,t)\right)=\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\left(a\otimes\sigma_i^{-1}(s)\right),\left(a\otimes\sigma_i^T(t)\right)\right)$ and choose an element $\left(a\otimes(s,t)\right)\in A_i\otimes_R\left((\Sigma S)_{n-i-1}\oplus T_{n-i-1}\right)$. Then,

$$\partial_n^{\mathrm{cone}\left(A\otimes_R^{\bowtie} f\right)}\tau_n\left(a\otimes\sigma_n^{\mathrm{cone}(f)}(s,t)\right)$$

$$=\partial_n^{\mathrm{cone}\left(A\otimes_R^{\bowtie} f\right)}\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\left(a\otimes\sigma_i^{-1}(s)\right),\left(a\otimes\sigma_i^T(t)\right)\right)$$

$$=\left[\partial_n^{\Sigma\left(A\otimes^{\bowtie}S\right)}\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}(a\otimes s)\right),\left(A_i\otimes_R(\Sigma f)_{n-1-i}\right)\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\right)^{n-1}\sigma_{n-1}^{A\otimes_R^{\bowtie}S}(a\otimes s)\right.$$

$$\left.+\partial_n^{A\otimes^{\bowtie}T}\left(a\otimes\sigma_i^T(t)\right)\right]$$

$$=\left[-\sigma_{n-2}^{A\otimes_R^{\bowtie}S}\partial_{n-1}^{A\otimes^{\bowtie}S}\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}\right)^{n-1}\left(\sigma_{n-1}^{A\otimes_R^{\bowtie}S}(a\otimes s)\right),\left(A_i\otimes_R(\Sigma f)_{n-1-i}\right)(a\otimes s)\right.$$

$$\left.+\partial_i^A(a)\otimes\sigma_i^T(t)+(-1)^i a\otimes\partial_{n-i}^{\Sigma T}\sigma_i^T(t)\right]$$

$$=\left[-\sigma_{n-2}^{A\otimes_R^{\bowtie}S}\partial_{n-1}^{A\otimes^{\bowtie}S}(a\otimes s),\left(a\otimes_R(\Sigma f)_{n-1-i}(s)\right)\right.$$

$$\left.+\partial_i^A(a)\otimes\sigma_i^T(t)+(-1)^i a\otimes\partial_{n-i}^{\Sigma S}\sigma_i^T(t)\right]$$

$$=\left[-\sigma_{n-2}^{A\otimes_R^{\bowtie}S}\left(\partial_i^A(a)\otimes s+(-1)^i a\otimes\partial_{n-i-1}^{\Sigma S}(s)\right),\left(a\otimes_R(\Sigma f)_{n-1-i}(s)\right)\right.$$

$$\left.+\partial_i^A(a)\otimes\sigma_i^T(t)+(-1)^i a\otimes\partial_{n-i}^{\Sigma T}\sigma_i^T(t)\right].$$

And

$$\tau_{n-1}\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}(f)}\left(a\otimes(s,t)\right)$$

$$=\tau_{n-1}\left[\partial_i^A(a)\otimes(s,t)+(-1)^i a\otimes\partial_{n-i}^{\mathrm{cone}(f)}(s,t)\right]$$

$$=\tau_{n-1}\left[\partial_i^A(a)\otimes(s,t)+(-1)^i a\otimes\left(\partial_{n-i}^{\Sigma S}(s),(\Sigma f)_{n-i-1}\left(\sigma_{n-i-1}^S\right)^{-1}(s)+\partial_{n-i}^T(t)\right)\right]$$

$$=\left[-\sigma_{n-2}^{A\otimes_R^{\bowtie}S}\left(\partial_i^A(a)\otimes s+(-1)^i a\otimes\partial_{n-i}^{\Sigma S}(s)\right),(-1)^i a\otimes(\Sigma f)_{n-i}\left(\sigma_{n-i}^S\right)^{-1}(s)\right.$$

$$\left.+\partial_i^A(a)\otimes\sigma_i^T(t)+\sigma_{n-i-1}^T\partial_{n-i}^T(t)\right].$$

Therefore, $\partial_n^{\text{cone}\left(A\otimes_R^{\bowtie} f\right)}\tau_n = \tau_{n-1}\partial_n^{A\otimes_R^{\bowtie}\text{cone}(f)}$, which is what we wanted to show. □

Remark 4.2. In Theorem 4.1 we note that we will not have an isomorphism between $A\otimes_R^{\bowtie}\text{cone}(f)$ and $\text{cone}\left(A\otimes_R^{\bowtie} f\right)$ because $\left(C\otimes_R^{\bowtie}\text{cone}(f)\right)_n$ has one more term than $\left(\text{cone}\left(A\otimes_R^{\bowtie} f\right)\right)_n$ for $n>0$ and vice versa for $n<0$.

Theorem 4.3. *Let* A *be an R-complex and* $f:S\to T$ *be a morphism of complexes of R-modules. Then there exist a morphism from* $\text{cone}\left(A\otimes_R^{\bowtie} f\right)$ *to* $A\otimes_R^{\bowtie}\text{cone}(f)$.

Proof. The proof is similar to that of Theorem 4.1. □

5. Conclusion

To conclude, the mapping cone stands as a potent instrument within algebraic topology, boasting myriad vital applications. Its inherent properties facilitate the establishment of intriguing connections among homotopy groups of spaces, playing a pivotal role in substantiating numerous key theorems. The mapping cone's versatility extends its utility across a wide array of subjects, spanning from homotopy groups to fiber bundles. For algebraic topologists, it remains an indispensable tool, frequently employed to tackle the challenging computation of homotopy groups. Notably, the pinched mapping cone serves as a testament to the robustness and value of algebraic topology in the realm of mathematics. As referenced in the central theorem, we substantiate the existence of morphisms from one cone to another, contributing significantly to the expansion of homotopy group properties. Future research avenues may delve even deeper into the intricacies of the mapping cone, providing more extensive exploration of its applications and investigating its pertinence in other mathematical domains.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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On the Construction and Classification of the Common Invariant Solutions for Some $P(1,4)$-Invariant Partial Differential Equations

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Abstract

We consider the following (1 + 3)-dimensional $P(1,4)$-invariant partial differential equations (PDEs): the Eikonal equation, the Euler-Lagrange-Born-Infeld equation, the homogeneous Monge-Ampère equation, the inhomogeneous Monge-Ampère equation. The purpose of this paper is to construct and classify the common invariant solutions for those equations. For this aim, we have used the results concerning construction and classification of invariant solutions for the (1 + 3)-dimensional $P(1,4)$-invariant Eikonal equation, since this equation is the simplest among the equations under investigation. The direct checked allowed us to conclude that the majority of invariant solutions of the (1 + 3)-dimensional Eikonal equation, obtained on the base of low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$, satisfy all the equations under investigation. In this paper, we present obtained common invariant solutions of the equations under study as well as the classification of those invariant solutions.

Keywords

Symmetry Reduction, Classification of Invariant Solutions, Common Invariant Solutions, The Eikonal Equations, The Euler-Lagrange-Born-Infeld Equations, The Monge-Ampère Equations, Classification of Lie Algebras, Nonconjugate Subalgebras, Poincaré Group $P(1,4)$

1. Introduction

A solution of many problems of the geometric optics, theories of anisotropic media, theory of minimal surfaces, nonlinear electrodynamics, theories of gravity, geometry, unified field theory, string theories, black holes, cosmology, etc. is reduced to the investigation of the Eikonal equations [1] [2] [3] [4] [5], the Euler-Lagrange equations [6]-[12], the Born-Infeld equations [13]-[22], the Monge-Ampère equations [23]-[40] in the spaces of different dimensions and different types (see also the references therein).

Nowadays, there exist a lot of methods for the construction exact solutions of linear and nonlinear partial differential equations (PDEs). More details on this theme can be found in [41]-[46] (see also the references therein).

We consider the following (1 + 3)-dimensional $P(1,4)$-invariant PDEs:

- the Eikonal equation,
- the Euler-Lagrange-Born-Infeld equation,
- the homogeneous Monge-Ampère equation,
- the inhomogeneous Monge-Ampère equation.

From the results obtained by Fushchich W.I., Shtelen W.M. and Serov N.I. [40], it follows, in particular, that the common symmetry group of those equations is the generalized Poincaré group $P(1,4)$. Therefore, in the natural way arises the following question: what is the relationship between invariant solutions of the equations under study? In particular, whether those equations have common invariant solutions?

The purpose of this paper is to try to construct and classify the common invariant solutions for the equations under consideration. It is known that the (1 + 3)-dimensional $P(1,4)$-invariant Eikonal equation is the simplest one among the equations under study. Therefore, we can use this fact for constructing the common invariant solutions. At the present time, we have constructed invariant solutions for the (1 + 3)-dimensional $P(1,4)$-invariant Eikonal equation obtained on the base of low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$, by using classical Lie-Ovsiannikov approach [41] [42] [43] [44]. This method, in particular, allows us to perform the symmetry reduction of the many-dimensional PDEs with non-trivial symmetry groups to differential equations with a fewer number of independent variables as well as to construct solutions, invariant with respect to nonconjugate subgroups of the symmetry groups, of the equations under study. According to this method, reduced equations (invariant solutions) should be classified with respect to the ranks of the corresponding nonconjugate subalgebras of the Lie algebras of the symmetry groups of the equations under study.

Our contribution in classical Lie-Ovsiannikov method consists in the suggestion to use, for the classification of symmetry reductions (invariant solutions) of PDEs with non-trivial symmetry groups, not only ranks of nonconjugate subalgebras, but also their structural property. Some details on this theme can be found in [47] [48].

In our paper, we have performed the suggestion for the classification of the common invariant solutions of some $P(1, 4)$-invariant PDEs by using the structural property of the low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1, 4)$.

The direct checks allowed us to conclude that the majority of invariant solutions of the (1 + 3)-dimensional Eikonal equation, obtained on the base of low-dimensional ($dimL \le 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$, satisfy all the equations under investigation. In this paper, we present obtained common invariant solutions of the equations under study as well as the classification of those invariant solutions.

To present the results obtained, we give some information about the Lie algebra of the Poincaré group $P(1,4)$ and its nonconjugate subalgebras.

2. The Lie Algebra of the Poincaré Group $P(1,4)$ and Its Nonconjugate Subalgebras

The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. It is the smallest group, which contains, as subgroups, the extended Galilei group $\tilde{G}(1,3)$ [49] (the symmetry group of classical physics) and the Poincaré group $P(1,3)$ (the symmetry group of relativistic physics).

The Lie algebra of the group $P(1,4)$ is generated by 15 bases elements $M_{\mu\nu} = -M_{\nu\mu}\left(\mu,\nu = 0,1,2,3,4\right)$ and $P_\mu\left(\mu = 0,1,2,3,4\right)$, which satisfy the commutation relations

$$\left[P_\mu, P_\nu\right] = 0, \quad \left[M_{\mu\nu}, P_\sigma\right] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu, \tag{1}$$

$$\left[M_{\mu\nu}, M_{\rho\sigma}\right] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}, \tag{2}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \ne \nu$.

In this paper, we consider the following representation [40] of the Lie algebra of the group $P(1,4)$:

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = -\frac{\partial}{\partial x_1}, \quad P_2 = -\frac{\partial}{\partial x_2}, \quad P_3 = -\frac{\partial}{\partial x_3}, \tag{3}$$

$$P_4 = -\frac{\partial}{\partial u}, \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \ x_4 \equiv u. \tag{4}$$

In the following, we will use the next bases elements:

$$G = M_{04}, \quad L_1 = M_{23}, \quad L_2 = -M_{13}, \quad L_3 = M_{12}, \tag{5}$$

$$P_a = M_{a4} - M_{0a}, \quad C_a = M_{a4} + M_{0a}, \quad \left(a = 1,2,3\right), \tag{6}$$

$$X_0 = \frac{1}{2}\left(P_0 - P_4\right), \quad X_k = P_k \left(k = 1,2,3\right), \quad X_4 = \frac{1}{2}\left(P_0 + P_4\right). \tag{7}$$

The Lie algebra of the extended Galilei group $\tilde{G}(1,3)$ is generated by the following bases elements:

$$L_1, \quad L_2, \quad L_3, \quad P_1, \quad P_2, \quad P_3, \quad X_0, \quad X_1, \quad X_2, \quad X_3, \quad X_4. \tag{8}$$

The classification of all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of dimensions ≤ 3 was performed in [50].

3. On the Construction and Classification of the Common Invariant Solutions for Some (1 + 3)-Dimensional $P(1,4)$-Invariant PDEs

In this Section, We Consider the Following PDEs

- the Eikonal equation

$$u_0^2 - u_1^2 - u_2^2 - u_3^2 = 1;$$

- the Euler-Lagrange-Born-Infeld equation

$$\Box u\left(1 - u_\nu u^\nu\right) + u^\mu u^\nu u_{\mu\nu} = 0;$$

- the homogeneous Monge-Ampère equation

$$\det\left(u_{\mu\nu}\right) = 0;$$

- the inhomogeneous Monge-Ampère equation

$$\det\left(u_{\mu\nu}\right) = \lambda\left(1 - u_\nu u^\nu\right)^3, \quad \lambda \neq 0,$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_\mu \equiv \frac{\partial u}{\partial x^\mu}$, $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x^\mu \partial x^\nu}$, $u^\mu = g^{\mu\nu} u_\nu$, $g_{\mu\nu} = (1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu = 0,1,2,3$, $\Box$ is the d'Alembert operator.

Here, and in what follows, $M(1,3)$ is a four-dimensional Minkowski space, $R(u)$ is a real number axis of the depended variable *u*.

From the results obtained by Fushchich W.I., Shtelen W.M. and Serov N.I. [40] it follows, in particule, that the common symmetry group of those equations is the generalised Poincaré group $P(1,4)$.

In this section we present obtained common invariant solutions of the equations under study as well as the classification of those invariant solutions. To obtain those results, we used the nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$, structural properties of its low-dimensional ($dim L \leq 3$) nonconjugate subalgebras as well as the results of the classification of symmetry reductions of the eikonal equation. More details on this theme can be found in [47] [48].

Bellow we present the results obtained.

3.1. Classification of the Common Invariant Solutions for the Equations under Study Using One-Dimensional Nonconjugate Subalgebras of the Lie Algebra of the Group $P(1,4)$

1) $\langle G \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - u^2\right)^{1/2} = -\left(1 - c_2^2 - c_3^2\right)^{1/2} x_1 + c_2 x_2 + c_3 x_3 + c_1,$$

where c_1, c_2 and c_3 are arbitrary real constants.

2) $\langle G + \alpha X_1, \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$\alpha \ln\left(\frac{2\alpha\left(\sqrt{\left(c_1^2+c_2^2+1\right)\left(x_0^2-u^2\right)+\alpha^2}+\alpha\right)}{x_0-u}\right)$$
$$-\sqrt{\left(c_1^2+c_2^2+1\right)\left(x_0^2-u^2\right)+\alpha^2}-x_1+c_1x_2+c_2x_3+c_3,$$

where c_1, c_2 and c_3 are arbitrary real constants.

3) $\langle L_3 \rangle$:

The common invariant solution for the equations under study:

$$u=\left(c_2^2+c_3^2+1\right)^{1/2}x_0+c_2x_3+c_3\left(x_1^2+x_2^2\right)^{1/2}+c_1,$$

where c_1, c_2 and c_3 are arbitrary real constants.

4) $\langle L_3+\alpha\left(X_0+X_4\right), \alpha>0\rangle$:

The common invariant solution for the equations under study:

$$u=i\alpha c_2 \operatorname{arctanh}\frac{c_2\alpha}{\left(\left(c_1^2-c_2^2+1\right)\left(x_1^2+x_2^2\right)+c_2^2\alpha^2\right)^{1/2}}$$
$$-i\left(\left(c_1^2-c_2^2+1\right)\left(x_1^2+x_2^2\right)+c_2^2\alpha^2\right)^{1/2}+c_2\left(x_0-\alpha\arctan\frac{x_1}{x_2}\right)+c_1x_3+c_3.$$

5) $\langle L_3+\alpha X_3, \alpha>0\rangle$:

The common invariant solution for the equations under study:

$$u=\sqrt{\left(c_1^2-c_2^2-1\right)\left(x_1^2+x_2^2\right)-\alpha^2c_2^2}+c_2\alpha\arctan\frac{x_1}{x_2}$$
$$-c_2\alpha\arctan\left(\frac{\sqrt{\left(c_1^2-c_2^2-1\right)\left(x_1^2+x_2^2\right)-\alpha^2c_2^2}}{c_2\alpha}\right)+c_2x_3+c_1x_0+c_3.$$

6) $\langle L_3+2X_4\rangle$:

The common invariant solution for the equations under study:

$$x_0-u+2\arctan\frac{x_2}{x_1}$$
$$=i\sqrt{\left(c_2^2+4c_1\right)\left(x_1^2+x_2^2\right)+4}-2i\operatorname{arctanh}\left(\frac{2}{\sqrt{\left(c_2^2+4c_1\right)\left(x_1^2+x_2^2\right)+4}}\right)$$
$$+c_1\left(x_0+u\right)+c_2x_3+c_3.$$

7) $\langle P_3-2X_0\rangle$:

The common invariant solution for the equations under study:

$$\frac{1}{6}\left(x_0+u\right)^3+x_3\left(x_0+u\right)+x_0-u$$
$$=-\frac{1}{6}\left(\left(x_0+u\right)^2+4x_3-c_1^2-c_2^2\right)^{3/2}+c_1x_1+c_2x_2+c_3,$$

where c_1, c_2 and c_3 are arbitrary real constants.

8) $\langle X_0+X_4\rangle$:

The common invariant solution for the equations under study:

$$u = i\left(c_2^2 + c_3^2 + 1\right)^{1/2} x_1 + c_2 x_2 + c_3 x_3 + c_1 ,$$

where c_1, c_2 and c_3 are arbitrary real constants.

9) $\langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$x_3 = -i\left(c_2^2 + 1\right)^{1/2} x_1 + c_2 x_2 + c_1 + f\left(x_0 + u\right),$$

where: c_1, c_2 are arbitrary real constants, f is an arbitrary smooth function.

3.2. Classification of the Common Invariant Solutions for the Equations under Study Using Two-Dimensional Nonconjugate Subalgebras of the Lie Algebra of the Group $P(1,4)$

3.2.1. Lie Algebras of the Type $2A_1$

1) $\langle G \rangle \oplus \langle L_3 \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - u^2\right)^{1/2} = \left(1 - c_2^2\right)^{1/2} x_3 + c_2\left(x_1^2 + x_2^2\right)^{1/2} + c_1,$$

where c_1, c_2 are arbitrary real constants.

2) $\langle G + \alpha X_3, \alpha > 0 \rangle \oplus \langle L_3 \rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} & x_3 - \alpha \ln\left(x_0 + u\right) \\ & = -\sqrt{\left(c_1^2 + 1\right)\left(x_0^2 - u^2\right) + \alpha^2} + \alpha \ln\left(\frac{2\alpha\left(\sqrt{\left(c_1^2 + 1\right)\left(x_0^2 - u^2\right) + \alpha^2} + \alpha\right)}{x_0^2 - u^2}\right) \\ & \quad + c_1\sqrt{x_1^2 + x_2^2} + c_2, \end{aligned}$$

where c_1, c_2 are arbitrary real constants.

3) $\langle G \rangle \oplus \langle L_3 + \alpha X_3, \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} & x_3 + \alpha \arctan\frac{x_1}{x_2} \\ & = \alpha \arctan\frac{\alpha}{\sqrt{\left(c_2^2 - 1\right)\left(x_1^2 + x_2^2\right) - \alpha^2}} + c_2\left(x_0^2 - u^2\right)^{1/2} \\ & \quad + \sqrt{\left(c_2^2 - 1\right)\left(x_1^2 + x_2^2\right) - \alpha^2} + c_1, \end{aligned}$$

where c_1, c_2 are arbitrary real constants.

4) $\langle G \rangle \oplus \langle X_1 \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - u^2\right)^{1/2} = \varepsilon\left(1 - c_2^2\right)^{1/2} x_2 + c_2 x_3 + c_1, \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

5) $\langle G + \alpha X_2, \alpha > 0 \rangle \oplus \langle X_1 \rangle$:

The common invariant solution for the equations under study:

$$x_3+\sqrt{\left(c_2^2+1\right)\left(x_0^2-u^2\right)+\alpha^2c_2^2}$$
$$=\alpha c_2\operatorname{arctanh}\frac{\alpha c_2}{\sqrt{\left(c_2^2+1\right)\left(x_0^2-u^2\right)+\alpha^2c_2^2}}+\frac{\alpha c_2}{2}\ln\frac{x_0-u}{x_0+u}+c_2x_2+c_1,$$

where c_1, c_2 are arbitrary constants.

6) $\langle L_3\rangle\oplus\langle P_3+C_3\rangle$:

The common invariant solution for the equations under study:

$$\left(u^2+x_3^2\right)^{1/2}=\left(c_2^2+1\right)^{1/2}x_0+c_2\left(x_1^2+x_2^2\right)^{1/2}+c_1,$$

where c_1, c_2 are arbitrary constants.

7) $\langle L_3+\alpha\left(X_0+X_4\right),\alpha>0\rangle\oplus\langle P_3+C_3\rangle$:

The common invariant solution for the equations under study:

$$x_0-\alpha\arctan\frac{x_1}{x_2}$$
$$=\alpha\arctan\frac{\alpha}{\sqrt{\left(1-c_2^2\right)\left(x_1^2+x_2^2\right)-\alpha^2}}+c_2\sqrt{u^2+x_3^2}$$
$$+\sqrt{\left(1-c_2^2\right)\left(x_1^2+x_2^2\right)-\alpha^2}+c_1,$$

where c_1, c_2 are arbitrary constants.

8) $\langle L_3\rangle\oplus\langle X_0+X_4\rangle$:

The common invariant solution for the equations under study:

$$u=i\varepsilon\left(c_2^2+1\right)^{1/2}\left(x_1^2+x_2^2\right)^{1/2}+c_2x_3+c_1,\ \varepsilon=\pm1,$$

where c_1, c_2 are arbitrary constants.

9) $\langle L_3+\alpha\left(X_0+X_4\right),\alpha>0\rangle\oplus\langle X_4\rangle$:

The common invariant solution for the equations under study:

$$u=\alpha\arctan\frac{x_1}{x_2}+i\sqrt{c_1^2\left(x_1^2+x_2^2\right)+\alpha^2}$$
$$-i\alpha\operatorname{arctanh}\frac{\alpha}{\sqrt{c_1^2\left(x_1^2+x_2^2\right)+\alpha^2}}-x_0+c_1x_3+c_2,$$

where c_1, c_2 are arbitrary constants.

10) $\langle L_3+\alpha X_3,\alpha>0\rangle\oplus\langle X_0+X_4\rangle$:

The common invariant solution for the equations under study:

$$u=\frac{\alpha}{c_1}\arctan\left(\frac{x_1\sqrt{\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+\alpha^2}-i\alpha x_2}{x_2\sqrt{\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+\alpha^2}+i\alpha x_1}\right)$$
$$+\frac{i}{c_1}\sqrt{\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+\alpha^2}+\frac{x_3}{c_1}+c_2,\ c_1\neq0.$$

11) $\langle L_3+2X_4\rangle\oplus\langle X_3\rangle$:

The common invariant solution for the equations under study:

$$x_0-u+2\arctan\frac{x_2}{x_1}$$

$$= 2i\operatorname{arctanh}\frac{1}{\sqrt{c_1\left(x_1^2+x_2^2\right)+1}} - 2i\sqrt{c_1\left(x_1^2+x_2^2\right)+1} + c_1\left(x_0+u\right) + c_2,$$

where c_1, c_2 are arbitrary constants.

12) $\langle L_3 - P_3 + 2\alpha X_0, \alpha \neq 0\rangle \oplus \langle X_4\rangle$:

The common invariant solution for the equations under study:

$$x_0 + u - 2\alpha\arctan\frac{x_1}{x_2}$$
$$= 2i\alpha\varepsilon\sqrt{4c_2^2\left(x_1^2+x_2^2\right)+1} - 2i\alpha\varepsilon\operatorname{arctanh}\frac{1}{\sqrt{4c_2^2\left(x_1^2+x_2^2\right)+1}}$$
$$+ c_2\left(\left(x_0+u\right)^2 + 4\alpha x_3\right) + c_1, \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

13) $\langle L_3\rangle \oplus \langle P_3 - 2X_0\rangle$:

The common invariant solution for the equations under study:

$$\frac{1}{6}\left(x_0+u\right)^3 + x_3\left(x_0+u\right) + x_0 - u = c_1\sqrt{x_1^2+x_2^2} - \frac{1}{6}\left(\left(x_0+u\right)^2 + 4x_3 - c_1^2\right)^{3/2} + c_2,$$

where c_1, c_2 are arbitrary constants.

14) $\langle P_1\rangle \oplus \langle P_2\rangle$:

The common invariant solution for the equations under study:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = 0.$$

15) $\langle P_1 - X_3\rangle \oplus \langle P_2\rangle$:

The common invariant solution for the equations under study:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = 0.$$

16) $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2 - \beta X_3, \beta > 0, \gamma > 0\rangle$:

The common invariant solution for the equations under study:

$$\frac{x_1\left(x_1 - 2c_2\right)}{x_0+u} + \frac{\left(x_2 - \beta c_2\right)^2 + c_2^2}{x_0+u+\gamma} + \frac{\gamma c_2^2}{\left(x_0+u\right)\left(x_0+u+\gamma\right)}$$
$$-\left(c_2^2+1\right)\left(x_0+u\right) + 2c_2x_3 + 2u + c_1 = 0,$$

where c_1, c_2 are arbitrary constants.

17) $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2, \gamma > 0\rangle$:

The common invariant solution for the equations under study:

$$\frac{\left(x_1 - c_2\right)^2}{x_0+u} + \frac{x_2^2}{x_0+u+\gamma} + 2u = \left(c_2^2+1\right)\left(x_0+u\right) - 2c_2x_3 + c_1,$$

where c_1, c_2 are arbitrary constants.

18) $\langle P_1\rangle \oplus \langle P_2 - X_2 - \beta X_3, \beta > 0\rangle$:

The common invariant solution for the equations under study:

$$\frac{x_1^2}{x_0+u} + 2u = \left(\frac{c_2^2}{4}+1\right)\left(x_0+u\right) - \frac{\left(\beta c_2 + 2x_2\right)^2}{4\left(x_0+u+1\right)} + c_2x_3 + c_1,$$

where c_1, c_2 are arbitrary constants.

19) $\langle P_1 \rangle \oplus \langle P_2 - X_2 \rangle$:

The common invariant solution for the equations under study:

$$\frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u+1} + 2u = \left(\frac{c_2^2}{4}+1\right)(x_0+u) + c_2 x_3 + c_1,$$

where c_1, c_2 are arbitrary constants.

20) $\langle P_3 - 2X_0 \rangle \oplus \langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$u = \pm\sqrt{c_2 x_2 - 4x_3 - i\sqrt{c_2^2+16}\, x_1 + c_1} - x_0,$$

where c_1, c_2 are arbitrary constants.

21) $\langle P_3 - 2X_0 \rangle \oplus \langle X_1 \rangle$:

The common invariant solution for the equations under study:

$$\frac{1}{6}(x_0+u)^3 + x_3(x_0+u) + x_0 - u = \varepsilon c_1 x_2 - \frac{\varepsilon}{6}\left((x_0+u)^2 + 4x_3 - c_1^2\right)^{3/2} + c_2,$$

where c_1, c_2 are arbitrary constants.

22) $\langle L_3 \rangle \oplus \langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$\left(x_1^2 + x_2^2\right)^{1/2} = i\varepsilon x_3 + f(x_0+u),\ \varepsilon = \pm 1,$$

where f is an arbitrary smooth function.

23) $\langle L_3 + \alpha X_3, \alpha > 0 \rangle \oplus \langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$x_3 + \alpha \arctan\frac{x_1}{x_2}$$
$$= i\varepsilon\sqrt{x_1^2 + x_2^2 + \alpha^2} - i\varepsilon\alpha \operatorname{arctanh}\frac{\alpha}{\sqrt{x_1^2+x_2^2+\alpha^2}} + f(x_0+u),\ \varepsilon = \pm 1,$$

where f is an arbitrary smooth function.

24) $\langle P_3 - X_1 \rangle \oplus \langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$x_1 - \frac{x_3}{x_0+u} = i\varepsilon x_2 \sqrt{\frac{1}{(x_0+u)^2} + 1} + f(x_0+u),\ \varepsilon = \pm 1,$$

where f is an arbitrary smooth function.

25) $\langle P_3 \rangle \oplus \langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$x_1 = i\varepsilon x_2 + f(x_0+u),\ \varepsilon = \pm 1,$$

where f is an arbitrary smooth function.

26) $\langle X_1 \rangle \oplus \langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$x_3 = i\varepsilon x_2 + f(x_0+u),\ \varepsilon = \pm 1,$$

where f is an arbitrary smooth function.

3.2.2. Lie Algebras of the Type A_2

1) $\langle -G, P_3 \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - x_3^2 - u^2\right)^{1/2} = \varepsilon\sqrt{1-c_2^2}\,x_1 + c_2 x_2 + c_1,\ \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

2) $\left\langle -G - \frac{1}{\lambda}L_3,\ X_4,\ \lambda > 0 \right\rangle$:

The common invariant solution for the equations under study:

$$\ln\left(x_0 + u\right) = i\lambda \operatorname{arctanh}\frac{\lambda}{\sqrt{c_1^2\left(x_1^2 + x_2^2\right) + \lambda^2}} - i\sqrt{c_1^2\left(x_1^2 + x_2^2\right) + \lambda^2}$$
$$- \lambda \arctan\frac{x_1}{x_2} + c_1 x_3 + c_2,$$

where c_1, c_2 are arbitrary constants.

3) $\langle -G - \alpha X_1,\ X_4,\ \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$x_1 - \alpha\ln\left(x_0 + u\right) = i\varepsilon\left(c_2^2 + 1\right)^{1/2} x_2 + c_2 x_3 + c_1,\ \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

4) $\left\langle -\frac{1}{\lambda}\left(L_3 + \lambda G + \alpha X_3\right),\ X_4,\ \alpha > 0, \lambda > 0 \right\rangle$:

The common invariant solution for the equations under study:

$$\ln\left(x_0 + u\right) = i\varepsilon\sqrt{c_2^2\left(x_1^2 + x_2^2\right) + \left(\alpha c_2 - \lambda\right)^2}$$
$$- i\varepsilon\left(\alpha c_2 - \lambda\right)\operatorname{arctanh}\frac{\alpha c_2 - \lambda}{\sqrt{c_2^2\left(x_1^2 + x_2^2\right) + \left(\alpha c_2 - \lambda\right)^2}}$$
$$+ \left(\alpha c_2 - \lambda\right)\arctan\frac{x_1}{x_2} + c_2 x_3 + c_1,\ \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

5) $\langle -G - \alpha X_1,\ P_3,\ \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$x_1 - \alpha\ln\left(x_0 + u\right) = \alpha\ln\left(2\alpha\frac{\sqrt{\left(c_1^2 + 1\right)\left(x_0^2 - x_3^2 - u^2\right) + \alpha^2} + \alpha}{x_0^2 - x_3^2 - u^2}\right)$$
$$- \sqrt{\left(c_1^2 + 1\right)\left(x_0^2 - x_3^2 - u^2\right) + \alpha^2} + c_1 x_2 + c_2,$$

where c_1, c_2 are arbitrary constants.

3.3. Classification of the Common Invariant Solutions for the Equations under Study Using Three-Dimensional Nonconjugate Subalgebras of the Lie Algebra of the Group $P(1,4)$

3.3.1. Lie Algebras of the Type $3A_1$

1) $\langle P_1 - \gamma X_3, \gamma > 0 \rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta \neq 0 \rangle \oplus \langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$(x_0+u)^4+2(x_0+u)^3+(\gamma^2+\delta^2+1)(x_0+u)^2+2\gamma^2(x_0+u)+\gamma^2=0.$$

2) $\langle P_1\rangle\oplus\langle P_2-X_2-\delta X_3,\delta>0\rangle\oplus\langle X_4\rangle$:

The common invariant solution for the equations under study:

$$(x_0+u)^2+2(x_0+u)+\delta^2+1=0.$$

3) $\langle P_1\rangle\oplus\langle P_2\rangle\oplus\langle X_3\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_1^2-x_2^2-u^2=c(x_0+u),$$

where c is an arbitrary constant.

4) $\langle P_3\rangle\oplus\langle X_1\rangle\oplus\langle X_2\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_3^2-u^2=c(x_0+u),$$

where c is an arbitrary constant.

5) $\langle P_1\rangle\oplus\langle P_2\rangle\oplus\langle P_3\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_1^2-x_2^2-x_3^2-u^2=c(x_0+u),$$

where c is an arbitrary constant.

6) $\langle P_1\rangle\oplus\langle P_2-X_2\rangle\oplus\langle X_3\rangle$:

The common invariant solution for the equations under study:

$$\frac{x_0^2-x_1^2-u^2}{x_0+u}-\frac{x_2^2}{x_0+u+1}=c,$$

where c is an arbitrary constant.

7) $\langle P_1\rangle\oplus\langle P_2-\alpha X_2,\alpha>0\rangle\oplus\langle P_3-\gamma X_3,\gamma\neq 0\rangle$:

The common invariant solution for the equations under study:

$$2u+\frac{x_1^2}{x_0+u}+\frac{x_2^2}{x_0+u+\alpha}+\frac{x_3^2}{x_0+u+\gamma}=x_0+u+c,$$

where c is an arbitrary constant.

8) $\langle P_1\rangle\oplus\langle P_2-\alpha X_2,\alpha>0\rangle\oplus\langle P_3\rangle$:

The common invariant solution for the equations under study:

$$2u+\frac{x_1^2+x_3^2}{x_0+u}+\frac{x_2^2}{x_0+u+\alpha}=x_0+u+c,$$

where c is an arbitrary constant.

9) $\langle G\rangle\oplus\langle X_2\rangle\oplus\langle X_1\rangle$:

The common invariant solution for the equations under study:

$$(x_0^2-u^2)^{1/2}=\varepsilon x_3+c,\ \varepsilon=\pm 1,$$

where c is an arbitrary constant.

10) $\langle G\rangle\oplus\langle L_3\rangle\oplus\langle X_3\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2-u^2\right)^{1/2}=\varepsilon\left(x_1^2+x_2^2\right)^{1/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

11) $\langle P_3-2X_0\rangle\oplus\langle X_1\rangle\oplus\langle X_2\rangle$:

The common invariant solution for the equations under study:

$$\frac{1}{6}\left(x_0+u\right)^3+x_3\left(x_0+u\right)+x_0-u=\frac{\varepsilon}{6}\left(\left(x_0+u\right)^2+4x_3\right)^{3/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

12) $\langle G+\alpha X_3,\alpha>0\rangle\oplus\langle X_1\rangle\oplus\langle X_2\rangle$:

The common invariant solution for the equations under study:

$$x_3-\alpha\ln\left(x_0+u\right)$$
$$=\varepsilon\left(\alpha^2+x_0^2-u^2\right)^{1/2}-\frac{\alpha}{2}\ln\left(x_0^2-u^2\right)-\varepsilon\alpha\operatorname{arctanh}\frac{\left(\alpha^2+x_0^2-u^2\right)^{1/2}}{\alpha}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

13) $\langle L_3\rangle\oplus\langle P_3+C_3\rangle\oplus\langle X_0+X_4\rangle$:

The common invariant solution for the equations under study:

$$\left(x_3^2+u^2\right)^{1/2}=i\varepsilon\left(x_1^2+x_2^2\right)^{1/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

14) $\langle L_3+\alpha\left(X_0+X_4\right),\alpha>0\rangle\oplus\langle X_3\rangle\oplus\langle X_4\rangle$:

The common invariant solution for the equations under study:

$$x_0+u+\alpha\arctan\frac{x_2}{x_1}=i\frac{\varepsilon\alpha}{2}\ln\left(x_1^2+x_2^2\right)+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

15) $\langle P_3-2X_0\rangle\oplus\langle X_1\rangle\oplus\langle X_4\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0+u\right)^2+4x_3=4i\varepsilon x_2+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

16) $\langle L_3\rangle\oplus\langle -P_3+2X_0\rangle\oplus\langle 2X_4\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0+u\right)^2+4x_3=4i\varepsilon\left(x_1^2+x_2^2\right)^{1/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

3.3.2. Lie Algebras of the Type $A_2\oplus A_1$

1) $\langle -G,P_3\rangle\oplus\langle X_1\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2-x_3^2-u^2\right)^{1/2}=\varepsilon x_2+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

2) $\langle -G, P_3 \rangle \oplus \langle L_3 \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - x_3^2 - u^2\right)^{1/2} = \varepsilon\left(x_1^2 + x_2^2\right)^{1/2} + c,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3) $\langle -(G + \alpha X_2), P_3, \alpha > 0 \rangle \oplus \langle X_1 \rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} & x_2 - \alpha \ln\left(x_0 + u\right) \\ & = \varepsilon\left(x_0^2 - x_3^2 - u^2 + \alpha^2\right)^{1/2} - \frac{\alpha}{2}\ln\left(x_0^2 - x_3^2 - u^2\right) \\ & \quad - \varepsilon\alpha \operatorname{arctanh}\frac{\alpha}{\left(x_0^2 - x_3^2 - u^2 + \alpha^2\right)^{1/2}} + c,\ \varepsilon = \pm 1, \end{aligned}$$

where c is an arbitrary constant.

4) $\left\langle -\frac{1}{\lambda} L_3 - G, 2X_4, \lambda > 0 \right\rangle \oplus \langle X_3 \rangle$:

The common invariant solution for the equations under study:

$$\ln\left(x_0 + u\right) + \lambda \arctan\frac{x_1}{x_2} = i\frac{\varepsilon\lambda}{2}\ln\left(x_1^2 + x_2^2\right) + c,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

5) $\langle -(G + \alpha X_3), X_4, \alpha > 0 \rangle \oplus \langle L_3 + \beta X_3, \beta > 0 \rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} & x_3 - \alpha \ln\left(x_0 + u\right) + \beta \arctan\frac{x_1}{x_2} \\ & = -i\varepsilon\beta \operatorname{arctanh}\frac{\beta}{\left(x_1^2 + x_2^2 + \beta^2\right)^{1/2}} + i\varepsilon\left(x_1^2 + x_2^2 + \beta^2\right)^{1/2} + c,\ \varepsilon = \pm 1, \end{aligned}$$

where c is an arbitrary constant.

6) $\langle -(G + \alpha X_3), X_4, \alpha > 0 \rangle \oplus \langle L_3 \rangle$:

The common invariant solution for the equations under study:

$$x_3 - \alpha \ln\left(x_0 + u\right) = i\varepsilon\left(x_1^2 + x_2^2\right)^{1/2} + c,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3.3.3. Lie Algebras of the Type $A_{3,1}$

1) $\langle 4X_4, P_1 - X_2 - \gamma X_3, P_2 + X_1 - \mu X_2 - \delta X_3, \gamma > 0, \delta \neq 0, \mu > 0 \rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} & \left(x_0 + u\right)^4 + 2\mu\left(x_0 + u\right)^3 + \left(\gamma^2 + \mu^2 + \delta^2 + 2\right)\left(x_0 + u\right)^2 \\ & + 2\mu\left(\gamma^2 + 1\right)\left(x_0 + u\right) + \left(\gamma\mu - \delta\right)^2 + \gamma^2 + 1 = 0. \end{aligned}$$

2) $\langle 2\mu X_4, P_3 - 2X_0, X_1 + \mu X_3, \mu > 0 \rangle$:

The common invariant solution for the equations under study:

$$u = 2\left(i\varepsilon x_2\sqrt{\mu^2 + 1} + \mu x_1 - x_3 + c\right)^{1/2} - x_0,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3) $\langle 2X_4, P_3 - L_3 - 2\alpha X_0, X_3, \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$u = 2\alpha \arctan \frac{x_1}{x_2} + i\varepsilon\alpha \ln\left(x_1^2 + x_2^2\right) - x_0 + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

4) $\langle -2\beta X_4, L_3 + \beta X_3, P_3 - 2X_0, \beta > 0 \rangle$:

The common invariant solution for the equations under study:

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2$$
$$= i\varepsilon\sqrt{x_1^2 + x_2^2 + \beta^2} - i\varepsilon\beta \operatorname{arctanh} \frac{\beta}{\sqrt{x_1^2 + x_2^2 + \beta^2}} - x_3 + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

5) $\langle 2X_4, P_3, X_3 \rangle$:

The common invariant solution for the equations under study:

$$x_2 = i\varepsilon x_1 + f(x_0 + u), \varepsilon = \pm 1,$$

where f is an arbitrary smooth function.

3.3.4. Lie Algebras of the Type $A_{3,2}$

1) $\left\langle 2\alpha X_4, \lambda P_3, \frac{1}{\lambda} L_3 + G + \frac{\alpha}{\lambda} X_3, \alpha > 0, \lambda > 0 \right\rangle$:

The common invariant solution for the equations under study:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = i\varepsilon \frac{\lambda}{2} \ln\left(x_1^2 + x_2^2\right) + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3.3.5. Lie Algebras of the Type $A_{3,3}$

1) $\langle P_1, P_2, G \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - x_1^2 - x_2^2 - u^2\right)^{1/2} = \varepsilon x_3 + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

2) $\langle P_1, P_2, G + \alpha X_3, \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$x_3 - \alpha \ln(x_0 + u)$$
$$= \varepsilon\left(x_0^2 - x_1^2 - x_2^2 - u^2 + \alpha^2\right)^{1/2} - i\varepsilon\alpha \arctan \frac{\left(x_0^2 - x_1^2 - x_2^2 - u^2 + \alpha^2\right)^{1/2}}{i\alpha}$$
$$- \frac{\alpha}{2} \ln\left(x_0^2 - x_1^2 - x_2^2 - u^2\right) + c, \quad \varepsilon = \pm 1.$$

3) $\left\langle P_3, X_4, \frac{1}{\lambda} L_3 + G, \lambda > 0 \right\rangle$:

The common invariant solution for the equations under study:

$$\ln\left(x_0+u\right)+\lambda\arctan\frac{x_1}{x_2}=i\varepsilon\frac{\lambda}{2}\ln\left(x_1^2+x_2^2\right)+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

3.3.6. Lie Algebras of the Type $A_{3,6}$

1) $\langle P_1-X_1, P_2-X_2, -P_3+L_3\rangle$:

The common invariant solution for the equations under study:

$$\frac{x_1^2+x_2^2}{x_0+u+1}+\frac{x_3^2}{x_0+u}+2u=x_0+u+c,$$

where c is an arbitrary constant.

2) $\langle P_1, -P_2, -\left(L_3+\alpha X_3\right), \alpha>0\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_1^2-x_2^2-u^2=c\left(x_0+u\right),$$

where c is an arbitrary constant.

3) $\langle P_1, P_2, -P_3+L_3\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_1^2-x_2^2-x_3^2-u^2=c\left(x_0+u\right),$$

where c is an arbitrary constant.

4) $\langle X_1, -X_2, P_3-L_3-2\alpha X_0, \alpha>0\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0+u\right)^3+6\alpha x_3\left(x_0+u\right)+6\alpha^2\left(x_0-u\right)=\varepsilon\left(\left(x_0+u\right)^2+4\alpha x_3\right)^{3/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

5) $\left\langle X_1, -X_2, -L_3-\frac{1}{2}\left(P_3+C_3\right)-\alpha\left(X_0+X_4\right), \alpha>0\right\rangle$:

The common invariant solution for the equations under study:

$$\alpha\arctan\frac{x_3}{u}-x_0=\varepsilon\sqrt{x_3^2+u^2-\alpha^2}+\varepsilon\alpha\arctan\frac{\alpha}{\sqrt{x_3^2+u^2-\alpha^2}}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

6) $\left\langle X_1, X_2, L_3+\frac{\lambda}{2}\left(P_3+C_3\right)+\alpha\left(X_0+X_4\right), \alpha>0, 0<\lambda<1\right\rangle$:

The common invariant solution for the equations under study:

$$\alpha\arctan\frac{x_3}{u}-\lambda x_0$$
$$=\varepsilon\sqrt{\lambda^2\left(x_3^2+u^2\right)-\alpha^2}+\varepsilon\alpha\arctan\frac{\alpha}{\sqrt{\lambda^2\left(x_3^2+u^2\right)-\alpha^2}}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

7) $\langle X_1, X_2, L_3+\lambda G+\alpha X_3, \alpha>0, \lambda>0\rangle$:

The common invariant solution for the equations under study:

$$\lambda x_3 - \alpha \ln(x_0 + u) = \varepsilon\sqrt{\lambda^2(x_0^2 - u^2) + \alpha^2} - \varepsilon\alpha \operatorname{arctanh}\frac{\alpha}{\sqrt{\lambda^2(x_0^2 - u^2) + \alpha^2}} - \frac{\alpha}{2}\ln(x_0^2 - u^2) + c,\ \varepsilon = \pm 1.$$

where c is an arbitrary constant.

8) $\langle P_1, P_2, L_3 \rangle$:

The common invariant solution for the equations under study:

$$x_3 = c_1 \ln(x_0 + u) - \varepsilon(x_0^2 - x_1^2 - x_2^2 - u^2 + c_1^2)^{1/2} + \varepsilon c_1 \operatorname{arctanh}\frac{\sqrt{x_0^2 - x_1^2 - x_2^2 - u^2 + c_1^2}}{c_1} - \frac{c_1}{2}\ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c_2,\ \varepsilon = \pm 1,\ c_1 \neq 0.$$

3.3.7. Lie Algebras of the Type $A_{3,7}^{a}$

1) $\langle P_1, P_2, L_3 + \lambda G, \lambda > 0 \rangle$:

The common invariant solution for the equations under study:

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varepsilon x_3 + c,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

2) $\langle P_1, P_2, L_3 + \lambda G + \alpha X_3, \alpha > 0, \lambda > 0 \rangle$:

The common invariant solution for the equations under study:

$$\lambda x_3 - \alpha \ln(x_0 + u) = \varepsilon(\lambda^2(x_0^2 - x_1^2 - x_2^2 - u^2) + \alpha^2)^{1/2} - \varepsilon\alpha \operatorname{arctanh}\frac{\sqrt{\lambda^2(x_0^2 - x_1^2 - x_2^2 - u^2) + \alpha^2}}{\alpha} - \frac{\alpha}{2}\ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3.3.8. Lie Algebras of the Type $A_{3,8}$

$\langle P_3, G, -C_3 \rangle$:

The common invariant solution for the equations under study:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varepsilon(1 - c_2^2)^{1/2} x_1 + c_2 x_2 + c_1,\ \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

3.3.9. Lie Algebras of the Type $A_{3,9}$

1) $\left\langle -\frac{1}{2}\left(L_3 + \frac{1}{2}(P_3 + C_3)\right), \frac{1}{2}\left(L_2 + \frac{1}{2}(P_2 + C_2)\right), \frac{1}{2}\left(L_1 + \frac{1}{2}(P_1 + C_1)\right) \right\rangle$:

The common invariant solution for the equations under study:

$$(x_1^2 + x_2^2 + x_3^2 + u^2)^{1/2} = \varepsilon x_0 + c,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

2) $\langle -L_3, -L_2, -L_1 \rangle$:

The common invariant solution for the equations under study:

$$u = \varepsilon \left(c_2^2 + 1\right)^{1/2} x_0 + c_2 \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2} + c_1, \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

4. Conclusions

In this paper, we have presented obtained common invariant solutions of the following (1 + 3)-dimensional equations: the Eikonal equations, the Euler-Lagrange-Born-Infeld equation, the homogeneous Monge-Ampère equation and the inhomogeneous Monge-Ampère equation. We have used the structural properties of the low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the same ranks of the Lie algebra of the Poincaré group $P(1,4)$ for classification of the obtained common invariant solutions.

Since the group $P(1,4)$ contains, as subgroups, the extended Galilei group $\tilde{G}(1,3)$ [49] (the symmetry group of classical physics) and the Poincaré group $P(1,3)$ (the symmetry group of relativistic physics), the results obtained can be useful in construction and investigation of corresponding physical models.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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On the Spectral Properties of Graphs with Rank 4

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Abstract

Let G be a graph and $A(G)$ the adjacency matrix of G. The spectrum of G is the eigenvalues together with their multiplicities of $A(G)$. Chang *et al.* (2011) characterized the structures of all graphs with rank 4. Monsalve and Rada (2021) gave the bound of spectral radius of all graphs with rank 4. Based on these results as above, we further investigate the spectral properties of graphs with rank 4. And we give the expressions of the spectral radius and energy of all graphs with rank 4. In particular, we show that some graphs with rank 4 are determined by their spectra.

Keywords

Spectral Radius, Energy, Cospectral Graphs, Rank

1. Introduction

All graphs considered in this paper are undirected, finite and simple. Let $G=(V(G),E(G))$ be a graph with n vertices and m edges. For convenience, the path, cycle and complete graph of order n are denoted by P_n, C_n and K_n, respectively. Let C be a set, and the number of elements in C is denoted by $|C|$, let $c_i(G)$ denote the number of cycles of length i.

The adjacency matrix of G is denoted by $A(G)$. The polynomial $\phi(G,x)=\det|xI-A(G)|$ is called the characteristic polynomial of a graph G, where I is the identity matrix of order n. The spectrum of G consists of the eigenvalues together with their multiplicities of $A(G)$. The spectral radius of G, denoted by $\rho(G)$, is the maximum eigenvalue of graph G. The nullity of G, denoted by $\eta(G)$, is the multiplicity of zeros in the spectrum of G. Let $r(G)$ be the rank of $A(G)$. Obviously, $\eta(G)=n-r(G)$. Two graphs G and H are said to be cospectral (denoted by $G\sim H$) if they share the same spectrum. A graph G is said to be determined by its spectrum (DS for short) if for any graph H, $\phi(G,x)=\phi(H,x)$ implies that H is isomorphic to G.

The energy of G first is defined by Gutman in 1978 as the sum of the absolute values of its eigenvalues. That is,

$$E(G)=\sum_{i=1}^{n}|\lambda_i|.$$

The theory of graph energy is well developed nowadays; its details can be found in the book [1] and reviews [2] [3].

Definition 1.1. ([4]) *Given a graph G with the set of vertices* $V(G)=\{v_1,v_2,\cdots,v_p\}$ *and a vector of positive integers* $m=(m_1,m_2,\cdots,m_p)$, *denote by* $Gom(m_1,m_2,\cdots,m_p)$ (*Gom for short*) *the graph obtained from G by replacing each vertex* v_i *of G with an independent set of* m_i *vertices* $v_i^1,v_i^2,\cdots,v_i^{m_i}$ *and joining* v_i^s *with* v_j^t *if and only if* v_i *and* v_j *are adjacent in G. The resulting graph Gom is said to be obtained from G by multiplication of vertices. For graph* $G_1,G_2,\cdots,G_k$, *we denote by* $N(G_1,G_2,\cdots,G_k)$ *the class of all graphs that can be obtained from one of the graphs in* $\{G_1,G_2,\cdots,G_k\}$ *by multiplication of vertices.*

By Definition 2.1, Chang *et al.* [4] characterized all connected graphs with rank 4. That is, if G is a connected graph with rank 4, then $G\in N(G_2,\cdots,G_9)$, the resulting graph, see **Figure 1**. Wu *et al.* [5] studied further the spectral properties of graphs with rank 4. They computed the characteristic polynomials of all graphs with rank 4. And they showed that some graphs with rank 4 are determined by their spectra. In particular, they proposed a problem: Which graphs with rank 4 are determined by their spectra? Recently, Monsalve and Rada [6] characterized spectral radius of all connected graphs with rank 4. A natural problem is: How to characterize the spectral radius of all graphs with rank 4?

In this paper, we intend to solve these two problems. Preliminaries are presented in Section 2. And we give the expressions of the spectral radius and energy of all graphs with rank 4 in Section 3. In Section 4, we consider which graphs

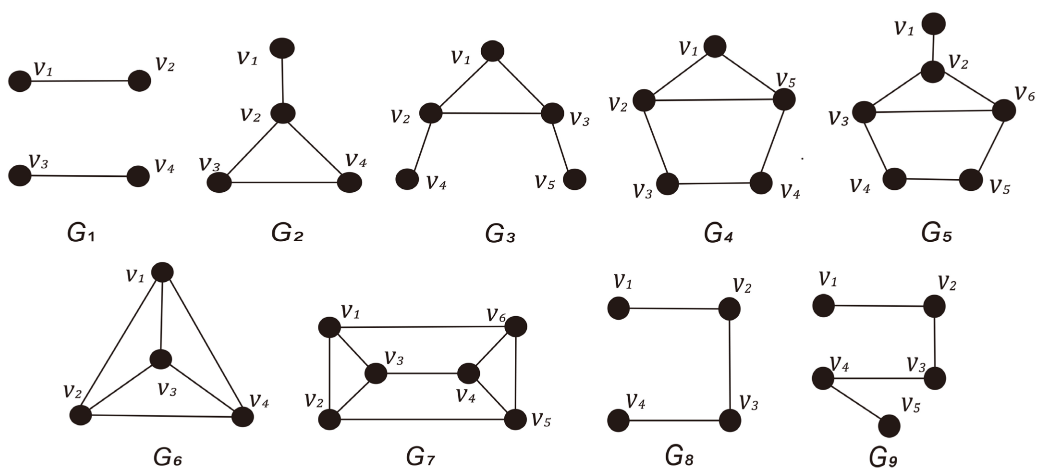

Figure 1. $G_1,G_2,G_3,G_4,G_5,G_6,G_7,G_8,G_9$.

with rank 4 are *DS*. More precisely, we prove that two classes of graphs with rank 4 are *DS*. And some cospectral graphs with rank 4 are presented.

2. Some Lemmas

Several lemmas are of importance to the description and proof of our results later, and we list them below.

By the properties of vertex multiplication, Wu *et al.* [5] computed the characteristic polynomials with rank 4 as follows.

Lemma 2.1. ([5]) *Let G be a simple graph on n vertices and* $n \geq 4$. *Then* $r(G)=4$ *if an only if* $G \in (G_1, G_2, \cdots, G_9)$, *where the graphs* $G_1, \cdots, G_9$ *are depicted in* **Figure 1**.

Lemma 2.2. ([7]) *Let G be a graph. For the adjacency matrix the following can be deduced from the spectrum*:

(i) *The number of vertices.*

(ii) *The number of edges.*

(iii) *Whether G is regular.*

(iv) *Whether G is regular with any fixed girth.*

(v) *The number of closed walk of any length.*

(vi) *Whether G is bipartite.*

Lemma 2.3. ([8]) *Let G be a simple bipartite graph with e edges. Then*

$$\rho(G) \leq \sqrt{e}$$

with equality if G is a disjoint union of a complete bipartite graph and isolated vertices.

Lemma 2.4. ([5]) *Suppose that* $G_i' = G_i om\left[m_1, m_2, \cdots, m_p\right]$, *where* G_i *is depicted in* **Figure 1**, $i = 1, 2, \cdots, 9$, $p = \left|V(G_i) \leq 6\right|$, $|m_1| = a$, $|m_2| = b$, $|m_3| = c$, $|m_4| = d$, $|m_5| = e$, $|m_6| = f$. *Then each of the following holds.*

(i) $\phi(G_1', x) = x^{a+b+c+d-4}\left[x^4 - (ab+cd)x^2 + abcd\right]$.

(ii) $\phi(G_2', x) = x^{a+b+c+d-4}\left[x^4 - (ab+bc+bd+cd)x^2 - 2bcdx + abcd\right]$.

(iii) $$\phi(G_3', x) = x^{a+b+c+d+e-4}\left[x^4 - (ab+ac+bc+bd+ce)x^2 - 2abcx + abdc + abec + dbec\right].$$

(iv) $$\phi(G_4', x) = x^{a+b+c+d+e-4}\left[x^4 - (ab+ae+be+bc+cd+de)x^2 - 2abex + abed + aecd + abcd + abec\right].$$

(v) $$\phi(G_5', x) = x^{a+b+c+d+e+f-4}\left[x^4 - (ab+bc+bf+cf+cd+de+ef)x^2 - 2bcfx + abed + abcd + bdec + abcf + abef + bcef + fbcd + fbed\right].$$

(vi) $$\phi(G_6', x) = x^{a+b+c+d-4}\left[x^4 - (ab+ac+ad+bc+bd+cd)x^2 - 2(abc+abd+acd+bcd)x - 3abcd\right].$$

(vii)

$$\phi(G_7', x) = x^{a+b+c+d+e+f-4}\left[x^4 - (ab+ac+af+bc+be+cd+df+de+fe)x^2 - 2(abc+def)x + abed + abdf + acbe + aced + acef + afed + abcd + fcde + abcf + bcef + fbcd + fbed\right].$$

(viii) $\phi(G_8', x) = x^{a+b+c+d-4}\left[x^4 - (ab+bc+cd)x^2 + abcd\right]$.

(ix) $\phi(G_9', x) = x^{a+b+c+d+e-4}\left[x^4 - (ab+bc+cd+de)x^2 + abcd + bcde + abde\right]$.

3. The Spectral Radii and Energies of Graphs with Rank 4

In this section, we give the spectral radii and energies of graphs with rank 4. All the notation in this paragraph is followed in Lemma 2.4.

Theorem 3.1. *Let* $G \in N(G_1, G_2, \cdots, G_9)$. *Then the spectral radius of graph G as follows*:

(i) *If* $G \in N(G_1)$. *Then*

$$\rho(G) = \left(\left\lfloor \frac{n-2}{2} \right\rfloor \left\lceil \frac{n-2}{2} \right\rceil\right)^{\frac{1}{2}}.$$

(ii) *If* $G \in N(G_2)$ *and let* $c' = ab + bc + bd + cd, d' = bcd, e' = abcd$. *Then*

$$\rho(G) = \frac{1}{2}\left[\left(\frac{2}{3}c' + \Delta\right)^{\frac{1}{2}} + \left(\frac{4}{3}c' - \Delta + 4d' / \left(\frac{2}{3}c' + \Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right],$$

where

$$\Delta = \left[\sqrt[3]{4}\left((c')^2 + 12e'\right) + \left(-2(c')^3 + 108(d')^2 + 72e'c' + A^{\frac{1}{2}}\right)^{\frac{2}{3}}\right]$$
$$/\left[3\sqrt[3]{2}\left(-2(c')^3 + 108(d')^2 + 72e'c' + A^{\frac{1}{2}}\right)^{\frac{1}{3}}\right],$$

where

$$A = -432e'(c')^4 + 3456(e')^2(c')^2 - 432(d')^2(c')^3 + 15552e'c'(d')^2$$
$$-6912(e')^3 + 11664(d')^4.$$

(iii) *If* $G \in N(G_3)$ *and let* $c' = ab + ac + bc + bd + ce$, $d' = abc$, $e' = abdc + abec + dbec$. *Then*

$$\rho(G) = \frac{1}{2}\left[\left(\frac{2}{3}c' + \Delta\right)^{\frac{1}{2}} + \left(\frac{4}{3}c' - \Delta + 4d' / \left(\frac{2}{3}c' + \Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right],$$

where

$$\Delta = \left[\sqrt[3]{4}\left((c')^2 + 12e'\right) + \left(-2(c')^3 + 108(d')^2 + 72e'c' + A^{\frac{1}{2}}\right)^{\frac{2}{3}}\right]$$
$$/\left[3\sqrt[3]{2}\left(-2(c')^3 + 108(d')^2 + 72e'c' + A^{\frac{1}{2}}\right)^{\frac{1}{3}}\right],$$

where

$$A=-432e'(c')^4+3456(e')^2(c')^2-432(d')^2(c')^3+15552e'c'(d')^2$$
$$-6912(e')^3+11664(d')^4.$$

(iv) *If* $G\in N(G_4)$ *and let* $c'=ab+ae+be+bc+cd+de$, $d'=abe$, $e'=abed+aecd+abcd+abec$. *Then*

$$\rho(G)=\frac{1}{2}\left[\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}+\left(\frac{4}{3}c'-\Delta+4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right],$$

where

$$\Delta=\left[\sqrt[3]{4}\left((c')^2+12e'\right)+\left(-2(c')^3+108(d')^2+72e'c'+A^{\frac{1}{2}}\right)^{\frac{2}{3}}\right]$$
$$/\left[3\sqrt[3]{2}\left(-2(c')^3+108(d')^2+72e'c'+A^{\frac{1}{2}}\right)^{\frac{1}{3}}\right],$$

where

$$A=-432e'(c')^4+3456(e')^2(c')^2-432(d')^2(c')^3+15552e'c'(d')^2$$
$$-6912(e')^3+11664(d')^4.$$

(v) *If* $G\in N(G_5)$ *and let* $c'=ab+bc+bf+cf+cd+de+ef$, $d'=bcf$, $e'=abed+abcd+bdec+abcf+abef+bcef+fbcd+fbed$. *Then*

$$\rho(G)=\frac{1}{2}\left[\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}+\left(\frac{4}{3}c'-\Delta+4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right],$$

where

$$\Delta=\left[\sqrt[3]{4}\left((c')^2+12e'\right)+\left(-2(c')^3+108(d')^2+72e'c'+A^{\frac{1}{2}}\right)^{\frac{2}{3}}\right]$$
$$/\left[3\sqrt[3]{2}\left(-2(c')^3+108(d')^2+72e'c'+A^{\frac{1}{2}}\right)^{\frac{1}{3}}\right],$$

where

$$A=-432e'(c')^4+3456(e')^2(c')^2-432(d')^2(c')^3+15552e'c'(d')^2$$
$$-6912(e')^3+11664(d')^4.$$

(vi) *If* $G\in N(G_6)$ *and let* $c'=ab+ac+ad+bc+bd+cd$, $d'=abc+abd+acd+bcd$, $e'=abcd$. *Then*

$$\rho(G)=\frac{1}{2}\left[\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}+\left(\frac{4}{3}c'-\Delta+4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right],$$

where

$$\Delta=\left[\sqrt[3]{4}\left((c')^2-36e'\right)+\left(-2(c')^3+108(d')^2-216e'c'+A^{\frac{1}{2}}\right)^{\frac{2}{3}}\right]$$
$$/\left[3\sqrt[3]{2}\left(-2(c')^3+108(d')^2-216e'c'+A^{\frac{1}{2}}\right)^{\frac{1}{3}}\right],$$

where

$$A=1296e'(c')^4+31104(e')^2(c')^2-432(d')^2(c')^3-46656e'c'(d')^2$$
$$+1886624(e')^3+11664(d')^4.$$

(vii) *If* $G\in N(G_7)$ *and let* $c'=ab+ac+af+bc+be+cd+df+de+fe$, $d'=abc+def$, $e'=abed+abdf+acbe+aced+acef+afed+abcd+fcde+abcf+bcef+fbcd+fbed$. *Then*

$$\rho(G)=\frac{1}{2}\left[\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}+\left(\frac{4}{3}c'-\Delta+4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right],$$

where

$$\Delta=\left[\sqrt[3]{4}\left((c')^2+12e'\right)+\left(-2(c')^3+108(d')^2+72e'c'+A^{\frac{1}{2}}\right)^{\frac{2}{3}}\right]$$
$$/\left[3\sqrt[3]{2}\left(-2(c')^3+108(d')^2+72e'c'+A^{\frac{1}{2}}\right)^{\frac{1}{3}}\right],$$

where

$$A=-432e'(c')^4+3456(e')^2(c')^2-432(d')^2(c')^3+15552e'c'(d')^2$$
$$-6912(e')^3+11664(d')^4.$$

(viii) *If* $G\in N(G_8)$. *Then*

$$\rho(G)=\frac{1}{2}\left[2ab+2bc+2cd+2\left((ab+bc+cd)^2-4abcd\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

(ix) *If* $G\in N(G_9)$. *Then*

$$\rho(G)=\frac{1}{2}\left[2ab+2bc+2cd+2de+2\left((ab+bc+cd+de)^2-4(abcd+bcde+abde)\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

Proof. Here we only consider the cases $G\in N(G_1)$, $G\in N(G_2)$. The proof of other cases is quite similar to $G \in N(G_2)$ and is thus omitted.

Let $G \in N(G_1)$. By Lemma 2.3, directly yields $\rho(G) = \left(\left\lfloor \frac{n-2}{2} \right\rfloor \left\lceil \frac{n-2}{2} \right\rceil\right)^{\frac{1}{2}}$.

Let $G \in N(G_2)$. By Theorem 2.4 (ii), there exist 4 nonzero eigenvalues and all other eigenvalues are 0. So we only need to input polynomial $x^4 - (ab+bc+bd+cd)x^2 - 2bcdx + abcd$ in *maple* 13.0, we can get the nonzero eigenvalues of the graph G as follows.

$$\begin{aligned}\lambda_1 = \frac{1}{2}\Bigg[&\left(\frac{2}{3}(ab+bc+bd+cd)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F\Big/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\\ &+\Bigg(\frac{4}{3}(ab+bc+bd+cd)-\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}+3F\Big/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\\ &+4bcd\Big/\left(\frac{2}{3}(ab+bc+bd+cd)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F\Big/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\Bigg)^{\frac{1}{2}}\Bigg],\end{aligned}$$

$$\begin{aligned}\lambda_2 = \frac{1}{2}\Bigg[&\left(\frac{2}{3}(ab+bc+bd+cd)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F\Big/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\\ &-\Bigg(\frac{4}{3}(ab+bc+bd+cd)-\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}+3F\Big/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\\ &+4bcd\Big/\left(\frac{2}{3}(ab+bc+bd+cd)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F\Big/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\Bigg)^{\frac{1}{2}}\Bigg],\end{aligned}$$

$$\begin{aligned}\lambda_3 = -\frac{1}{2}\Bigg[&\left(\frac{2}{3}(ab+bc+bd+cd)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F\Big/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\\ &+\Bigg(\frac{4}{3}(ab+bc+bd+cd)-\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}+3F\Big/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\\ &-4bcd\Big/\left(\frac{2}{3}(ab+bc+bd+cd)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F\Big/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\Bigg)^{\frac{1}{2}}\Bigg],\end{aligned}$$

$$\lambda_4=-\frac{1}{2}\left[\left(\frac{2}{3}(ab+bc+bd+cd)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\right.$$

$$-\left(\frac{4}{3}(ab+bc+bd+cd)-\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}+3F/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right.$$

$$\left.\left.-4bcd/\left(\frac{2}{3}(ab+bc+bd+cd)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right].$$

where

$$\begin{aligned}D=&-a^3b^3-b^3c^3-b^3d^3-c^3d^3+33a^2b^2cd+30ab^2c^2d+30ab^2cd^2\\&+33abc^2d^2-6ab^3cd-3a^2b^3c-3a^2b^3d-3ab^3c^2-3ab^3d^2\\&-3b^3c^2d-3b^3cd^2-3b^2c^3d-3b^2cd^3-3bc^3d^2-3bc^2d^3+48b^2c^2d^2,\end{aligned}$$

$$\begin{aligned}E=&-3a^5b^5cd-12a^4b^5c^2d-12a^4b^5cd^2+12a^4b^4c^2d^2-18a^3b^5c^3d\\&-39a^3b^5c^2d^2-18a^3b^5cd^3+12a^3b^4c^3d^2+12a^3b^4c^2d^3-18a^3b^3c^3d^3\\&-12a^2b^5c^4d-45a^2b^5c^3d^2-45a^2b^5c^2d^3-12a^2b^5cd^4-12a^2b^4c^4d^2\\&+75a^2b^4c^3d^3-12a^2b^4c^2d^4+12a^2b^3c^4d^3+12a^2b^3c^3d^4+12a^2b^2c^4d^4\\&-3ab^5c^5d-21ab^5c^4d^2-36ab^5c^3d^3-21ab^5c^2d^4-3ab^5cd^5-12ab^4c^5d^2\\&+54ab^4c^4d^3+54ab^4c^3d^4-12ab^4c^2d^5-18ab^3c^5d^3+63ab^3c^4d^4\\&-18ab^3c^3d^5-12ab^2c^5d^4-12ab^2c^4d^5-3abc^5d^5-3b^5c^5d^2-9b^5c^4d^3\\&-9b^5c^3d^4-3b^5c^2d^5-9b^4c^5d^3+63b^4c^4d^4-9b^4c^3d^5-9b^3c^5d^4\\&-9b^3c^4d^5-3b^2c^5d^5,\end{aligned}$$

$$\begin{aligned}F=&-\frac{14}{9}abcd-\frac{1}{9}a^2b^2-\frac{2}{9}ab^2d-\frac{1}{9}b^2c^2-\frac{2}{9}b^2cd\\&-\frac{1}{9}b^2d^2-\frac{2}{9}c^2bd-\frac{2}{9}bcd^2-\frac{1}{9}c^2d^2.\end{aligned}$$

Due to $\lambda_3,\lambda_4<0$, $\lambda_1,\lambda_2>0$ and $\lambda_1>\lambda_2$, we can obviously get λ_1 is the spectral radius of graph *G*. Let $c'=ab+bc+bd+cd$, $d'=bcd$, $e'=abcd$, due to

$$\begin{aligned}D=&-a^3b^3-b^3c^3-b^3d^3-c^3d^3+33a^2b^2cd+30ab^2c^2d+30ab^2cd^2\\&+33abc^2d^2-6ab^3cd-3a^2b^3c-3a^2b^3d-3ab^3c^2-3ab^3d^2\\&-3b^3c^2d-3b^3cd^2-3b^2c^3d-3b^2cd^3-3bc^3d^2-3bc^2d^3+48b^2c^2d^2\\=&-(ab+bc+bd+cd)^3+54(bcd)^2+36abcd(ab+bc+bd+cd)\\=&-(c')^3+54(d')^2+36e'c'\end{aligned}$$

$$\begin{aligned}E=&-3a^5b^5cd-12a^4b^5c^2d-12a^4b^5cd^2+12a^4b^4c^2d^2-18a^3b^5c^3d\\&-39a^3b^5c^2d^2-18a^3b^5cd^3+12a^3b^4c^3d^2+12a^3b^4c^2d^3-18a^3b^3c^3d^3\\&-12a^2b^5c^4d-45a^2b^5c^3d^2-45a^2b^5c^2d^3-12a^2b^5cd^4-12a^2b^4c^4d^2\end{aligned}$$

$$
\begin{aligned}
&+75a^2b^4c^3d^3-12a^2b^4c^2d^4+12a^2b^3c^4d^3+12a^2b^3c^3d^4+12a^2b^2c^4d^4\\
&-3ab^5c^5d-21ab^5c^4d^2-36ab^5c^3d^3-21ab^5c^2d^4-3ab^5cd^5-12ab^4c^5d^2\\
&+54ab^4c^4d^3+54ab^4c^3d^4-12ab^4c^2d^5-18ab^3c^5d^3+63ab^3c^4d^4\\
&-18ab^3c^3d^5-12ab^2c^5d^4-12ab^2c^4d^5-3abc^5d^5-3b^5c^5d^2-9b^5c^4d^3\\
&-9b^5c^3d^4-3b^5c^2d^5-9b^4c^5d^3+63b^4c^4d^4-9b^4c^3d^5-9b^3c^5d^4\\
&-9b^3c^4d^5-3b^2c^5d^5\\
=&-3abcd\left(ab+bc+bd+cd\right)^4+24\left(abcd\right)^2\left(ab+bc+bd+cd\right)^2\\
&-3\left(bcd\right)^2\left(ab+bc+bd+cd\right)^3+108abcd\left(ab+bc+bd+cd\right)\left(bcd\right)^2\\
&-48\left(abcd\right)^3+81\left(bcd\right)^4\\
=&-3e'\left(c'\right)^4+24\left(e'\right)^2\left(c'\right)^2-3\left(d'\right)^2\left(c'\right)^3+108e'c'\left(d'\right)^2-48\left(e'\right)^3+81\left(d'\right)^4\\
=&\Big[-432e'\left(c'\right)^4+3456\left(e'\right)^2\left(c'\right)^2-432\left(d'\right)^2\left(c'\right)^3+15552e'c'\left(d'\right)^2\\
&-6912\left(e'\right)^3+11664\left(d'\right)^4\Big]/144\\
=&\,A/144
\end{aligned}
$$

$$
\begin{aligned}
F=&-\frac{14}{9}abcd-\frac{1}{9}a^2b^2-\frac{2}{9}ab^2d-\frac{1}{9}b^2c^2-\frac{2}{9}b^2cd\\
&-\frac{1}{9}b^2d^2-\frac{2}{9}c^2bd-\frac{2}{9}bcd^2-\frac{1}{9}c^2d^2\\
=&-\frac{1}{9}\left(\left(ab+bc+bd+cd\right)^2+12abcd\right)\\
=&-\frac{1}{9}\left(\left(c'\right)^2+12e'\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\\
=&\frac{1}{3}\left(-\left(c'\right)^3+54\left(d'\right)^2+36e'c'+6\left(A/144\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}\\
&-3\left(-\frac{1}{9}\left(\left(c'\right)^2+12e'\right)\right)/\left(-\left(c'\right)^3+54\left(d'\right)^2+36e'c'+6\left(A/144\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}\\
=&\left[\sqrt[3]{4}\left(\left(c'\right)^2+12e'\right)+\left(-2\left(c'\right)^3+108\left(d'\right)^2+72e'c'+A^{\frac{1}{2}}\right)^{\frac{2}{3}}\right]\\
&/\left[3\sqrt[3]{2}\left(-2\left(c'\right)^3+108\left(d'\right)^2+72e'c'+A^{\frac{1}{2}}\right)^{\frac{1}{3}}\right]\\
=&\,\Delta
\end{aligned}
$$

So we get

$$
\lambda_1=\frac{1}{2}\left[\left(\frac{2}{3}\left(ab+bc+bd+cd\right)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\right.
$$

$$+\left(\frac{4}{3}(ab+bc+bd+cd)-\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}+3F/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right.$$

$$\left.\left.+4bcd/\left(\frac{2}{3}(ab+bc+bd+cd)+\frac{1}{3}\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}-3F/\left(D+6E^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]$$

$$=\frac{1}{2}\left[\left(\frac{2}{3}(ab+bc+bd+cd)+\Delta\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(\frac{4}{3}(ab+bc+bd+cd)-\Delta+4bcd/\left(\frac{2}{3}(ab+bc+bd+cd)+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]$$

$$=\frac{1}{2}\left[\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}+\left(\frac{4}{3}c'-\Delta+4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]$$

$$=\rho(G)$$

It is consistent with the spectral radius obtained as above.

This completes the proof. □

Example 1. *Solve the spectral radius of graph* $G=G_4o(5,6,3,4,6)$.

By employing maple 13.0 *to calculate, we can get that* 1.5808, −5.3747, −9.5359, 13.3297 *are the nonzero eigenvalues of the graph G. By comparison, it is obvious that* 13.3297 *is the spectral radius of the graph G.*

Theorem 3.2. *Let* $G\in N(G_1,G_2,\cdots,G_9)$. *Then the energy of graph G as follows, where the notations is defined as same as above Theorem.*

(i) *If* $G\in N(G_1)$. *Then*

$$E(G)=2(ab)^{\frac{1}{2}}+2(cd)^{\frac{1}{2}}.$$

(ii) *If* $G\in N(G_2)$. *Then*

$$E(G)=2\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}.$$

(iii) *If* $G\in N(G_3)$. *Then*

$$E(G)=2\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}.$$

(iv) *If* $G\in N(G_4)$. *Then*

$$E(G)=2\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}.$$

(v) *If* $G\in N(G_5)$. *Then*

$$E(G)=2\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}.$$

(vi) *If* $G\in N(G_6)$. *Then*

$$E(G)=\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}+\left(\frac{4}{3}c'-\Delta+4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

(vii) *If* $G\in N(G_7)$. *Then*

$$E(G)=2\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}.$$

(viii) *If* $G\in N(G_8)$. *Then*

$$E(G)=\left[2ab+2bc+2cd+2\left((ab+bc+cd)^2-4abcd\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$
$$+\left[2ab+2bc+2cd-2\left((ab+bc+cd)^2-4abcd\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

(ix) *If* $G\in N(G_9)$. *Then*

$$E(G)=\left[2ab+2bc+2cd+2de+2\left((ab+bc+cd+de)^2-4(abcd+bcde\right.\right.$$
$$\left.\left.+abde)\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}+\left[2ab+2bc+2cd+2de-2\left((ab+bc+cd+de)^2\right.\right.$$
$$\left.\left.-4(abcd+bcde+abde)\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

Proof. Here we only consider the cases $G\in N(G_2)$. The proof of other cases is quite similar to $G\in N(G_2)$ and is thus omitted.

The proof of Theorem 3.2 follows from Theorem 3.1. So we have

$$\lambda_1=\frac{1}{2}\left[\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}+\left(\frac{4}{3}c'-\Delta+4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right],$$

$$\lambda_2=\frac{1}{2}\left[\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}-\left(\frac{4}{3}c'-\Delta+4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right],$$

$$\lambda_3=-\frac{1}{2}\left[\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}+\left(\frac{4}{3}c'-\Delta-4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right],$$

$$\lambda_4=-\frac{1}{2}\left[\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}-\left(\frac{4}{3}c'-\Delta-4d'/\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right].$$

Due to $\lambda_3,\lambda_4<0$, $\lambda_1,\lambda_2>0$, we get

$$\begin{aligned}E(G)&=|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4|\\&=\lambda_1+\lambda_2-\lambda_3-\lambda_4\\&=2\left(\frac{2}{3}c'+\Delta\right)^{\frac{1}{2}}\end{aligned}$$

It is consistent with the energy obtained as above.

This completes the proof. □

4. The Spectral Characterization of Graphs with Rank 4

In this section, we will investigate which graph $G\in N(G_1,G_2,\cdots,G_9)$ is *DS* and find some cospectral graphs.

Theorem 4.1. *Let* $G=G_4'\cup rK_1$, *where* $a=b=e=1$ *in* G_4'. *Then G is DS.*

Proof. Suppose that G has $3+c'+d'+r$ vertices. Checking G, we note that it only contains one triangle. This implies, by Lemma 2.2 (v), that if graph H is cospectral with G, then H must contain one triangle. By Lemma 2.1 and Lemma 2.4, $G_2'\cup gK_1$ (here $b=c=d=1$ in G_2'), $G_3'\cup mK_1$ (here $a=b=c=1$ in G_3') and $G_5'\cup wK_1$ (here $b=c=f=1$ in G_5') contains one triangle, respectively. It has been proved that $G_2'\cup gK_1$ (here $b=c=d=1$ in G_2') is *DS*. In the following we consider two cases.

Case 1. Assume that G and $G_3'\cup mK_1$ are cospectral and let $1\le d\le e$, $1\le c'\le d'$. Therefore, G and $G_3'\cup mK_1$ have the same vertices and coefficients of their characteristic polynomials. By Lemma 2.4 (iv) and (iii), we have

$$\begin{cases}3+c'+d'+r=3+d+e+m\\3+c'+d'+c'd'=3+d+e\\c'+d'+2c'd'=d+e+de\end{cases}$$

Solving the equation system as above, we obtain that $r-m=c'd'=de$, which implies $d=c'd'/e$ and $r-m>0$. By $3+c'+d'+r=3+d+e+m$ and $r-m>0$, we can obtain that $d+e>c'+d'$. Taking $d=c'd'/e$ into $d+e>c'+d'$, we obtain that $e^2-(c'+d')e+c'd'>0$. Solving this equation, we obtain that $e>d'$ or $e<c'$. However, by $c'd'=de$ and $1\le d\le e$, $1\le c'\le d'$, we obtain that $c'\le d\le e\le d'$ or $d\le c'\le d'\le e$, which in contradict with $e>d'$ or $e<c'$. Hence G and $G_3'\cup mK_1$ are not cospectral.

Case 2. Assume that G and $G_5'\cup wK_1$ are cospectral. Therefore, G and $G_5'\cup wK_1$ have the same vertices and coefficients of their characteristic polynomials. By Lemma 2.4 (iv) and (v), we obtain that

$$\begin{cases}3+c'+d'+r=3+a+d+e+w\\3+c'+d'+c'd'=3+a+d+e+ed\\c'+d'+2c'd'=a+d+e+ad+ae+2ed+aed\end{cases}$$

Solving the equation system as above, we have

$$\begin{cases} w-r=ed-c'd' \\ c'd'=ad+ae+ed+aed \\ c'+d'+c'd'=a+d+e+ed \end{cases}$$

By $c'd'=ad+ae+ed+aed$, we obtain that $c'=(ad+ae+ed+aed)/d'$. Taking $c'=(ad+ae+ed+aed)/d'$ into $c'+d'+c'd'=a+d+e+ed$, we obtain that $d'^2+(ad+ae+aed-a-d-e)d'+ad+ae+ed+aed=0$. Supposing the roots of the equation as above are d_{11},d_{12}, we have

$$\begin{cases} d_{11}+d_{12}=a(1-d)+d(1-ae)+e(1-a) \\ d_{11}d_{12}=ad+ae+ed+aed \end{cases}$$

By the definition of G_5', one has $a,e,d\ge 1$, which implies that $d_{11}+d_{12}\le 0$, $d_{11}d_{12}>0$. By $d_{11}d_{12}>0$, we know that d_{11},d_{12} are nonzero and have the same sign. However, by $d_{11}+d_{12}\le 0$, we know that $d_{11},d_{12}<0$. This contradicts the fact $d_{11},d_{12}>0$. Thus, G and $G_5'\cup wK_1$ are not cospectral.

Next, assume that $G_4o(1,1,c',d',1)$ and $G_4o(1,1,c'',d'',1)$ are cospectral and $c''<c'\le d'<d''$. Therefore, they have the same vertices and coefficients of their characteristic polynomials. By Lemma 2.4 (iv), we get that

$$\begin{cases} 3+c''+d''=3+c'+d' \\ 3+c''+d''+c''d''=3+c'+d'+c'd' \\ c''+d''+2c''d''=c'+d'+2c'd' \end{cases}$$

Solving the equation system as above, we obtain that $c''+d''=c'+d'$, $c''d''=c'd'$. By $c''d''=c'd'$, we obtain that $c''=c'd'/d''$. Taking $c''=c'd'/d''$ into $c''+d''=c'+d'$, we have $d''^2-(c'+d')d''+c'd'=0$. Solving this equation, we obtain that $d''=c'$ or $d''=d'$. If $d''=c'$, $c''<d'$, then we have $c''+d''<c'+d'$, a contradiction; If $d''=d'$, $c'>c''$, then we have $c''+d''<c'+d'$, a contradiction. Thus, $G_4o(1,1,c',d',1)$ and $G_4o(1,1,c'',d'',1)$ are not cospectral.

From the argument above, we obtain that G is *DS*. □

Theorem 4.2. *Let* $G=G_3'$, *where* $a=b=c=1$ *in* G_3'. *Then G is DS if and only if* $w\ne ed$ *or* $e'^2-(a+d+e+de)e'+ad+ae+de+aed=0$ *has no positive integer solution.*

Proof. Suppose that G has $3+d'+e'$ vertices. By Lemma 2.2 (v), we know if graph H is cospectral with graph G, then H must contain one triangle. By Lemma 2.1 and Theorem 2.4, $G_2'\cup gK_1$ (here $b=c=d=1$), $G_4'\cup rK_1$ (here $a=b=e=1$) and $G_5'\cup wK_1$ (here $b=c=f=1$) contain one triangle, respectively. It has been proved that $G_2'\cup gK_1$ (here $b=c=d=1$) is *DS*. In the following we consider two cases.

Case 1. Assume that G and $G_4'\cup rK_1$ are cospectral and let $1\le d'\le e'$, $1\le c\le d$. Therefore, G and $G_4'\cup rK_1$ have the same vertices and coefficients of their characteristic polynomials. By Lemma 2.4 (iii) and (iv), we have

$$\begin{cases} 3+d'+e'=3+c+d+r \\ 3+d'+e'=3+c+d+cd \\ s+e'+d'e'=c+d+2cd \end{cases}$$

Solving the equation system as above, we obtain that $r=cd=d'e'$, which implies that $d'=cd/e'$ and $r>0$. By $3+d'+e'=3+c+d+r$ and $r>0$, we can obtain that $d'+e'>c+d$. Taking $d'=cd/e'$ into $d'+e'>c+d$, we obtain that $e'^2-(c+d)e'+cd>0$. Solving the equation as above, we obtain that $e'>d$ or $e'<c$. However, by $cd=d'e'$, $1\le d'\le e'$, $1\le c\le d$, we obtain that $c\le d'\le e'\le d$ or $d'\le c\le d\le e'$, which in contradict with $e'>d$ or $e'<c$. Thus, G and $G_4'\cup rK_1$ are not cospectral.

Case 2. Assume that G and $G_5'\cup wK_1$ are cospectral. Therefore, G and $G_5'\cup wK_1$ have the same vertices and coefficients of their characteristic polynomials. By Lemma 2.4 (iii) and (v), we have

$$\begin{cases} 3+d'+e'=3+a+d+e+w \\ 3+d'+e'=3+a+d+e+ed \\ s+d'+d'e'=a+d+e+ad+ae+2ed+aed \end{cases}$$

Solving the equation system as above, we have

$$\begin{cases} w=ed \\ d'e'=ad+ae+ed+aed \\ d'+e'=a+d+e+2ed \end{cases}$$

By $d'e'=ad+ae+ed+aed$, we obtain that $d'=(ad+ae+ed+aed)/e'$. Taking $d'=(ad+ae+ed+aed)/e'$ into $d'+e'=a+d+e+2ed$, we obtain that $e'^2-(a+d+e+de)e'+ad+ae+de+aed=0$. If $w=ed$ and $e'^2-(a+d+e+de)e'+ad+ae+de+aed=0$ has positive integer solution are satisfied at the same time, then G and $G_5'\cup wK_1$ are cospectral. On the contrary, G and $G_5'\cup wK_1$ are not cospectral.

Then, assume that $G_3o(1,1,1,d',e')$ and $G_3o(1,1,1,d'',e'')$ are cospectral and let $d''<d'\le e'<e''$. Therefore, they have the same vertices and coefficients of their characteristic polynomials. By Lemma 2.4 (iii), we have

$$\begin{cases} 3+d'+e'=3+d''+e'' \\ d'+e'+d'e'=d''+e''+d''e'' \end{cases}$$

Solving the equation system as above, we obtain that $d'+e'=d''+e''$, $d'e'=d''e''$. By $d'e'=d''e''$, we have $d'=d''e''/e'$. Taking $d'=d''e''/e'$ into $d'+e'=d''+e''$, we have $e'^2-(d''+e'')e'+d''e''=0$. Solving the equation as above, we obtain that $e'=d''$ or $e'=e''$. If $e'=d''$, $d'<e''$. Then we have $d'+e'<d''+e''$, which in contradict with $d'+e'=d''+e''$; When $e'=e''$, $d'>d''$. Then we have $d'+e'>d''+e''$, which in contradict with $d'+e'=d''+e''$. Thus, $G_3o(1,1,1,d',e')$ and $G_3o(1,1,1,d'',e'')$ are not cospectral.

In conclusion, we obtain that G is DS if and only if $w\ne ed$ or $e'^2-(a+d+e+de)e'+ad+ae+de+aed=0$ has no positive integer solution. □

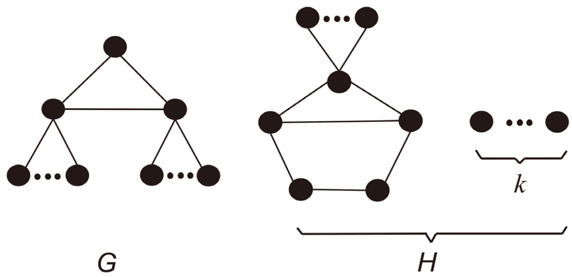

Figure 2. G and H.

Corollary 4.3. *Let* $G = G_3'$ *where* $a = b = c = 1$ *in* G_3' *and* $H = G_5' \cup kK_1$ *where* $b = c = d = e = f = 1$ *in* G_5'. *They are cospectral if and only if* $k = 1$ *and* $e'^2 - (a+3)e' + 3a + 1 = 0$ *has positive integer solution, where the graphs* G, H *are depicted in* Figure 2.

Proof. By Theorem 4.2, we can obtain it obviously. □

5. Conclusion

In this paper, we give the expressions of the spectral radius and energy of all graphs with rank 4. At the same time, we investigate some graph $G \in N(G_1, G_2, \cdots, G_9)$ is *DS* and find some cospectral graphs.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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A Class of New Optimal Ternary Cyclic Codes over $\mathbb{F}_{3^m}$ with Minimum Distance 4

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Abstract

As a branch of applied mathematics, coding theory plays an important role. Among them, cyclic codes have attracted much attention because of their good algebraic structure and easy analysis performance. In this paper, we will study one class of cyclic codes over $\mathbb{F}_3$. Given the length and dimension, we show that it is optimal by proving its minimum distance is equal to 4, according to the Sphere Packing bound.

Keywords

Cyclic Code, Minimal Distance, Sphere Packing Bound

1. Introduction

Information transmission is an important means of human communication, and with the development of technology, coding theory has also been established. In the information age, cyberspace security is a very important issue, and cryptography and encoding play important roles in it. Coding theory is a technique for encoding information. During the process of transmitting information, it is inevitable that information may be distorted due to some reasons. In this process, information cannot correct errors on its own. Therefore, a self-correcting code space has been studied, which is called the error-correcting-codes.

Among error-correction-codes, linear codes are widely studied due to their excellent algebraic structure and other characteristics, and cyclic codes are the most important among them. Due to their excellent algebraic structure and cyclic properties, they can be easily studied and obtained through algebraic methods, and are widely used in various information security systems.

Let $\mathbb{F}_{p^m}$ be a finite field with p^m elements, where p is a prime. A linear code $\mathcal{C}$ with parameters $[n,k,d]$ over $\mathbb{F}_p$ is a linear subspace of $\mathbb{F}_{p^m}^n$, which has the length *n*, dimension *k* and minimum Hamming distance *d*. We say the linear code $\mathcal{C}$ is a cyclic code if for any codewords $c=(c_0,c_1,\cdots,c_{n-1})\in\mathcal{C}$, the cyclic shift of the codeword $(c_{n-1},c_0,\cdots,c_{n-2})\in\mathcal{C}$. Now we use the polynomial ring $\mathbb{F}_p[x]$ and the quotient ring $\mathbb{F}_p[x]/(x^n-1)$ to describe the cyclic code. We define a linear code as a cyclic code if for any codeword $f(x)\in\mathcal{C}$, which can be identified with a polynomial

$$c_0+c_1x+c_2x^2+\cdots+c_{n-1}x^{n-1}\in\mathbb{F}_p[x]/(x^n-1),$$

the codeword $xf(x)\in\mathcal{C}$. So we can easily know that an nonempty set $\mathcal{C}$ in $\mathbb{F}_p^n$ is a cyclic code if and only if $\mathcal{C}$ is a principal ring in $\mathbb{F}_p^n$. We denote any cyclic code $\mathcal{C}$ as $\mathcal{C}=\langle g(x)\rangle$, and $g(x)$ is called the generator polynomial of $\mathcal{C}$.

The study of cyclic codes has been the focus of attention in recent years. Because of its excellent characteristics, it has been widely used in lots of fields. We always hope that a cyclic code has better error correction ability. The error correction ability is closely related to the minimum distance. The larger minimum distance it has, the better error correction ability it gets. Therefore, we are very interested in the minimum distance of a code.

Let $p=3$, we consider the cyclic code $\mathcal{C}_{(u,v)}$ over the finite field $\mathbb{F}_3$. Ding and Helleseth [1] state the theory of the APN monomials and used some of these to construct many classes of optimal ternary cyclic codes in 2013. In 2019, by giving a new ternary power mapping, Yan and Han [2] considered a related optimal ternary cyclic code which $u=1,v=(3^m-3)/4$ in some conditions. Zha and Hu [3] proved some new classes of optimal ternary cyclic codes with minimum distance 4 for some given parameters *v*, $u=1$ in 2020. For the given $u=(3^m+1)/2$, Ding and Zhou [4] studied the cyclic code is optimal when $v=(3^s+1)/2$ in some conditions. Similarly, Fan, Zhou and Li [5] proved that the cyclic code $\mathcal{C}_{\left(\frac{3^m+1}{2},2\cdot 3^{(m-1)/2}+1\right)}$ is optimal when *m* is odd in 2016. They also discussed the weight distribution of the dual of this code. In 2020, Liu, Cao and Lu [6] studied the code $\mathcal{C}_{(2,v)}$, which is constructed by using monomials x^2 and x^v. For $v=(3^m-1)/2+2(3^k+1)$, $\mathcal{C}_{(2,v)}$ is optimal by choosing suitable *m* and *k*. Recently, by choosing proper *u* and *v*, Zha, Hu, Liu and Cao [7] show that $\mathcal{C}_{\left(\frac{3^m+1}{2},\frac{3^m-1}{2}+v\right)}$ and $\mathcal{C}_{(1,v)}$ have the same optimality.

In previous studies, there are not many studies on cyclic codes with parameter $\mathcal{C}_{(u,v)}$, $u=\dfrac{3^m+1}{2}$. In this paper, we study the cyclic code $\mathcal{C}_{(u,v)}$ with the parameters which is $\mathcal{C}_{\left(\frac{3^m+1}{2},\frac{3^{m+1}+7}{8}\right)}$. We show that the minimum distance of this cyclic code is equal to 4 for the given $n=3^m-1$ and $k=3^m-1-2m$, according to the Sphere Packing bound. It is optimal. Therefore, in the coding theory, we can obtain a new class of ternary cyclic codes whose minimum distance can reach the theoretical maximum for the given length and dimension. It can achieve the best error correction effect and ensure that the information is not distorted as much as possible in the transmission process. These cyclic codes will have important applications in radar, satellite communications and other communication fields.

2. Preliminaries

- *The notation we use in this paper*

(1) p is prime, and is an odd. Let $p=3$.

(2) s,r,m,k are positive integers, m is odd.

(3) Let SQ be the set of square in $\mathbb{F}_{3^m}$, NSQ be the set of the nonsquare in $\mathbb{F}_{3^m}$.

(4) In $\mathbb{F}_{3^m}$, we have $\alpha^{\frac{3^m+1}{2}}=\alpha$ if $\alpha\in SQ$ and $\alpha^{\frac{3^m+1}{2}}=-\alpha$ if $\alpha\in NSQ$.

- *The p-cyclotomic cosets modulo n*, $n=p^m-1$

We define the p-cyclotomic coset modulo n containing j as

$$C_j=\left\{j,pj,p^2j,\cdots,p^{k-1}j\right\}$$

and k is the smallest positive integer such that $p^kj\equiv j(modn)$. In this paper, let $p=3$. The cyclic code of length $n=3^m-1$. The dimension of this code $\mathcal{C}_{(u,v)}$ is determined by k, where $k=|C_v|$. The dimension of $\mathcal{C}_{(u,v)}$ is equal to $n-(m+k)$. We now consider the case that $v\notin C_1$ and $k=m$, so the dimension is equal to 3^m-1-2m.

Theorem 2.1. [8] (***Sphere Packing Bound***) *$A_p(n,d)$ is the maximum number of codewords in a code over $\mathbb{F}_p$ of length n and minimum distance at least d, or we use p^k to represent it. Then*

$$A_p(n,d)\le\frac{p^n}{\sum_{i=0}^{t}\binom{n}{i}(p-1)^i}$$

where $t=\left[(d-1)/2\right]$.

We can see that by using Sphere Packing Bound, we can get a bound of the minimum distance of a cyclic code. Taking the cyclic code to be studied in this paper as an example, let $p=3$, and when $n=3^m-1$, $k=3^m-1-2m$, the minimum distance of this cyclic codes can be obtained no more than 4. We obtain the upper bound of the minimum distance of this cyclic code. Therefore, we only need to prove that the minimum distance of this cyclic code can reach this upper bound, and it can be shown that it is optimal.

The distance d between two codewords $c,\bar{c}\in\mathcal{C}$ is defined to be the number of coordinates in which $c,\bar{c}$ are different. The minimum distance of a code $\mathcal{C}$ is the smallest distance between distinct codewords. The weight $wt(c)$ of a codeword c is the number of the nonzero coordinates in c. It has $d(c,\bar{c})=wt(c-\bar{c})$ [9]. If $\mathcal{C}$ is a linear code, the minimum distance equal to the minimum weight of the nonzero codewords of $\mathcal{C}$. The parity check matrices of a code is a matrices H which satisfied

$$Hc^{\mathrm{T}}=0$$

$c \in \mathcal{C}$. The parity check matrices of a code are the generator matrices of its dual code. From the definition of dual codes, the parity check matrices of the code $\mathcal{C}_{(u,v)}$ is define as

$$\begin{pmatrix} \pi^{u} & \pi^{2u} & \cdots & \pi^{(3^m-1)u} \\ \pi^{v} & \pi^{2v} & \cdots & \pi^{(3^m-1)v} \end{pmatrix}$$

π is a generator of $\mathbb{F}_{3^m}^*$.

If a linear code $\mathcal{C}$ has minimum distance *d*, there exist two distinct codewords $c, \overline{c} \in \mathcal{C}$, $Hc^{\mathrm{T}} = 0$, $H\overline{c}^{\mathrm{T}} = 0$, satisfied

$$\begin{cases} \tilde{c}_1 x_1^u + \tilde{c}_2 x_2^u + \cdots + \tilde{c}_d x_d^u = 0 \\ \tilde{c}_1 x_1^v + \tilde{c}_2 x_2^v + \cdots + \tilde{c}_d x_d^v = 0 \end{cases}$$

$c_j, \overline{c}_j$ is the coordinates of the codeword $c, \overline{c}$, respectively. $\tilde{c}_i = c_j - \overline{c}_j$, $c_j \neq \overline{c}_j$, $x_i = \pi^j$, $1 \le j \le 3^m - 1$, $i = 1, 2, \cdots, d$.

If the code has minimum distance *d*, the equations above has solution, if the code has not minimum distance *d*, the equations above has not solution. So we can discuss the solution of the equations to find if the code has the codeword of weight *d*. According to the minimum distance $d \le 4$ given by the sphere packing bound, we can prove that $d = 4$.

Lemma 2.2. *Let* $u = (3^m + 1)/2$, v *be an odd,* $v \notin C_1$, *and* $\ell_v = |C_v| = m$. *Cyclic code* $\mathcal{C}_{(u,v)}$ *has parameters* $[3^m - 1, 3^m - 1 - 2m, 4]$ *if and only if the following equations:*

$$1 + x^v = \pm(1 + x)^v \tag{1}$$

$$1 + x^v = \pm(1 - x)^v \tag{2}$$

$$x^v - 1 = \pm(1 - x)^v \tag{3}$$

and the equation

$$x^v - 1 = \pm(1 + x)^v \tag{4}$$

have no solution in $\mathbb{F}_{3^m} \setminus \{0, 1\}$.

Proof. It is clear that the distance of the code cannot be 1. The code $\mathcal{C}_{(u,v)}$ has a codeword of Hamming weight 2 if and only if there exist two elements $c_1, c_2 \in \mathbb{F}_3^*$ and two distinct elements $x_1, x_2 \in \mathbb{F}_{3^m}^*$ such that

$$\begin{cases} c_1 x_1^u + c_2 x_2^u = 0 \\ c_1 x_1^v + c_2 x_2^v = 0 \end{cases}$$

Case 1: $c_1 = c_2 = 1$ If $x_1 \in SQ$, $x_2 \in SQ$, the first equation becomes to $x_1 + x_2 = 0$, which is impossible because x_1, x_2 are all *SQ*. If $x_1 \in NSQ$, $x_2 \in NSQ$, the first equation becomes to $x_1 = -x_2$, let $x_1 = -a^2$, then we have $-a^2 = -x_2$, $a^2 = x_2$ but x_2 is a *NSQ*, which is also impossible. If $x_1 \in SQ$, $x_2 \in NSQ$ or $x_1 \in NSQ$, $x_2 \in SQ$, the first equation becomes to $x_1 = x_2$, which is also a contradiction..

Case 2: $c_1 = 1, c_2 = -1$ If $x_1 \in SQ$, $x_2 \in SQ$ or $x_1 \in NSQ$, $x_2 \in NSQ$, the first equation becomes to $x_1 = x_2$, which is a contradiction. If $x_1 \in SQ$, $x_2 \in NSQ$ or $x_1 \in NSQ$, $x_2 \in SQ$, the first equation becomes to $x_1 = -x_2$. Taking it into the second equation we will get $2x_1 = 0$, which is a contradiction.

Thus it does not have a codeword of Hamming weight 2.

The code $\mathcal{C}_{(u,v)}$ has a codeword of Hamming weight 3 if and only if there exist three elements $c_1, c_2, c_3 \in \mathbb{F}_3^*$ and three distinct elements $x_1, x_2, x_3 \in \mathbb{F}_{3^m}^*$ such that

$$\begin{cases} c_1 x_1^u + c_2 x_2^u + c_3 x_3^u = 0 \\ c_1 x_1^v + c_2 x_2^v + c_3 x_3^v = 0 \end{cases} \quad (5)$$

Case 1: $c_1 = c_2 = c_3 = 1$. In this case, let $y_1 = x_2/x_1, y_2 = x_3/x_1$. It follows from (5) that

$$\begin{cases} y_1^u + y_2^u + 1 = 0 \\ y_1^v + y_2^v + 1 = 0 \end{cases} \quad (6)$$

$y_1, y_2 \notin \{0,1\}$. If $y_1, y_2 \in SQ$ or $y_1 \in SQ, y_2 \in NSQ$, (6) becomes to

$$y_1^v + 1 = \pm(1 + y_1)^v$$

If $y_1, y_2 \in NSQ$ or $y_1 \in NSQ, y_2 \in SQ$, (6) becomes to

$$y_1^v + 1 = \pm(1 - y_1)^v$$

Case 2: $c_1 = c_2 = 1, c_3 = -1$. Similarly, we arrive at

$$\begin{cases} y_1^u + y_2^u - 1 = 0 \\ y_1^v + y_2^v - 1 = 0 \end{cases} \quad (7)$$

$y_1, y_2 \notin \{0,1\}$. If $y_1, y_2 \in SQ$ or $y_1 \in SQ, y_2 \in NSQ$, (7) becomes to

$$y_1^v - 1 = \pm(1 - y_1)^v$$

If $y_1, y_2 \in NSQ$ or $y_1 \in NSQ, y_2 \in SQ$, (7) becomes to

$$y_1^v - 1 = \pm(1 + y_1)^v$$

So if the four equations have no solutions in $\mathbb{F}_{3^m}^*$, we get $d \geq 4$, according to the Sphere Packing bound, the minimal distance of any linear code with length $3^m - 1$ and the dimension $3^m - 1 - 2m$ should be less than or equal 4. Hence $d = 4$. □

The following Lemma will be used in the sequel of the proof.

Lemma 2.3. [10] Let $f(x)$ be a irreducible polynomial with degree r over $\mathbb{F}_p$. If $f(x)$ has a root in $\mathbb{F}_{p^m}$, then $r \mid m$.

3. A Class of Optimal Ternary Cyclic Codes

In this section, we construct a class of optimal ternary cyclic codes $\mathcal{C}_{(u,v)}$ with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$.

Theorem 3.1. *Let m is odd,* $m \geq 3$, $u = (3^m + 1)/2$, $v = (3^{m+1} + 7)/8$. $m \equiv 3 \pmod 4$, $9 \nmid m$, $5 \nmid m$. *The cyclic code* $\mathcal{C}_{(u,v)}$ *is an optimal ternary cyclic code with parameters* $[3^m - 1, 3^m - 1 - 2m, 4]$.

Proof. It is easy to prove that the minimal distance of the code $d \geq 2$. By lemma 2.2 we can know that it does not have a codeword of Hamming weight 2, which means $d \geq 3$. Now we prove that the minimal distance of the code $d = 4$.

Now, we prove that the code has no codewords of Hamming weight 3. It has a codeword of Hamming weight 3 if and only if there exist three elements $c_1, c_2, c_3 \in \mathbb{F}_3^*$ and three distinct elements $x_1, x_2, x_3 \in \mathbb{F}_{3^m}^*$ such that

$$\begin{cases} c_1 x_1^u + c_2 x_2^u + c_3 x_3^u = 0 \\ c_1 x_1^v + c_2 x_2^v + c_3 x_3^v = 0 \end{cases} \tag{8}$$

Case 1: $c_1 = c_2 = c_3 = 1$. In this case, let $y_1 = x_2/x_1$, $y_2 = x_3/x_1$ It follows from (8) that

$$\begin{cases} y_1^u + y_2^u + 1 = 0 \\ y_1^v + y_2^v + 1 = 0 \end{cases} \tag{9}$$

$$y_1, y_2 \notin \{0,1\}.$$

Now we consider the following four cases.

Case 1.1: When $y_1, y_2 \in SQ$, (9) follows to

$$\begin{cases} y_1 + y_2 + 1 = 0 \\ y_1^v + y_2^v + 1 = 0 \end{cases} \tag{10}$$

The Equation (10) leads to

$$1 + y_1^v = (1 + y_1)^v$$

Let $y_1 = a^2$, Then we have

$$1 + a^{\frac{3^{m+1}+7}{4}} = (1 + a^2)^{\frac{3^{m+1}+7}{8}}$$

By taking the eight power of both sides of the equation, we can get

$$\left(1 + a^{\frac{3^{m+1}+7}{4}}\right)^8 = (1 + a^2)^{3^{m+1}+7}$$

If $a \in SQ$, let $a = t^2$, and if t also $\in SQ$, we have

$$(1 + t^4)^{3^{m+1}+7} = \left(1 + t^{\frac{3^{m+1}+7}{2}}\right)^8$$

It becomes to

$$(1 + t^4)^{10} = (1 + t^5)^8$$

Expand it, we can get

$$t^{36} + t^{35} + 2t^{30} + t^{25} + 2t^{20} + t^{15} + 2t^{10} + t^5 + t^4 = 0$$

Because $t \neq 0$, we have

$$t^{32} + t^{31} + 2t^{26} + t^{21} + 2t^{16} + t^{11} + 2t^6 + t + 1 = 0$$

It can be factorized over $\mathbb{F}_3$ by Magma to

$$(t-1)^2(t^2+1)(t^5-t^2+t+1)(t^5+t^4-t^3+1)(t^9-t^6+t^4+t^3+t-1)$$
$$\times(t^9-t^8-t^6-t^5+t^3-1)=0$$

if $t=1$, it means $y_1=1$, is impossible, and by lemma 2.3, we get it has no root in $\mathbb{F}_{3^m}\setminus\{0,1\}$.

If $a\in SQ$, let $a=t^2$, and if $t\in NSQ$, we have

$$(1+t^4)^{10}=(1-t^5)^8$$

By following the same steps, we get

$$t^{32}-t^{31}-t^{26}-t^{21}-t^{16}-t^{11}-t^6-t+1=0$$

It can be factorized over $\mathbb{F}_3$ by Magma to

$$(t+1)^2(t^2+1)(t^5+t^2+t-1)(t^5-t^4-t^3-1)(t^9+t^6-t^4+t^3+t+1)$$
$$\times(t^9+t^8+t^6-t^5+t^3+1)=0$$

By the same reason, it has no root in $\mathbb{F}_{3^m}\setminus\{0,1\}$.

If $a\in NSQ$, $a=-t^2$ and $t\in SQ$ or $t\in NSQ$, it is similar to the above case, we omit it here.

Case 1.2: When $y_1\in NSQ, y_2\in NSQ$, (8) follows to

$$\begin{cases} y_1+y_2+1=0 \\ y_1^v+y_2^v+1=0 \end{cases} \tag{11}$$

The Equation (11) leads to

$$1+y_1^v=-(1-y_1)^v$$

Let $y_1=-a^2$, Then we have

$$1-a^{\frac{3^{m+1}+7}{4}}=-(1+a^2)^{\frac{3^{m+1}+7}{8}}$$

By taking the eight power of both sides of the equation, we can get

$$\left(1-a^{\frac{3^{m+1}+7}{4}}\right)^8=(1+a^2)^{3^{m+1}+7}$$

If $a\in SQ$, let $a=t^2$, and if t also $\in SQ$, by the similar steps, we directly obtain

$$(1+t^4)^{10}=(1-t^5)^8$$

As before, with the help of Magma, we get the same equation

$$(t+1)^2(t^2+1)(t^5+t^2+t-1)(t^5-t^4-t^3-1)(t^9+t^6-t^4+t^3+t+1)$$
$$\times(t^9+t^8+t^6-t^5+t^3+1)=0$$

if $t=-1$, it means $y_1=-1$, but $y_1=-1$ is not the solution of $1+y_1^v=-(1-y_1)^v$, and by lemma 2.3, we get it has no root in $\mathbb{F}_{3^m}\setminus\{0,1\}$

If $a\in SQ$, let $a=t^2$, and if $t\in NSQ$, we have

$$\left(1+t^4\right)^{10}=\left(1+t^5\right)^8$$

By following the same steps, we get

$$\left(t-1\right)^2\left(t^2+1\right)\left(t^5-t^2+t+1\right)\left(t^5+t^4-t^3+1\right)\left(t^9-t^6+t^4+t^3+t-1\right)$$
$$\times\left(t^9-t^8-t^6-t^5+t^3-1\right)=0$$

By the same reason as before, it has no root in $\mathbb{F}_{3^m}\setminus\{0,1\}$.

If $a\in NSQ$, $a=-t^2$ and $t\in SQ$ or $t\in NSQ$, it is similar to the above case, we omit it here.

Case 1.3: When $y_1\in NSQ, y_2\in SQ$. It is similar as before, we omit it here.

Case 1.4: When $y_1\in SQ, y_2\in NSQ$. It is similar as before, we omit it here.

Case 2: $c_1=c_2=1, c_3=-1$. By the similar calculation as Case 1, we can prove that the equation

$$\begin{cases} y_1^u+y_2^u-1=0 \\ y_1^v+y_2^v-1=0 \end{cases} \tag{12}$$

also has no solution in $\mathbb{F}_{3^m}\setminus\{0,1\}$. We omit the details of the proof.

By the Lemma 2.2, we have finished the proof. □

Example

Let $m=3$, $u=\left(3^m+1\right)/2$, $v=\left(3^{m+1}+7\right)/8$. Let α be the generator of $\mathbb{F}_{3^m}^*$ with $\alpha^3+2\alpha+1=0$. Then $\mathcal{C}_{(u,v)}$ is a ternary cyclic code with parameters [26, 20, 4] and generator polynomial $x^6+x^5+x^4+2x^3+2$.

4. Conclusions

In this paper, based on the Sphere Packing Bound, we show that for the fixed length and dimension, with the help of factorization by Magma, by discussing the solutions of some correlative equations on $\mathbb{F}_{3^m}$, the ternary cyclic code $\mathcal{C}_{\left(\frac{3^m+1}{2},\frac{3^{m+1}+7}{8}\right)}$ has the minimum distance 4, according to the Sphere Packing bound. It is optimal.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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