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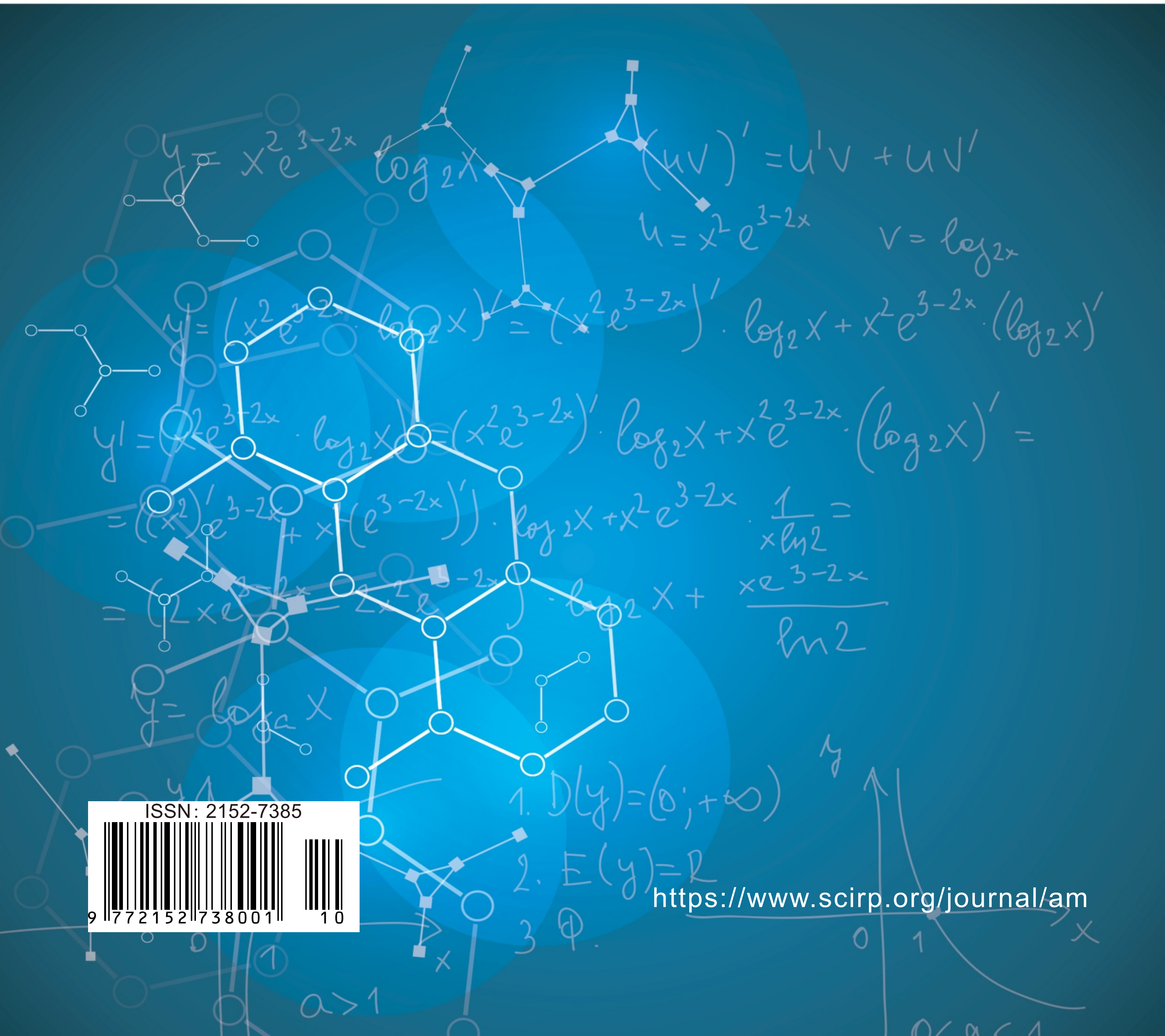

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On the Well-Posedness of a Class of Hybrid Weakly Singular Integro-Differential Equations

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Abstract

In this study, a revised version of some numerical methods for a class of hybrid integro-differential equations with weakly singular kernels (Abel types) is presented. These equations were developed from a class of integro-differential equations of first kind originating from an aeroelasticity problem. By manipulating the bounds of initial conditions with random variations, this study numerically demonstrated the well-posedness properties of the equations. Finally, an assumption of separating variables, allowed for linear splines to be chosen as a basis and for the differentiation and integration of the integro-differential part to be interchanged; hence, a numerical scheme was constructed.

Keywords

Well-Posedness, Hybrid, Weakly Singular, Integro-Differential Equations

1. Introduction

The aeroelastic dynamical model is governed by a class of integro-differential equations with weakly singular kernels [1]. In [2], Burns and Ito examined the well-posedness of the first kind equations in weighted product L_2-spaces with singular kernels as weights. In this study, we numerically investigated the well-posedness property of hybrid equations. For hybrid type equations, especially for the integro-differential parts, we followed the works in previous studies [3] [4]. For the derivative parts, we revised the results outlined in [5] with second order accuracy difference methods for different boundary conditions. Thereafter, we introduced randomly perturbed noises with different bounds in the initial conditions, and compared the corresponding solutions to solutions without initial perturbations. By setting reasonable tolerance of deviation, we successfully demonstrated the well-posedness property. This paper is organized as follows: Section 1 introduces dynamical systems, Section 2 explains the development of the numerical methods, Section 3 presents examples with numerical results and Section 4 summarizes the study.

2. Dynamical Systems

Let us consider weakly singular integro-differential equations of hybrid types with the general normalized form

$$G(x,t,\dot{x})+\frac{\mathrm{d}}{\mathrm{d}t}D_t(x)+L_t(x)=f(t),\ \ t>0, \tag{1}$$

and initial condition $x(s)=\varphi(s)$, for $s\le 0$.

Here, $G(x,t,\dot{x})$ is a function of state, time and time derivative of state. The other terms are such that

$$D_t(x)=\int_{-1}^{0}g(s)x(t+s)\mathrm{d}s\,, \tag{2}$$

and

$$L_t(x)=\int_{-1}^{0}b(s)x(t+s)\mathrm{d}s\,. \tag{3}$$

The kernel $g(s)$ belongs to a weakly singular type. In particular, the Abel type $g(s)=|s|^{-p}$ is considered, where $0<p<1$. The kernel $b(s)$ is assumed to be a smooth function for $-1\le s\le 0$.

3. Numerical Methods

To develop the numerical algorithms, we separately discretize two variables. For the first variable, $s\in[-1,0]$ is discretized as $-1=\tau_n<\tau_{n-1}<\cdots<\tau_1<\tau_0=0$. For the second variable $t\in[0,1]$, the nodes are $T^0,T^1,\cdots,T^m$, with $0=T^0<T^1<\cdots<T^m=1$. The typical equations we study are

$$G(x(t),t,\dot{x}(t))+\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{0}|s|^{-p}x(t+s)\mathrm{d}s+\int_{-1}^{0}b(s)x(t+s)\mathrm{d}s=f(t). \tag{4}$$

Because the derivative is respect to *t*, we interchange the differentiation and integration of the second term and then apply the property $\frac{\mathrm{d}}{\mathrm{d}t}x(t+s)=\frac{\mathrm{d}}{\mathrm{d}s}x(t+s)$. If we assume that $\kappa(t,s)=x(t+s)$, then

$$\frac{\partial\kappa(t,s)}{\partial t}=\frac{\partial\kappa(t,s)}{\partial s}. \tag{5}$$

Next, suppose that $\kappa(t,s)=\sum_{i=1}^{n}a_i(t)\beta_i(s)$, with the basis $\beta_i(s)$, $i=1,2,\cdots,n-1$, defined as:

$$\beta_i(s)=\begin{cases}\frac{1}{\delta_{i+1}}(s-\tau_{i+1}), & s\in[\tau_{i+1},\tau_i],\\ \frac{1}{\delta_i}(\tau_{i-1}-s), & s\in[\tau_i,\tau_{i-1}],\\ 0, & \text{otherwise,}\end{cases} \tag{6}$$

where $\delta_i = \tau_{i-1} - \tau_i > 0$, $i = 1,\cdots,n$. In particular, $\beta_0(s) = \frac{1}{\tau_0 - \tau_1}(\tau_0 - s)$, $s \in [\tau_1, \tau_0]$ and $\beta_n(s) = \frac{1}{\tau_{n-1} - \tau_n}(\tau_{n-1} - s)$, $s \in [\tau_n, \tau_{n-1}]$.

The semi-discretized form of Equation (4) becomes

$$\begin{aligned} &G\left(a_0(t), t, \dot{a}_0(t)\right) + \int_{-1}^{0} |s|^{-p} \left(\sum_{i=0}^{n} a_i(t) \frac{\mathrm{d}}{\mathrm{d}s} \beta_i(s) \right) \mathrm{d}s \\ &+ \int_{-1}^{0} b(s) \left(\sum_{i=0}^{n} a_i(t) \beta_i(s) \right) \mathrm{d}s = f(t). \end{aligned} \tag{7}$$

With the piecewise linear property of $\beta_i(s)$, the second term of Equation (7) can be further partitioned into $c_0 a_0(t) + c_1 a_1(t) + \cdots + c_n a_n(t)$, where c_k, $k = 0,1,\cdots,n$, depending on τ_l, $l = 0,1,\cdots,n$. Analogously, the third term of Equation (7) can be discretized as $d_0 a_0(t) + d_1 a_1(t) + \cdots + d_n a_n(t)$, and d_k, $k = 0,1,\cdots,n$, also depending on τ_l, $l = 0,1,\cdots,n$.

For a fully-discretized form, with a second-order finite difference approximating the derivative term, Equation (7) becomes the following

For $k = 1,\cdots,m-2$,

$$G\left(a_0^k, T^k, \frac{-a_0^{k+2} + 4a_0^{k+1} - 3a_0^k}{2\Delta_k} \right) + \sum_{i=0}^{n} a_i^k c_i + \sum_{i=0}^{n} a_i^k d_i = f\left(T^k\right), \tag{8}$$

for $k = m-1$, $G\left(a_0^{m-1}, T^{m-1}, \frac{a_0^m - a_0^{m-2}}{2\Delta_{m-1}} \right) + \sum_{i=0}^{n} a_i^{m-1} c_i + \sum_{i=0}^{n} a_i^{m-1} d_i = f\left(T^{m-1}\right)$,

and for $k = m$, $G\left(a_0^m, T^m, \frac{3a_0^m - 4a_0^{m-1} + a_0^{m-2}}{2\Delta_m} \right) + \sum_{i=0}^{n} a_i^m c_i + \sum_{i=0}^{n} a_i^m d_i = f\left(T^m\right)$,

with $\Delta_k = T^k - T^{k-1} > 0$, $k = 1,\cdots,m$.

To identify a_0^k, $k = 1,\cdots,m$, and apply uniform discretization in both *t* and *s*, we use the transit property $a_i^j = a_{i-1}^{j-1}$ and assume $m = n$. Therefore, we have $a_i^j = a_0^{j-i}$, for $j > i$. For $j \le i$, a_l^0, $l = 0,1,\cdots,n-1$, can be determined by the initial condition.

Without loss of generality, we use the special form $G\left(x(t), t, \dot{x}(t)\right) = \dot{x}(t) - x(t)$ and construct an $n \times n$ linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ for the system of algebraic equations, where

$$A = \begin{bmatrix} c_0 - d_0 - 1 - \frac{3}{2\Delta} & \frac{2}{\Delta} & \frac{-1}{2\Delta} & \cdots & \cdots & 0 \\ c_1 - d_1 & c_0 - d_0 - 1 - \frac{3}{2\Delta} & \frac{2}{\Delta} & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ c_{n-3} - d_{n-3} & \cdots & \cdots & c_0 - d_0 - 1 - \frac{3}{2\Delta} & \frac{2}{\Delta} & \frac{-1}{2\Delta} \\ c_{n-2} - d_{n-2} & \cdots & \vdots & c_2 - d_2 & c_0 - d_0 - 1 & \frac{1}{2\Delta} \\ c_{n-1} - d_{n-1} & \cdots & c_3 - d_3 & c_2 - d_2 + \frac{1}{2\Delta} & c_1 - d_1 - \frac{2}{\Delta} & c_0 - d_0 - 1 - \frac{3}{2\Delta} \end{bmatrix}_{n \times n},$$

$$\boldsymbol{x}=\begin{bmatrix} a_0^1 \\ \vdots \\ a_0^n \end{bmatrix}_{n\times 1} \text{ and } \boldsymbol{b}=\begin{bmatrix} f\left(T^1\right)-a_0^0\left[c_1-d_1\right]-a_1^0\left[c_2-d_2\right]-\cdots-a_{n-1}^0\left[c_n-d_n\right] \\ \vdots \\ f\left(T^{n-1}\right)-a_0^0\left[c_{n-1}-d_{n-1}\right]-\cdots-a_1^0\left[c_n-d_n\right] \\ f\left(T^n\right)-a_0^0\left[c_n-d_n\right] \end{bmatrix}_{n\times 1}.$$

4. Examples

In this section, we apply the methods derived from Section 2 to calculate the percentages of computed solutions that satisfy the infinity norm criteria. Computed solutions are obtained by randomly perturbations in each node of the initial condition but bounded within $[-\varepsilon,\varepsilon]$. The numbers in the tables indicate the percentages of successes satisfying the bounded criteria in each case. For n = 1000, the number of nodes and number of test cases, three results are provided for comparison.

In all cases, $p=0.5$. The bounded criterion of infinity norm between the computed and exact solutions is 0.1.

The following examples mainly show that for the hybrid type integro-differential equations, percentages of satisfying the infinity norm criteria between computed solutions and exact solution $x(t)$ are increasing by decreasing the perturbation bounds of initial conditions.

Example 1.

$$b(s)=1,\phi(s)=s^2,\ f(t)=-2t^2+7t-\frac{5}{3},x(t)=t^2.$$

Table 1 contains percentages that satisfy the infinity norm criteria.

Example 2.

$$b(s)=1,\phi(s)=s,\ f(t)=-2t+\frac{7}{2},x(t)=t.$$

Table 2 contains percentages that satisfy the infinity norm criteria.

Example 3.

$$b(s)=1,\phi(s)=0,\ f(t)=1+2t^{0.5}-t-\frac{t^2}{2},x(t)=t.$$

Table 3 contains percentages that satisfy the infinity norm criteria.

Example 4.

$$b(s)=s,\phi(s)=s,\ f(t)=-\frac{t}{2}+\frac{8}{3},x(t)=t.$$

Table 4 contains percentages that satisfy the infinity norm criteria.

Example 5.

$$b(s)=s,\phi(s)=0,\ f(t)=1+2t^{0.5}-t+\frac{t^3}{6},x(t)=t.$$

Table 5 contains percentages that satisfy the infinity norm criteria.

Example 6.

$$b(s)=s,\phi(s)=s^2,\ f(t)=-\frac{1}{12}+2t^{0.5}+\frac{7}{3}t-\frac{8}{3}t^{1.5}+\frac{t^2}{2}+\frac{t^3}{6}-\frac{t^4}{12},x(t)=t.$$

Table 6 contains percentages that satisfy the infinity norm criteria.

Table 1. Results of Example 1.

ε	0.05	0.005	0.0005	0.00005	0.000005
$n = 100$	0%	0%	77%	100%	100%
$n = 100$	0%	0%	70%	100%	100%
$n = 1000$	0%	0%	90%	100%	100%
$n = 1000$	0%	0%	100%	100%	100%
$n = 1000$	0%	0%	100%	100%	100%

Table 2. Results of Example 2.

ε	0.05	0.005	0.0005	0.00005	0.000005
$n = 100$	0%	0%	69%	100%	100%
$n = 100$	0%	0%	78%	100%	100%
$n = 1000$	0%	0%	100%	100%	100%
$n = 1000$	0%	0%	100%	100%	100%
$n = 1000$	0%	0%	100%	100%	100%

Table 3. Results of Example 3.

ε	0.05	0.005	0.0005	0.00005	0.000005
$n = 100$	0%	0%	68%	100%	100%
$n = 100$	0%	0%	57%	100%	100%
$n = 1000$	0%	0%	100%	100%	100%
$n = 1000$	0%	0%	100%	100%	100%
$n = 1000$	0%	0%	99.9%	100%	100%

Table 4. Results of Example 4.

ε	0.05	0.005	0.0005	0.00005	0.000005
$n = 100$	0%	0%	0%	99%	100%
$n = 100$	0%	0%	0%	99%	100%
$n = 1000$	0%	0%	92.5%	100%	100%
$n = 1000$	0%	0%	90.8%	100%	100%
$n = 1000$	0%	0%	92%	99.9%	100%

Table 5. Results of Example 5.

ε	0.05	0.005	0.0005	0.00005	0.000005
$n = 100$	0%	0%	0%	100%	100%
$n = 100$	0%	0%	0%	100%	100%
$n = 1000$	0%	0%	91.9%	100%	100%
$n = 1000$	0%	0%	91.8%	100%	100%
$n = 1000$	0%	0%	92.1%	100%	100%

Table 6. Results of Example 6.

ε	0.05	0.005	0.0005	0.00005	0.000005
$n = 100$	0%	0%	0%	100%	100%
$n = 100$	0%	0%	0%	99%	100%
$n = 1000$	0%	0%	92%	99%	100%
$n = 1000$	0%	0%	92.4%	100%	100%
$n = 1000$	0%	0%	92.4%	100%	100%

5. Conclusion

In this study, we investigated the well-posedness property of a class of hybrid integro-differential Equations by revising the numerical methods outlined in a previous study [5]. From the numerical examples, when the bounds ε of perturbation of the initial conditions approach 0, the percentages of the associated solutions fall into the envelopes of 0.1 bounded criteria compared with solutions without perturbation, which increase instead.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Selection of Coherent and Concise Formulae on Bernoulli Polynomials-Numbers-Series and Power Sums-Faulhaber Problems

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Abstract

Utilizing the translation operator to represent Bernoulli polynomials and power sums as polynomials of Sheffer-type, we obtain concisely almost all their known properties as so as many new ones, especially new recursion relations for calculating Bernoulli polynomials and numbers, new formulae for obtaining power sums of entire and complex numbers. Then by the change of arguments from z into $Z = z(z-1)$ and n into λ which is the 1st order power sum we obtain the Faulhaber formula for powers sums in term of polynomials in λ having coefficients depending on Z. Practically we give tables for calculating in easiest possible manners, the Bernoulli numbers, polynomials, the general powers sums.

Keywords

Bernoulli Numbers, Bernoulli Polynomials, Powers Sums, Zeta Function, Faulhaber Conjecture

1. Introduction

In many branches of mathematics the problem of Bernoulli numbers related to the millenary problem of power sums is probably the most studied since the publication of the book Ars Conjectandi by Euler in 1738 [1] as we can see on the net and, specially, in a didactical thesis of Coen [2], the explicative work of Raugh [3], Beardon [4], the bibliography of thousands of articles on Bernoulli numbers realized by Dilcher, Shula, Slavutskii [5], etc.

Concerning Bernoulli polynomials $B_m(z)$, classically defined from a generating function, there had not so much properties, the most remarkable is its representation by a hyper-differential operator, the Hurwitz expansion of them into Fourier series, the Roman formula for $B_m(nz)$, the Euler-McLaurin formula, etc.

As for the power sums on real and complex numbers, including the famous Faulhaber conjecture, there has no valuable formula linking them with Bernoulli polynomials until only some years ago [6].

Regarding the situation, we would like to perform a selection of as many as possible known and new interesting properties of Bernoulli polynomials then of Bernoulli numbers in a coherent way, *i.e.*, by only one approach, which utilizes principally operator calculus lying on the couple of operators position and derivation, similar as the couple $\vec{r}, \nabla$ in quantum mechanics.

In Section 2, we will treat the problem of Bernoulli polynomials, from their representation by a hyper-differential operator to almost all of their algebraic properties to the fact that $B_m(n)$ is equal to the primitive of the power sums of natural integers. Afterward we show that the formula giving Bernoulli polynomials of a sum of two arguments $B_m(z+y)$ leads to two new recurrence relations for obtaining $B_m(z)$. We also give another approach for calculating without integrations, the Fourier series of Bernoulli polynomials and the Bernoulli series of functions, the relation of $B_m(z)$ with the Euler zeta function. Afterward we show an up-to-date procedure for obtaining $B_m(z)$ from and only from $B_{m-1}(z)$ leading to the rapid establishment of Table of Bernoulli polynomials and numbers. Finally, we show a new way for obtaining Fourier series of Bernoulli polynomials, Euler zeta function, and vice-versa, the series of functions in term of a set of Bernoulli polynomials.

In Section 3, we treat the problem of Bernoulli numbers B_m, from its initial definition by Jakob Bernoulli in 1713 who related them by conjecture with the power sums on natural numbers. By comparison of this relation with the preceding formula linking $B_m(n)$ with power sums, we may identify B_m with $B_m(0)$ then calculate B_m by a simple matrix method side-by-side with the method, more powerful, link with l, coming from the special recurrence formula coming from $B_m(z+y)$.

In Section 4, we prove by utilizing the translation operator $e^{a\partial_z}$, coming from the Newtonian binomial, that the power sums on complex numbers are simply related to those on natural numbers. On the other hand, we prove that they are also related very simply to Bernoulli polynomials, from that we get again the recurrence relation between Bernoulli polynomials.

Section 5 is devoted to the Faulhaber problem regarding power sums on complex numbers. Here we show that power sums on complex numbers may be calculated from sums of entire numbers somehow by writing $B_{2m}(z)$ in function of the new argument $Z = z(z-1)$.

2. Bernoulli Polynomials

2.1. Definition and Principal Properties

In 1738, Euler introduced the Bernoulli polynomials $B_m(z)$ via the generating function [1]

$$\frac{t}{e^t-1}e^{zt}=\sum_{m=0}^{\infty}\frac{1}{m!}B_m(z)t^m \tag{2.1}$$

which directly gives by identification

$$B_0(z)=1,\quad B_1(z)=z-\frac{1}{2},\quad B_2(0)=\frac{1}{6} \tag{2.2}$$

Utilizing the translation operator $e^{a\partial_z}$ coming from the Newtonian binomial

$$(x+a)^m=\sum_{k=0}^{m}\binom{m}{k}a^k x^{m-k}=\sum_{k=0}^{m}\frac{a^k}{k!}\partial_z^k x^m=e^{a\partial_z}x^m \tag{2.3}$$

and having the property

$$e^{a\partial_z}f(x)=f(x+a)$$

$$e^{\partial_z}e^{tz}=e^{t(z+1)}=e^t e^{tz} \tag{2.4}$$

$$\frac{\partial_z}{e^{\partial_z}-1}e^{tz}=\frac{t}{e^t-1}e^{tz} \tag{2.5}$$

we directly find from (2.1) that $B_m(z)$ is the transform of z^m via a differential operator

$$B_m(z)=\frac{\partial_z}{e^{\partial_z}-1}z^m,\ m>0 \tag{2.6}$$

From (2.6) we get the famous known formulae

$$B_m'(z)=mB_{m-1}(z) \tag{2.7}$$

$$B_m(z+1)-B_m(z)=\left(e^{\partial_z}-1\right)B_m(z)=\partial_z z^m=mz^{m-1} \tag{2.8}$$

$$B_m(1)-B_m(0)=\delta_{m1} \tag{2.9}$$

and the following formula which gives e^{inxz} as series of Bernoulli polynomials.

$$\sum_{m=0}^{\infty}\frac{t^m B_m(z)}{m!}=\frac{\partial_z}{e^{\partial_z}-1}e^{tz}=\frac{t}{e^t-1}e^{tz} \tag{2.10}$$

From (2.8) we get the formula given by Roman [7]

$$B_{m+1}\left(\frac{z}{y}+N\right)-B_{m+1}\left(\frac{z}{y}\right)=(m+1)\sum_{n=0}^{N-1}\left(\frac{z}{y}+n\right)^m \tag{2.11}$$

From (2.10) we get the formulae on relations of Bernoulli polynomials versus trigonometric functions, especially the Castellanos formula [8]

$$\sum_{m=2}^{\infty}\frac{(2ix)^m B_m(0)}{m!}=\frac{x\cos x}{\sin x}-1 \tag{2.12}$$

The formulae (2.7) and (2.8) give the important formulae

$$\int_0^1 B_{m-1}(z)\,\mathrm{d}z=\frac{1}{m}\left(B_m(1)-B_m(0)\right)=\delta_{m1} \tag{2.13}$$

$$B_m(n)-B_m(0)=m\left(0^{m-1}+1^{m-1}+\cdots+(n-1)^{m-1}\right) \tag{2.14}$$

and the Taylor expansion

$$B_m(z)=B_m(a)+\cdots+\binom{m}{k}(z-a)^k B_{m-k}(a)+\cdots+(z-a)^m B_0(a)$$

which may be put under symbolic form

$$B_m(z+a)=:\left(B(a)+z\right)^m \tag{2.15}$$

where undefined symbols $B^k(a)$ are to be replaced with $B_k(a)$.

Exploring now the inter-relations between Bernoulli polynomials.

From (2.4) and (2.7) we get the complementary of (2.15)

$$\begin{aligned} B_m(z+a) &= \mathrm{e}^{a\partial_z}B_m(z)=\left(1+\cdots+\frac{a^k}{k!}\partial^k+\cdots+\frac{a^m}{m!}\partial^m\right)B_m(z) \\ &= B_m(z)+\cdots+\binom{m}{k}B_{m-k}(z)a^k+\cdots+B_0(z)a^m \\ &=: \left(B(z)+a\right)^m, \quad 0^0=1 \end{aligned} \tag{2.16}$$

From (2.13)

$$\int_z^{z+1}B_m(y)\mathrm{d}y=\int_0^1 B_m(z+y)\mathrm{d}y=z^m\int_0^1 B_0(y)\mathrm{d}y=z^m$$

$$\int_0^1 B_m(y)\mathrm{d}y=0^m$$

$$\int_0^n B_m(y)\mathrm{d}y=0^m+1^m+\cdots+(n-1)^m \tag{2.17}$$

i.e.,

"The sum of powers of order *m* of *n* first entire numbers from 0 to $(n-1)$, denoted by $S_m(n)$, is equal to the simple primitive (without constant of integration) of the Bernoulli polynomial $B_m(n)$" and vice-versa,

"The Bernoulli polynomial $B_m(n)$ is equal to the derivative of the power sums $S_m(n)$"

As for $B_m(-z)$ we see that

$$\begin{aligned} B_m(-z) &= \frac{-\partial_z}{1-\mathrm{e}^{-\partial_z}}(-z)^m=(-1)^m\mathrm{e}^{\partial_z}\frac{\partial_z}{\mathrm{e}^{\partial_z}-1}z^m \\ &= (-1)^m B_m(z+1)=:(-1)^m\left(B(z)+1\right)^m \end{aligned} \tag{2.18}$$

which leads to

$$B_m\left(-z+\frac{1}{2}\right)=(-1)^m B_m\left(z+\frac{1}{2}\right) \tag{2.19}$$

i.e., to the theorem

"The graph of a Bernoulli polynomial is symmetric with respect to the axis $z=\frac{1}{2}$ if *m* is pair and anti-symmetric if *m* is impair".

Joint (2.19) with (2.9) we get the famous property [1]

$$B_{2m+1}(1) = -B_{2m+1}(0) = \frac{1}{2}\delta_{m0} \tag{2.20}$$

Now, by replacing in (2.6) z with $\frac{z}{n}$ so that ∂_z is with $n\partial_z$ we get

$$B_m\left(\frac{z}{n}\right) = \frac{n\partial_z}{e^{n\partial_z}-1}\left(\frac{z}{n}\right)^m = \frac{n\partial_z}{\left(e^{\partial_z}-1\right)\left(1+e^{\partial_z}+e^{2\partial_z}+\cdots+e^{(n-1)\partial_z}\right)}\left(\frac{z}{n}\right)^m$$

and the formula

$$\sum_{k=0}^{n-1} B_m\left(\frac{z+k}{n}\right) = n^{1-m} B_m(z) \tag{2.21}$$

saying that

$B_m(z)$ is n^{m-1} times the sum of $B_m\left(\frac{z+k}{n}\right), k<n$

For examples:

$$2^{1-m} B_m(z) = B_m\left(\frac{z}{2}\right) + B_m\left(\frac{z+1}{2}\right)$$

$$B_{2m+1}\left(\frac{1}{3}\right) + B_{2m+1}\left(\frac{2}{3}\right) = 0 = \left(3^{-2m}-1\right)B_{2m+1}(1)$$

By replacing in (2.6) z with nz and ∂_z with $\frac{1}{n}\partial_z$ we find again the formula given by Raabe [9] in 1851

$$B_m(nz) = n^{m-1}\left(B_m(z) + B_m\left(z+\frac{1}{n}\right) + \cdots + B_m\left(z+\frac{n-1}{n}\right)\right) \tag{2.22}$$

saying that

"$B_m(nz)$ is n^{m-1} times the sum of $B_m\left(z+\frac{k}{n}\right), k<n$."

For examples

$$B_m(2z) = 2^{m-1}\left(B_m(z) + B_m\left(z+\frac{1}{2}\right)\right)$$

$$B_m\left(\frac{1}{2}\right) = \left(2^{1-m}-1\right)B_m(0)$$

$$B_1(3z) = B_1(z) + B_1\left(z+\frac{1}{3}\right) + B_1\left(z+\frac{2}{3}\right)$$

$$5^{-m} B_m(0) = \left(1+(-1)^m\right)\left(B_m\left(\frac{1}{5}\right) + B_m\left(\frac{2}{5}\right)\right)$$

2.2. Bernoulli Polynomials of Sum of Two Arguments

From the following property of operators that we characterize fundamental [10]

$$f(\partial_z)g(z) \equiv g(z)f(\partial_z) + \frac{1}{1!}g'(z)f'(\partial_z) + \frac{1}{2!}g''(z)f''(\partial_z) + \cdots \tag{2.23}$$

we get

$$B_{m+1}(z)=\frac{\partial_z}{e^{\partial_z}-1}zz^m=z\frac{\partial_z}{e^{\partial_z}-1}z^m+\left(\frac{1}{e^{\partial_z}-1}-\frac{\partial_z e^{\partial_z}}{\left(e^{\partial_z}-1\right)^2}\right)z^m$$

$$=zB_m(z)+\frac{1}{e^{\partial_z}-1}z^m-\frac{e^{\partial_z}\partial_z}{\left(e^{\partial_z}-1\right)^2}z^m$$

$$\partial_z B_{m+1}(z)=\partial_z zB_m(z)+B_m(z)-\frac{e^{\partial_z}\partial_z^2}{\left(e^{\partial_z}-1\right)^2}z^m$$

$$(m-1)B_m(z)=z\partial_z B_m(z)-e^{\partial_z}\frac{\partial_z}{e^{\partial_z}-1}B_m(z)$$

Now, because

$$\partial_{z+y}f(z+y)=\partial_z f(z+y)=\partial_y f(z+y) \tag{2.24}$$

$$\begin{aligned}(m-1)B_m(z+y)&=m(z+y)B_{m-1}(z+y)-e^{\partial_y}\frac{\partial_y}{e^{\partial_y}-1}B_m(z+y)\\ &=: m(z+y)B_{m-1}(z+y)-\left(B(z)+B(y+1)\right)^m\end{aligned} \tag{2.25}$$

The above recurrence formula is to be compare with that given by Weisstein [11] without proof where there seems has a little mistake

$$(1-m)B_m(z+y)+m(z+y-1)B_{m-1}(z+y)=:\left(B(z)+B(y)\right)^m$$

From (2.25) and knowing that $B_k(1)=(-1)^k B_k(0)$ we obtain another type of recurrence formula for Bernoulli polynomials

$$(m-1)B_m(z)=mzB_{m-1}(z)-\left(B(z)+B(1)\right)^m \tag{2.26}$$

$$B_m(z)=B_1(z)B_{m-1}(z)-\frac{1}{m}\sum_{k=2}^{m}(-1)^k\binom{m}{k}B_k(0)B_{m-k}(z) \tag{2.27}$$

For examples, with $B_1(z)=z-\frac{1}{2}$, $B_2(0)=\frac{1}{6}$,

$$B_2(z)=B_1(z)B_1(z)-\frac{1}{2}B_2(0)B_0(z)=\left(z-\frac{1}{2}\right)^2-\frac{1}{12}=z^2-z+\frac{1}{6}$$

$$\begin{aligned}B_3(z)&=B_1(z)B_2(z)-\frac{1}{3}3B_2(0)B_1(z)\\ &=\left(z-\frac{1}{2}\right)\left(z^2-z+\frac{1}{6}-\frac{1}{6}\right)=z^3-\frac{3}{2}z^2+\frac{1}{2}z\end{aligned}$$

$$B_4(z)=B_1(z)B_3(z)-\frac{1}{4}\left(B_2(z)+B_4(0)\right)$$

2.3. The Fourier Series of Bernoulli Polynomials. Euler Zeta Function. Powers of pi

By successive integrations by parts and utilizing the formula (2.13) for $n,m\geq 1$ we get, knowing (2.9),

$$\begin{aligned}\int_0^1 B_n(z)B_m(z)\,\mathrm{d}z &= \frac{1}{m+1}\int_0^1 B_n(z)B'_{m+1}(z)\,\mathrm{d}z\\ &= \frac{1}{m+1}\left(B_n(z)B_{m+1}(z)\right)\Big|_0^1 - \frac{n}{m+1}\int_0^1 B_{n-1}(z)B_{m+1}(z)\,\mathrm{d}z\\ &= (-1)^{n-1}\frac{n!m!}{(m+n)!}\left(B_1(z)B_{m+n}(z)\right)\Big|_0^1\\ &= (-1)^{n-1}\frac{n!m!}{(m+n)!}B_{m+n}(0)\end{aligned} \tag{2.28}$$

Because of the factor $(-1)^{n-1}$ we may conclude that

$B_{2n+1}(0)=0$ for $n>0$ and $B_{2n+2}(0)$ has opposite sign with respect to $B_{2n}(0)$.

The same method also gives

$$\begin{aligned}\int_0^1 B_m(z)\mathrm{e}^{-2ik\pi z}\,\mathrm{d}z &= \frac{-1}{2\pi ik}\int_0^1 B_m(z)\left(\mathrm{e}^{-2ik\pi z}\right)'\mathrm{d}z\\ &= \frac{-1}{2\pi ik}\delta_{m1} - \frac{-m}{2\pi ik}\int_0^1 B_{m-1}(z)\mathrm{e}^{-2i\pi kz}\,\mathrm{d}z\\ &= \cdots = -\frac{m!}{(2\pi ik)^m}\end{aligned} \tag{2.29}$$

which provides us the following formula on Fourier series of $B_m(z)$ proven by Hurwitz in 1890 by another method [10]

$$B_m(z)=\sum_{k\in Z,k\neq 0}^{\infty}\left(\int_0^1 B_m(z)\mathrm{e}^{-2ik\pi z}\,\mathrm{d}z\right)\mathrm{e}^{i2\pi kz} = -\frac{m!}{(2i\pi)^m}\sum_{k\in Z,k\neq 0}^{\infty}\frac{1}{k^m}\mathrm{e}^{i2\pi kz},\ 0\le z\le 1 \tag{2.30}$$

2.4. Bernoulli Series of Functions

Let $f(z)$ be a periodic function defined on an interval $a\le z<b$ and has the period $P=b-a$. For expanding $f(z)$ into a Fourier series of exponentials

$$f(z)=\sum_{n\in Z}c(n)\mathrm{e}^{i2\pi n\frac{z}{P}},\quad a\le z<b=a+P \tag{2.31}$$

we firstly write

$$\int_a^{b^-}\mathrm{e}^{-i2\pi n_0\frac{z}{P}}f(z)\frac{\mathrm{d}z}{P}=\sum_{n\in Z}c(n)\int_a^{b^-}\mathrm{e}^{i2\pi(n-n_0)\frac{z}{P}}\frac{\mathrm{d}z}{P}$$

and see that the second member is equal uniquely to $c(n_0)$ so that

$$c(n_0)=\frac{1}{P}\int_a^{b^-}\mathrm{e}^{-i2\pi n_0\frac{z}{P}}f(z)\,\mathrm{d}z \tag{2.32}$$

The Fourier series of a function, if it exists, is then

$$f(z)=\frac{1}{P}\sum_{n\in Z}^{\infty}\mathrm{e}^{i2\pi n\frac{z}{P}}\int_a^{b^-}\mathrm{e}^{-i2\pi n\frac{z}{P}}f(z)\,\mathrm{d}z \tag{2.33}$$

To avoid integrations in the calculation, we may utilize the method of integrations by parts and get

$$c(n)=\frac{1}{P}\int_a^{b^-}f(z)\mathrm{e}^{-i2\pi n\frac{z}{P}}\mathrm{d}z$$

$$Pc(n)=\frac{-P}{2i\pi n}\left(f\left(b^-\right)e^{-i2\pi n\frac{b^-}{P}}-f(a)e^{-i2\pi n\frac{a}{P}}\right)+\frac{P}{2i\pi n}\int_a^{b^-}f'(z)e^{-i2\pi n\frac{z}{P}}dz$$

$$Pc(n)=-\sum_{k=0}^{\infty}\left(\frac{P}{2i\pi n}\right)^{k+1}\left(f^{(k)}\left(b^-\right)e^{-i2\pi n\frac{b^-}{P}}-f^{(k)}(a)e^{-i2\pi n\frac{a}{P}}\right)-0(k)$$

so that we may write down the Fourier series formula

$$f(z)=\frac{1}{P}\int_a^{b^-}f(z)dz-\frac{1}{P}\sum_{k=0}^{\infty}f^{(k)}(z)\Big|_a^{b^-}\sum_{n\in Z,n\neq 0}\left(\frac{P}{2i\pi n}\right)^{k+1}e^{i2\pi n\frac{z-a}{P}} \quad (2.34)$$

In the case $0\le z<1$, jointed the preceding formula written under the form

$$f(z)=\int_0^{1^-}f(z)dz-\sum_{n\in Z,n\neq 0}\sum_{k=0}^{\infty}f^{(k)}(z)\Big|_0^{1^-}\left(\frac{1}{2i\pi n}\right)^{k+1}e^{i2\pi nz}$$

with the Hurwitz formula we get the new and precious formula on expansion of derivable functions into series of Bernoulli polynomials

$$f(z)=\int_0^1 f(z)dz+\sum_{k=0}^{\infty}\left[f^{(k)}(1)-f^{(k)}(0)\right]\frac{1}{(k+1)!}B_{k+1}(z) \quad (2.35)$$

or

$$f(z)=\int_0^1 f(z)dz+\sum_{k=0}^{N}f^{(k)}(z)\Big|_0^1\frac{B_{k+1}(z)}{(k+1)!}-\sum_{\substack{n\in Z\\ n\neq 0}}\sum_{k=N+1}^{\infty}f^{(k)}(z)\Big|_0^1\left(\frac{1}{2i\pi n}\right)^{k+1}e^{i2\pi nz} \quad (2.36)$$

For examples, under matrix form

$$\begin{pmatrix} f(z)\\ 1\\ z\\ z^2\\ \vdots\\ z^m\end{pmatrix}=\begin{pmatrix}\left[\int f(z)\right]_0^1 & \left[f(z)\right]_0^1 & \left[f'(z)\right]_0^1 & \left[f''(z)\right]_0^1 & \cdots & \left[f^{(m-1)}(z)\right]_0^1\\ 1 & & & & & \\ 1/2 & 1 & & & & \\ 1/3 & 1 & 2 & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 1/(m+1) & 1 & m & m(m-1) & \cdots & m!\end{pmatrix}\begin{pmatrix}1\\ B_1(z)/1!\\ B_2(z)/2!\\ B_3(z)/3!\\ \vdots\\ B_m(z)/m!\end{pmatrix} \quad (2.37)$$

to be compared with

$$\begin{pmatrix} f(z)\\ 1\\ z\\ z^2\\ \vdots\\ z^m\end{pmatrix}=\begin{pmatrix}\left[\int f(z)\right]_0^1 & \left[f(z)\right]_0^1 & \left[f'(z)\right]_0^1 & \left[f''(z)\right]_0^1 & \cdots & \left[f^{(m-1)}(z)\right]_0^1\\ 1 & & & & & \\ 1/2 & 1 & & & & \\ 1/3 & 1 & 2 & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 1/(m+1) & 1 & m & m(m-1) & \cdots & m!\end{pmatrix}\begin{pmatrix}1\\ \sum_{n\neq 0}e^{2i\pi nz}/(2i\pi n)\\ \sum_{n\neq 0}e^{2i\pi nz}/(2i\pi n)^2\\ \sum_{n\neq 0}e^{2i\pi nz}/(2i\pi n)^3\\ \vdots\\ \sum_{n\neq 0}e^{2i\pi nz}/(2i\pi n)^m\end{pmatrix} \quad (2.38)$$

Formula (2.36) leads also to

$$f(0)=\int_0^1 f(z)dz+\sum_{k=0}^{\infty}\left[f^{(k)}(1)-f^{(k)}(0)\right]\frac{1}{(k+1)!}B_{k+1}(0) \quad (2.39)$$

$$f'(z)=\sum_{k=1}^{\infty}\left[f^{(k)}(1)-f^{(k)}(0)\right]\frac{1}{(k+1)!}B_{k+1}(z) \tag{2.40}$$

As first interesting applications

$$\mathrm{e}^{z}=(e-1)+(e-1)\sum_{k=0}^{\infty}\frac{B_{k+1}(z)}{(k+1)!}\qquad 0\le z<1$$

$$\frac{e-2}{e-1}=-\sum_{k=0}^{\infty}\frac{B_{k+1}(0)}{(k+1)!} \tag{2.41}$$

By (2.36) we also obtain a precious recurrence formula of Bernoulli polynomials

$$z^{m}=\int_0^1 z^{m}\mathrm{d}z+\sum_{k=1}^{m}\binom{m}{k-1}\frac{B_k(z)}{k}\qquad 0\le z<1 \tag{2.42}$$

i.e., under matrix form

$$\begin{pmatrix} z\\ z^2\\ z^3\\ z^4\\ \vdots\end{pmatrix}=\begin{pmatrix} 2^{-1}\\ 3^{-1}\\ 4^{-1}\\ 5^{-1}\\ \vdots\end{pmatrix}+\begin{pmatrix} 1 & & & & \cdots\\ 1 & 2 & & & \cdots\\ 1 & 3 & 3 & & \cdots\\ 1 & 4 & 6 & 4 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix} B_1(z)/1\\ B_2(z)/2\\ B_3(z)/3\\ B_4(z)/4\\ \vdots\end{pmatrix} \tag{2.43}$$

which may be resolved for $B_m(z)$ and $B_m(0)$ my matrix calculus.

2.5. Obtaining $B_m(z)$ from $B_{m-1}(z)$ and Table of Bernoulli Polynomials

Integrating two times as followed the Hurwitz formula on Fourier series of Bernoulli polynomials we get

$$\int_0^x B_m(z)\mathrm{d}z=-m!\sum_{\substack{n\in Z\\ n\ne 0}}^{\infty}\left(\frac{1}{2i\pi n}\right)^{m+1}\left(\mathrm{e}^{i2\pi nz}-1\right)=\frac{1}{m+1}B_{m+1}(z)+m!\sum_{\substack{n\in Z\\ n\ne 0}}^{\infty}\left(\frac{1}{2i\pi n}\right)^{m+1}$$

$$\int_0^1\mathrm{d}z\int_0^z B_m(x)\mathrm{d}x=m!\sum_{\substack{n\in Z\\ n\ne 0}}^{\infty}\left(\frac{1}{2i\pi n}\right)^{m+1}$$

$$B_{m+1}(z)=(m+1)\int_0^z B_m(x)\mathrm{d}x-(m+1)\int_0^1\mathrm{d}z\int_0^z B_m(x)\mathrm{d}x \tag{2.44}$$

i.e.,

$B_{m+1}(z)$ is equal to $(m+1)$ times the primitive of $B_m(z)$ minus the double primitive of $B_m(z)$ calculated for $z=1$. The second term is so equal to

$B_m(0)=(-1)^m B_m(1)$. (2.45)

This new algorithm for obtaining $B_{m+1}(z)$ from $B_m(z)$ and $B_m(0)$ is very easy to perform and may be utilized to establish Table of Bernoulli polynomials.

For examples:

$$B_0(x)=1$$

$$B_1(x) = x - \left.\frac{x^2}{2}\right|_{x=1} = x - \frac{1}{2}$$

$$B_2(x) = 2\left(\frac{x^2}{2} - \frac{x}{2}\right) - 2\left.\left(\frac{x^3}{6} - \frac{x^2}{4}\right)\right|_{x=1} = x^2 - x + \frac{1}{6}$$

$$B_3(x) = 3\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6}\right) - 3\left(\frac{1}{12} - \frac{1}{6} + \frac{1}{12}\right) = x^3 - \frac{3}{2}x^2 + \frac{x}{2}$$

$$B_4(x) = 4\left(\frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4}\right) - 4\left(\frac{1}{20} - \frac{1}{8} + \frac{1}{12}\right) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

$$B_5(x) = 5\left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30}\right) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$$

$$B_6(x) = 6\left(\frac{x^6}{6} - \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12}\right) + 6\left(\frac{1}{6.7} - \frac{1}{2.4} + \frac{5}{12.4} - \frac{1}{12.3}\right)$$

$$= x^6 - 3x^5 + \frac{5x^4}{2} - \frac{x^2}{2} + \frac{1}{42}$$

$$B_7(x) = 7\left(\frac{x^7}{7} - \frac{x^6}{2} + \frac{x^5}{2} - \frac{x^3}{6} + \frac{x}{42}\right) + 0 = x^7 - \frac{7}{2}x^6 + \frac{7x^5}{2} - \frac{7x^3}{6} + \frac{x}{6}$$

$$B_8(x) = 8\left(\frac{x^8}{8} - \frac{x^7}{2} + \frac{7x^6}{12} - \frac{7x^4}{24} + \frac{x^2}{12}\right) - 8\left(\frac{1}{8.9} - \frac{1}{2.8} + \frac{7}{8.6} - \frac{7}{24.5} + \frac{1}{12.3}\right)$$

$$= x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2 - \frac{1}{30}$$

$$B_9(x) = 9\left(\frac{x^9}{9} - \frac{x^8}{2} + \frac{2x^7}{3} - \frac{7x^5}{15} + \frac{2x^3}{9} - \frac{x}{30}\right) + 0$$

$$= x^9 - \frac{9x^8}{2} + 6x^7 - \frac{21x^5}{5} + 2x^3 - \frac{3x}{10}$$

This method for establishing a table of Bernoulli polynomials is extremely easier if we utilize the list of fifty Bernoulli numbers $B_m(0)$ conscientiously established by Coen [2]. For examples

$$B_{10}(x) = x^{10} - 5x^9 + \frac{15}{2}x^8 - 7x^6 + 5x^4 - \frac{3}{2}x^2 + \frac{5}{66}$$

$$B_{11}(x) = x^{11} - \frac{11}{2}x^{10} + \frac{55}{6}x^9 - 11x^7 + 11x^5 - \frac{11}{2}x^3 + \frac{5}{6}x$$

$$B_{12}(x) = x^{12} - 6x^{11} + 11x^{10} - \frac{33}{2}x^8 + 22x^6 - \frac{33}{2}x^4 + 5x^2 - \frac{691}{2730} \qquad (2.46)$$

2.6. Bernoulli Polynomials and Euler Zeta Function

From the Hurwitz formula

$$\frac{1}{k!}B_k(z) = -\frac{1}{(2i\pi)^k}\sum_{n\in Z, n\neq 0}^{\infty}\frac{1}{n^k}e^{i2\pi nz} \qquad 0 \le z \le 1$$

we get the Euler zeta function one may find references in Coen [2] and Raugh

[3]

$$\zeta(2m)=\sum_{k=1}^{\infty}\frac{1}{k^{2m}}=(-1)^{m+1}\frac{1}{(2m)!2}(2\pi)^{2m}B_{2m}(0) \tag{2.47}$$

as so as

$$(2\pi)^{2m}=(-1)^{m+1}\frac{(2m)!2}{B_{2m}(z)}\sum_{k=1}^{\infty}\frac{1}{k^{2m}}\cos 2\pi kz \tag{2.48}$$

$$(2\pi)^{2m+1}=(-1)^{m+1}\frac{(2m+1)!2}{B_{2m+1}(z)}\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}}\sin 2\pi kz \tag{2.49}$$

Moreover, by taking $z=0,\frac{1}{8},\frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},1$ in these formulae we get the known property

$$B_{2m+1}(0)=B_{2m+1}(1)=-(2m+1)!\sum_{\substack{k\in Z\\ k\neq 0}}^{\infty}\left(\frac{1}{2i\pi k}\right)^{2m+1}=0 \quad \text{for } m>0 \tag{2.50}$$

and the powers of pi.

For examples

$$(2\pi)^{2m}=(-1)^{m+1}\frac{(2m)!2}{B_{2m}(1/2)}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{2m}} \tag{2.51}$$

$$\pi^{2m}=(-1)^{m+1}\frac{(2m)!}{2^{2m-1}\left(1-2^{2m-1}\right)B_{2m}}\sum_{k=1}^{\infty}(-1)^k\frac{1}{k^{2m}} \tag{2.52}$$

$$(2\pi)^{2m}=(-1)^{m+1}\frac{(2m)!2}{B_{2m}(1/6)}\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{3}\right)\frac{1}{k^{2m}} \tag{2.53}$$

$$(2\pi)^{2m+1}=(-1)^{m+1}\frac{(2m+1)!2}{B_{2m+1}(1/4)}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi}{2}\right)\frac{1}{k^{2m+1}} \tag{2.54}$$

and

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

$$\left(\frac{\pi}{4}\right)^2=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

$$\pi^2=36\left(\frac{1}{2}\frac{1}{1^2}-\frac{1}{2}\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{2}\frac{1}{4^2}+\frac{1}{2}\frac{1}{5^2}+\frac{1}{6^2}\right)+\left(\frac{1}{2}\frac{1}{7^2}-\frac{1}{2}\frac{1}{8^2}\right)+\cdots$$

$$\pi^3=32\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots\right)$$

etc.

3. Bernoulli Numbers

3.1. Definition and Properties

In 1713, according to Jacob Bernoulli (1655-1705), was published the list of ten first sums of powers of entire numbers [3]

$$\sum n^m = 1^m + 2^m + \cdots + n^m \tag{3.1}$$

in terms of the numbers B_k which are conjectured to be the same for all m

$$\sum n^m = \frac{1}{m+1}\sum_{k=0}^{m}(-1)^k \binom{m+1}{k} B_k n^{m+1-k}. \tag{3.2}$$

Afterward, the B_k were baptized Bernoulli numbers.

By comparison of the relation coming from (3.2)

$$\begin{aligned}\partial_n \sum n^m &= \sum_{k=0}^{m}(-1)^k \frac{m!}{(m-k)!k!} B_k n^{m-k} =: (B-n)^m \\ &= B_0 n^m - mB_1 n^{m-1} + \frac{m(m-1)}{2} B_2 n^{m-2} + \cdots + (-1)^m B_m\end{aligned} \tag{3.3}$$

with the formula coming from (2.16), (2.17)

$$\begin{aligned}&\partial_n\left(1^m + \cdots + n^{m-1}\right) + \partial_n n^m = B_m(n) + \partial_n n^m =: \left(B(0)+n\right)^m + mn^{m-1} \\ &= B_0(0)n^m + m\left(1+B_1(0)\right)n^{m-1} + \binom{m}{2} B_2(0) n^{m-2} + \cdots + B_m(0)\end{aligned} \tag{3.4}$$

we get, combining with (2.20),

$$\begin{gathered}B_0 = B_0(0) \\ B_1 = -B_1(0) - 1 = -\frac{1}{2} \\ B_{2m} = B_{2m}(0) \\ B_{2m+1} = -B_{2m+1}(0) = B_{2m+1}(1) = \frac{1}{2}\delta_{m0} \\ B_m = B_m(0)\end{gathered} \tag{3.5}$$

i.e.

"The Bernoulli numbers B_m are equal to the values at origin of the Bernoulli polynomial $B_m(z)$".

3.2. Obtaining Bernoulli Numbers

The above formula (3.5) and the recurrence formula for Bernoulli polynomials (2.43) corresponding to $z = 0$

$$0^m = \frac{1}{m+1} + \sum_{k=1}^{m} \binom{m}{k-1} \frac{B_k(0)}{k} \tag{3.6}$$

lead to that for Bernoulli numbers

$$\frac{1}{m+1}B_0 + \binom{m}{0}\frac{B_1}{1} + \binom{m}{1}\frac{B_2}{2} + \binom{m}{2}\frac{B_3}{3} + \cdots + \binom{m}{m-1}\frac{B_m}{m} = 0, m > 0 \tag{3.7}$$

which, knowing $B_0 = B_m(0) = 1$, gives $B_1, B_2, B_4, \cdots, B_m$ according to following Table 1.

This matrix equation may be resolved by doing linear combinations over lines from the second one in order to replace them with lines containing only some non-zero numbers.

Table 1. Matrix equation for calculating B_m.

$$\begin{pmatrix} 1 & & & & \cdots & & \cdots \\ 1 & 2 & & & \cdots & & \cdots \\ 1 & 3 & 3 & & \cdots & & \cdots \\ 1 & 4 & 6 & 4 & \cdots & & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 1 & \binom{m}{1} & \binom{m}{2} & \cdots & \cdots & \binom{m}{m-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{m-1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}$$

For instance, for calculating successively $\{B_0, B_1, B_2, B_4, B_6, \cdots, B_{18}\}$ we may utilize the matrix equation (**Table 2**).

We remark that the last line of this matrix has replaced

$$\left\{\binom{19}{i}, i = 0,1,2,4,6,8,10,12,14,16,18\right\}.$$

The results are

$$B_0 = 1,\ B_0 + 2B_1 = 0,\ B_1 + 3B_2 = 0,\ B_2 + 5B_4 = 0,\ -B_2 + 7B_6 = 0$$

$$\frac{9}{5}B_2 + 9B_8 = 0.\ 5B_2 + 11B_{10} = 0,\ 3B_1 - \frac{61}{105}B_2 + 13B_{12} = 0,$$

$$35B_1 + 15B_{14} = 0,\ 240B_1 + 17B_3 + 17B_{16} = 0$$

$$2052B_1 - 775B_4 + 19B_{18} = 0 = \frac{43867}{798} + B_{18} \tag{3.8}$$

Another method, maybe more interesting, for establishing table of Bernoulli numbers is obtained from the formula (2.27). It is

$$(-1-2m)B_{2m} = \binom{2m}{2}B_{2m-2}B_2 + \binom{2m}{4}B_{2m-4}B_4 + \cdots + \binom{2m}{2}B_2B_{2m-2},\ m > 1$$

or, symbolically,

$$(1-m)B_m =: (B-B)^m \tag{3.9}$$

For examples

$$(1-2)B_2 =: (B-B)^2 = 2B_0B_2 - 2B_1B_1$$

$$-4B_3 =: -3B_1B_2 + 3B_2B_1 = 0$$

$$-5B_4 = \binom{4}{2}B_2B_2 = 6B_2B_2 \Rightarrow B_4 = \frac{-1}{30}$$

$$-7B_6 = 2\binom{6}{4}B_4B_2 = 30B_4B_2 = -\frac{1}{6} \Rightarrow B_6 = \frac{1}{42}$$

$$-9B_8 = 8\times7\times B_6B_2 + \binom{8}{4}B_4B_4 \Rightarrow B_8 = -\frac{1}{30}$$

$$-11B_{10} = 10\times9\times B_8B_2 + 2\binom{10}{6}B_6B_4 \Rightarrow B_{10} = \frac{5}{66}$$

Table 2. Simplified matrix equation for calculating B_m.

$$\begin{pmatrix} 1 &&&&&&&&&&& \\ 1 & 2 &&&&&&&&&& \\ 0 & 1 & 3 &&&&&&&&& \\ 0 & 0 & 1 & 5 &&&&&&&& \\ 0 & 0 & -1 & 0 & 7 &&&&&&& \\ 0 & 0 & 9/5 & 0 & 0 & 9 &&&&&& \\ 0 & 0 & -5 & 0 & 0 & 0 & 11 &&&&& \\ 0 & 3 & \frac{-61}{105} & 0 & 0 & 0 & 0 & 13 &&&& \\ 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 & 15 &&& \\ 0 & 240 & 0 & 17 & 0 & 0 & 0 & 0 & 0 & 17 && \\ 0 & 2052 & 0 & 0 & -775 & 0 & 0 & 0 & 0 & 0 & 19 \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_4 \\ B_6 \\ B_8 \\ B_{10} \\ \vdots \\ B_{12} \\ B_{14} \\ B_{16} \\ B_{18} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-13B_{12} = 12\times 11\times B_{10}B_2 + 2\binom{12}{8}B_8B_4 + \binom{12}{6}B_6B_6 \Rightarrow B_{12} = -\frac{691}{2730}$$

$$-15B_{14} = 14\times 13\times B_{12}B_2 + 2\binom{14}{10}B_{10}B_4 + 2\binom{14}{8}B_8B_6 \Rightarrow B_{14} = \frac{7}{6}$$

$$-17B_{16} = 16\times 15\times B_{14}B_2 + 2\binom{16}{10}B_{12}B_4 + \binom{16}{8}B_8B_8 \Rightarrow B_{16} = -\frac{3617}{510}$$

$$-19B_{18} = 18\times 17\times B_{16}B_2 + 2\binom{18}{14}B_{14}B_4 + 2\binom{18}{12}B_{12}B_6 + 2\binom{18}{10}B_{10}B_8$$

$$B_{18} = \frac{43867}{798},\quad B_{20} = -\frac{174611}{330},\ \text{etc.} \tag{3.10}$$

We see that B_{18} is a sum over only four terms $B_{16}B_2, B_{14}B_4, B_{12}B_6, B_{10}B_8$; B_{20} is over five, B_{40} over ten, B_{50} over twelve terms.

3.3. Obtaining Bernoulli Polynomials and Power Sums from Bernoulli Numbers

From the formula (2.15)

$$B_m(z+a) =: (B(a)+z)^m$$

we get the symbolic Lucas formula

$$B_m(z) =: (B+z)^m \tag{3.11}$$

for calculating Bernoulli polynomials $B_m(z)$ from the set of Bernoulli numbers.

For examples

$$B_1(z) =: (B+z) = B_0 z + B_1 z^0 = z - \frac{1}{2}$$

$$B_2(z) =: (B+z)^2 = B_0 z^2 + 2B_1 z + B_2 = z^2 - z + \frac{1}{6}$$

$$B_3(z) =: (B+z)^3 = B_0 z^3 + 3B_1 z^2 + 3B_2 z + B_3 = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z$$

As for the power sums $S_m(n)$ we begin by calculating the formula coming from (2.17)

$$\partial_n S_m(n) = B_m(n) =: (B+n)^m \tag{3.12}$$

then take the primitives of both members.

For examples

$$\partial_n S_1(n) = B_1(n) \Rightarrow S_1(n) = 0+1+\cdots+(n-1) = \int B_1(n) = \int\left(n-\frac{1}{2}\right) = \frac{n^2}{2} - \frac{n}{2}$$

$$S_2(n) = 0^2 + 1^2 + \cdots + (n-1)^2 = \int B_2(n) = \int\left(n^2 - n + \frac{1}{6}\right) = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$

$$S_3(n) = \int B_3(n) = \int\left(n^3 - \frac{3}{2}n^2 + \frac{n}{2}\right) = \frac{n^4}{4} - \frac{n^3}{2} + \frac{n^2}{4} = \frac{n^2(n-1)^2}{4}$$

3.4. Bernoulli Numbers and the Euler-McLaurin Formula

From the formula for expansion of derivable functions into series of Bernoulli polynomials

$f(z) = \int_0^1 f(z)\mathrm{d}z + \sum_{k=0}^{\infty}\left[f^{(k)}(1) - f^{(k)}(0)\right]\frac{1}{(k+1)!}B_{k+1}(z)$ which leads, for periodic functions $B_m(z)$ identical to $B_m(z)$ in the interval $(0,1)$, to

$$f(z+m) = \int_m^{m+1} f(z)\mathrm{d}z + \sum_{k=0}^{\infty}\left[f^{(k)}(m+1) - f^{(k)}(m)\right]\frac{1}{(k+1)!}B_{k+1}(z) \tag{3.13}$$

we get the formula

$$\begin{aligned} f(m) = \int_m^{m+1} f(z)\mathrm{d}z + \sum_{k=0}^{r-1}\left[f^{(k)}(m+1) - f^{(k)}(m)\right]\frac{B_{k+1}(0)}{(k+1)!} \\ + \frac{(-1)^{r+1}}{r!}\int_m^{m+1} f^{(r)}(z) B_r(z)\mathrm{d}z \end{aligned} \tag{3.14}$$

analogue to the Euler-McLaurin formula one may find in [11]

For example, with $f(z) = z^3$, $B_1(0) = -\frac{1}{2}$, $B_2(0) = \frac{1}{6}$, $B_3(0) = 0$ it is verified that

$$2^3 = \int_2^3 z^3\mathrm{d}z - \frac{1}{2}\left(3^3 - 2^3\right) + 3\left(3^2 - 2^2\right)\frac{1}{12} + 6\left(3^1 - 2^1\right)\frac{0}{3} = \frac{81-16}{4} - \frac{19}{2} + \frac{15}{12} = 8$$

4. Obtaining Powers Sums of Real and Complex Numbers

4.1. From Power Sums of Integers

From the definition of the power sums on real and complex numbers

$$S_m(z,n) = z^m + (z+1)^m + \cdots + \left(z+(n-1)\right)^m \tag{4.1}$$

we get, by utilizing the translation operator e^{∂_z} mentioned in (2.4),

$$S_m(z,n)=\left(1+e^{\partial_z}+\cdots+e^{(n-1)\partial_z}\right)z^m \tag{4.2}$$

and the formula for sums of geometric progressions, the compact formula

$$S_m(z,n)=\frac{e^{n\partial_z}-1}{e^{\partial_z}-1}z^m \tag{4.3}$$

From (4.3) and the fact that

$$\partial_{z+y}f(z+y)=\partial_z f(z+y)=\partial_y f(z+y) \tag{4.4}$$

we get the symbolic formula

$$S_m(z+y,n)=\frac{e^{n\partial_{z+y}}-1}{e^{\partial_{z+y}}-1}(z+y)^m=\frac{e^{n\partial_y}-1}{e^{\partial_y}-1}(z+y)^m=:(z+S(y,n))^m$$

leading to the very interesting new formula given powers sums of complex numbers from powers sums of integers

$$S_m(z,n)=(S(n)+z)^m \tag{4.5}$$

where the undefined symbol $S^k(n)$ is to be replaced with the power sums on integers (2.17)

$$S_k(n)=0^k+1^k+\cdots+(n-1)^k=\int B_k(n),\quad 0^0=1 \tag{4.6}$$

Another way, more shortly, to obtain (4.5) is by remarking that

$$(z+n)=e^{z\partial_n}(n)$$

so that

$$S_m(z,n)=e^{z\partial_n}S_m(n)=\sum_{k=0}^{m}\frac{z^k}{k!}\partial_n^k S_m(n)=\sum_{k=0}^{m}\binom{m}{k}z^k S_{m-k}(n)=(S(n)+z)^m$$

For examples

$$S_1(z,n)=S_1(n)z^0+S_0(n)z^1=\frac{n(n-1)}{2}+nz$$

$$S_2(z,n)=:S_2(n)+2S_1(n)z+S_0(n)z^2=\left(\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}\right)+(n^2-n)z+nz^2$$

$$S_3(z,n)=:(S(n)+z)^3=:\int B_3(n)+3z\int B_2(n)+3z^2\int B_1(n)+nz^3$$

4.2. From Bernoulli Polynomials

Now, because *n* may go until infinity, ∂_n is well defined so that

$$\partial_n S_m(z,n)=\partial_n\frac{e^{n\partial_z}-1}{e^{\partial_z}-1}z^m=\frac{e^{n\partial_z}\partial_z}{e^{\partial_z}-1}z^m=B_m(z+n) \tag{4.7}$$

On the other hand, from (2.18)

$$\partial_z S_m(z,n)=\left(e^{n\partial_z}-1\right)\frac{\partial_z}{e^{\partial_z}-1}z^m=B_m(z+n)-B_m(z) \tag{4.8}$$

so that we obtain the following beautiful important formula

$$(\partial_n-\partial_z)S_m(z,n)=B_m(z) \tag{4.9}$$

as so as the historic Jacobi conjectured formula

$$\partial_n S_m(n) = B_m(n) \tag{4.10}$$

Formula (4.9) leads to the formula giving $S_m(z,n)$ directly from $B_m(z)$

$$\begin{aligned} S_m(z,n) &= \frac{1}{\partial_n - \partial_z} B_m(z) = \left(1 + \frac{\partial_z}{\partial_n} + \left(\frac{\partial_z}{\partial_n}\right)^2 + \cdots\right) n B_m(z) \\ &= \frac{n}{1!} B_m(z) + \frac{n^2}{2!} B'_m(z) + \cdots + \frac{n^{m+1}}{(m+1)!} B_m^{(m)}(z) \end{aligned} \tag{4.11}$$

i.e., to the algorithm saying that

$S_m(z,n)$ is equal to $nB_m(z)$ plus $\frac{n^2}{2!} B'_m(z)$ and so all until $\frac{n^{m+1}}{(m+1)!} B_m^{(m)}(z)$

For examples

$$S_1(z,n) = nB_1(z) + \frac{n^2}{2!} B_0(z) = nz - \frac{n}{2} + \frac{n^2}{2}$$

$$S_2(z,n) = nB_2(z) + \frac{n^2}{2!} 2B_1(z) + \frac{n^3}{3!} 2B_0(z) = n\left(z^2 - z + \frac{1}{6}\right) + n^2\left(z - \frac{1}{2}\right) + \frac{n^3}{3}$$

$$S_3(z,n) = nB_3(z) + \frac{n^2}{2!} 3B_2(z) + n^3 B_1(z) + \frac{n^4}{4}$$

In particular, we get the recurrence relation between Bernoulli polynomials given by Roman [8]

$$S_m(z,1) = z^m = B_m(z) + \frac{m}{2!} B_{m-1}(z) + \cdots + B_1(z) + \frac{1}{m+1} B_0(z) \tag{4.12}$$

and the well-known ancient formula of Bernoulli (1713)

$$S_m(n) = \frac{n}{1!} B_m + \frac{n^2}{2!} m B_{m-1} + \cdots + \frac{n^m}{m!} B_1 + \frac{n^{m+1}}{m+1} B_0 \tag{4.13}$$

Lastly, because of (4.10)

$$n^m = S_m(n+1) - S_m(n) = \int_n^{n+1} B_m(n)\,\mathrm{d}n$$

we get

$$z^m = \left(\mathrm{e}^{\partial_z} - 1\right) S_m(z) = \int_z^{z+1} B_m(n)\,\mathrm{d}n$$

$$\frac{\mathrm{e}^{\partial_z} - 1}{\partial_z} B_m(z) = \int_z^{z+1} B_m(n)\,\mathrm{d}n$$

and, by expanding functions into Bernoulli series, the formula found in Wikipedia

$$\frac{\mathrm{e}^{\partial_z} - 1}{\partial_z} f(z) = \int_z^{z+1} f(n)\,\mathrm{d}n = \left(1 + \frac{\partial_z^2}{2!} + \frac{\partial_z^3}{3!} + \cdots\right) f(z) \tag{4.14}$$

We resuming the herein-before results of calculations in following Tables (**Tables 3-5**).

Table 3. Obtaining $B_m(z)$ and $S_m(n)$ from B_m.

B_m	$B_m(z)=m\int_0^z B_{m-1}(z)\mathrm{d}z+B_m$	$S_m(n)=0^m+1^m+\cdots+(n-1)^m=\int B_m(n)$
$B_0=1$	$B_0(z)=1$	$S_0(n)=n$
$B_1=-\frac{1}{2}$	$B_1(z)=z-\frac{1}{2}$	$S_1(n)=\frac{n^2}{2}-\frac{n}{2}$
$B_2=\frac{1}{6}$	$B_2(z)=z^2-z+\frac{1}{6}$	$S_2(n)=\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$
$B_3=0$	$B_3(z)=z^3-\frac{3}{2}z^2+\frac{1}{2}z$	$S_3(n)=\frac{n^4}{4}-\frac{n^3}{2}+\frac{n^2}{4}$
$B_4=\frac{-1}{30}$	$B_4(z)=z^4-2z^3+z^2-\frac{1}{30}$	$S_4(n)=\frac{n^5}{5}-\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$
$B_5=0$	$B_5(z)=z^5-\frac{5}{2}z^4+\frac{5}{3}z^3-\frac{z}{6}$	$S_5(n)=\frac{n^6}{6}-\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}$
$B_6=\frac{1}{42}$	$B_6(z)=z^6-3z^5+\frac{5}{2}z^4-\frac{z^2}{2}+\frac{1}{42}$	$S_6(n)=\frac{n^7}{7}-\frac{n^6}{2}+\frac{n^5}{2}-\frac{n^3}{6}+\frac{n}{42}$
$B_7=0$	$B_7(z)=z^7-\frac{7}{2}z^6+\frac{7}{2}z^5-\frac{7}{6}z^3+\frac{7}{42}z$	$S_7(n)=\frac{n^8}{8}-\frac{n^7}{2}+\frac{7}{12}n^6-\frac{7}{24}n^4+\frac{7}{84}n^2$

Table 4. Obtaining $S_m(z,n)$ from $B_m(z)$.

$B_m(z)$	$S_m(z,n)=B_m(z)n+B'_m(z)\frac{n^2}{2!}+\cdots+B_m^{(m)}(z)\frac{n^{m+1}}{(m+1)!}$
$B_0(z)=1$	$S_0(z,n)=n$
$B_1(z)=z-\frac{1}{2}$	$S_1(z,n)=\left(z-\frac{1}{2}\right)n+\frac{n^2}{2!}$
$B_2(z)=z^2-z+\frac{1}{6}$	$S_2(z,n)=\left(z^2-z+\frac{1}{6}\right)n+(2z-1)\frac{n^2}{2!}+2\frac{n^3}{3!}$
$B_3(z)=z^3-\frac{3z^2}{2}+\frac{z}{2}$	$S_3(z,n)=\left(z^3-\frac{3z^2}{2}+\frac{z}{2}\right)n+\left(3z^2-3z+\frac{1}{2}\right)\frac{n^2}{2!}+(6z-3)\frac{n^3}{3!}+6\frac{n^4}{4!}$
$B_4(z)=z^4-2z^3+z^2-\frac{1}{30}$	$S_4(z,n)=B_4(z)n+4B_3(z)\frac{n^2}{2!}+12B_2(z)\frac{n^3}{3!}+24B_1(z)\frac{n^4}{4!}+24B_0(z)\frac{n^5}{5!}$

Table 5. Obtaining $S_m(z,n)$ from $S_m(n)$.

$S_m(n)$	$S_m(z,n)=:(S(n)+z)^m$
$S_0(n)=n$	$S_0(z,n)=n$
$S_1(n)=\frac{n(n-1)}{2}$	$S_1(z,n)=nz+\frac{n(n-1)}{2}$
$S_2(n)=\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$	$S_2(z,n)=nz^2+n(n-1)z+\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$
$S_3(n)=\frac{n^4}{4}-\frac{n^3}{2}+\frac{n^2}{4}$	$S_3(z,n)=S_0(n)z^3+3S_1(n)z^2+3S_2(n)z+S_3(n)$

5. The Faulhaber Formulae on Power Sums of Complex Numbers

5.1. Powers Sums of Odd Order

Although the problems of powers sums and Faulhaber conjecture were treated by many authors for examples by Radermacher [12], by Tsao in (2008) [13], by Chen, Fu, Zhang in (2009) [14], etc., nevertheless we would like to present hereafter one new approach about the problems.

In $S_m(z,n)$ let us replace the arguments z and n by

$$Z = z(z-1) \quad \text{and} \quad \lambda = S_1(z,n) = B_1(z)n + \frac{n^2}{2} \tag{5.1}$$

Because

$$\frac{\mathrm{d}Z}{\mathrm{d}z} = 2B_1(z), \quad \frac{\mathrm{d}Z}{\mathrm{d}n} = 0, \quad \frac{\mathrm{d}\lambda}{\mathrm{d}n} = B_1(z)+n, \quad \frac{\mathrm{d}\lambda}{\mathrm{d}z} = n \tag{5.2}$$

and consequently

$$\begin{aligned} \partial_n &\equiv \frac{\mathrm{d}Z}{\mathrm{d}n}\partial_Z + \frac{\mathrm{d}\lambda}{\mathrm{d}n}\partial_\lambda = \left(B_1(z)+n\right)\partial_\lambda \\ \partial_z &\equiv \frac{\mathrm{d}Z}{\mathrm{d}z}\partial_Z + \frac{\mathrm{d}\lambda}{\mathrm{d}z}\partial_\lambda = 2B_1(z)\partial_Z + n\partial_\lambda \\ \partial_n - \partial_z &\equiv B_1(z)\left(\partial_\lambda - 2\partial_Z\right) \end{aligned} \tag{5.3}$$

we have, regarding (4.9),

$$\left(\partial_\lambda - 2\partial_Z\right)S_m(z,n) = B_1^{-1}(z)\left(\partial_n - \partial_z\right)S_m(z,n) = B_1^{-1}(z)B_m(z) \tag{5.4}$$

and the form of the formula for general power sums

$$2S_m(z,n) = \left(2\lambda + \frac{(2\lambda)^2}{2!}\partial_Z + \cdots + \frac{(2\lambda)^m}{m!}\partial_Z^{m-1}\right)B_1^{-1}(z)B_m(z) \tag{5.5}$$

that may be calculated by the following considerations.

From the property

$$\begin{aligned} (2k+2)B_{2k+1}(z) &= \partial_z B_{2k+2}(z) = \frac{\mathrm{d}Z}{\mathrm{d}z}\partial_Z B_{2k+2}(z) \\ &= (2z-1)\partial_Z B_{2k+2}(z) = 2B_1(z)\partial_Z B_{2k+2}(z) \end{aligned} \tag{5.6}$$

we get, for utilization in (5.5),

$$B_1^{-1}(z)B_{2k+1}(z) = \frac{1}{k+1}\partial_Z B_{2k+2}(z) \tag{5.7}$$

and, finally,

$$\begin{aligned} 2S_{2k+1}(z,n) &= \left(2\lambda + \frac{(2\lambda)^2}{2!}\partial_Z + \cdots + \frac{(2\lambda)^{2k+1}}{(2k+1)!}\partial_Z^{2k}\right)\frac{1}{k+1}\partial_Z B_{2k+2}(z) \\ &= \sum_{j=1}^{2k+1}\frac{(2\lambda)^j}{j!}\partial_Z^{j-1}\left(\partial_Z \frac{B_{2k+2}(z)}{k+1}\right) \end{aligned} \tag{5.8}$$

All the problem is reduced to the calculations of $\partial_Z B_{2m+2}(z)$ in function of Z

which are not so difficult.

For examples:

$$2B_1^{-1}(z)B_3(z)=\partial_Z B_4(z)=\partial_Z\left(z^4-2z^3+z^2-\frac{1}{30}\right)$$
$$=\partial_Z\left(z^2-z\right)^2=\partial_Z Z^2=2Z$$

$$2S_3(z,n)=(2\lambda)Z+\frac{(2\lambda)^2}{2!}=2\lambda Z+2\lambda^2,\quad \lambda=S_1(z,n)$$

$$2S_3(n)=\frac{(2\lambda)^2}{2!}=2\lambda^2$$

$$3B_1^{-1}(z)B_5(z)=\partial_Z B_6(z)=\partial_Z\left(z^6-3z^5+\frac{5}{2}z^4-\frac{1}{2}z^2+\frac{1}{42}\right)$$
$$=\partial_Z\left(z^6-3z^5+3z^4-z^3\right)-\frac{1}{2}z^4+z^3-\frac{1}{2}z^2$$
$$=\partial_Z\left(Z^3-\frac{1}{2}Z^2\right)=3Z^2-Z$$

$$S_5(z,n)=\frac{1}{2}Z\left(Z-\frac{1}{3}\right)\frac{2\lambda}{1!}+\frac{1}{2}\left(2Z-\frac{1}{3}\right)\frac{(2\lambda)^2}{2!}+\frac{1}{2}(2)\frac{(2\lambda)^3}{3!}$$
$$=\left(Z^2-\frac{1}{3}Z\right)\lambda+\left(2Z-\frac{1}{3}\right)\lambda^2+4\frac{\lambda^3}{3}$$

$$S_5(n)=-\frac{1}{3}S_1^2(n)+\frac{4}{3}S_1^3(n)$$

$$4B_1^{-1}(z)B_7(z)=\partial_Z B_8(z)=\partial_Z\left(z^8-4z^7+\frac{14}{3}z^6-\frac{7}{3}z^4+\frac{2}{3}z^2\right)$$
$$=\partial_Z\left(Z^4-\frac{4}{3}Z^3+\frac{2}{3}Z^2\right)=4Z^3-4Z^2+\frac{4}{3}Z$$

$$2S_7(z,n)=\left(Z^3-Z^2+\frac{1}{3}Z\right)\frac{2\lambda}{1!}+\left(3Z^2-2Z+\frac{1}{3}\right)\frac{(2\lambda)^2}{2!}+(6Z-2)\frac{(2\lambda)^3}{3!}$$

and so all.

As corollary of the calculations of $\partial_Z B_{2k+2}(z)$ we may state that "All $B_{2k}(z)$ and all $B_1^{-1}(z)B_{2k+1}(z)$ are polynomials of order k in Z".

5.2. Faulhaber Formula for Even Power Sums $S_{2k}(z,n)$

By differentiating both members of (5.7) and remarking that $\partial_z Z=Z'=(2z-1)$, $\lambda'=n$ we obtain the formula giving $S_{2m}(z,n)$

$$(2m+1)S_{2m}(z,n)=\sum_{k=1}^{2m+1}\frac{(2\lambda)^k}{k!}B_1(z)\partial_Z^k\left(\frac{1}{m+1}\partial_Z B_{2m+2}(z)\right)+\sum_{k=1}^{2m+1}\frac{(2\lambda)^{k-1}2n}{(k-1)!}\partial_Z^{k-1}\left(\frac{1}{m+1}\partial_Z B_{2m+2}(z)\right)\tag{5.9}$$

For examples

$$S_3(z,n) = Z\lambda + \lambda^2$$

$$S_5(z,n) = \left(\frac{1}{2}Z^2 - \frac{1}{6}Z\right)\frac{2\lambda}{1!} + \left(Z - \frac{1}{6}\right)\frac{(2\lambda)^2}{2!} + \frac{(2\lambda)^3}{3!}$$

$$5S_4(z,n) = Z'\left(Z - \frac{1}{6}\right)\frac{2\lambda}{1!} + \left(\frac{1}{2}Z^2 - \frac{1}{6}Z\right)2n + Z'\frac{(2\lambda)^2}{2!} + \left(Z - \frac{1}{6}\right)4\lambda n + 4\lambda^2 n$$

The arrangement into polynomials with respect to (2λ) is immediate.

Remarks and Conclusions

We subjectively think that this work is a real and effective contribution to the knowledge of Bernoulli polynomials, Bernoulli numbers and Sums of powers of entire and complex numbers, as indicated in Introduction.

The main particularity of this work is the use of the translation or shift operator $e^{a\partial_z}$ that is curiously let apart by quasi all authors although this is seen to be very useful and easy to utilize.

By the utilization of many new properties on $B_m(z)$ such as

$$S_m(n) = \int B_m(n)$$

$$B_{m+1}(z) = (m+1)\int_0^z B_m(x)\mathrm{d}x - (m+1)\int_0^1 \mathrm{d}z\int_0^z B_m(x)\mathrm{d}x$$

$$B_m(z+y) =: m(z+y)B_{m-1}(z+y) - (B(z) + B(y+1))^m$$

$$(z+n) = e^{z\partial_n}(n)$$

we easily get the new key formulae

$S_m(z,n) =: (S(n)+z)^m$ together with $(\partial_n - \partial_z)S_m(z,n) = B_m(z)$

for obtaining $S_m(z,n)$.

We find also the miraculous symbolic formula for calculating rapidly the Bernoulli numbers

$$(1-m)B_m =: (B-B)^m$$

which together with the Lucas symbolic formula

$$B_m(z) =: (B+z)^m$$

give easily $B_m(z)$.

Afterward by a change of arguments from z into $Z = z(z-1)$ and n into $\lambda = S_1(z,n)$ we get the relation $(\partial_n - \partial_z) = B_1(z)(\partial_{2\lambda} - \partial_Z)$ which together with the proof that $B_1^{-1}(z)B_{2k+1}(z)$ and $B_{2k}(z)$ are polynomials in Z gives simply rise to the Faulhaber form of $S_m(z,n)$.

Operator calculus, which is very different from Heaviside operational calculus thus merits to be known. Moreover, it has a solid foundation and many interesting applications in the domains of Special functions, Differential equations, Fourier and other transforms, quantum mechanics [10].

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Because there are thousands of works about Bernoulli numbers and polynomials during centuries we surely have omitted to cite many references, we apologize for this and would like to receive comments from researchers in order to correct this work. Before, we thank you very much for that.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Dynamics Analysis of an Aquatic Ecological Model with Allee Effect

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Abstract

In this paper, based on the dynamic relationship between algae and protozoa, an aquatic ecological model with Allee effect was established to investigate how some ecological environment factors affect coexistence mode of algae and protozoa. Mathematical derivation works mainly gave some key conditions to ensure the existence and stability of all possible equilibrium points, and to induce the occurrence of transcritical bifurcation and Hopf bifurcation. The numerical simulation works mainly revealed ecological relationship change characteristics of algae and protozoa with the help of bifurcation dynamics evolution process. Furthermore, it was also worth emphasizing that Allee effect had a strong influence on the dynamic relationship between algae and protozoa. In a word, it was hoped that the research results could provide some theoretical support for algal bloom control, and also be conducive to the rapid development of aquatic ecological models.

Keywords

Algae, Protozoa, Allee Effect, Bifurcation, Relationship

1. Introduction

As everyone knows, lake eutrophication is a natural process and a stage of lake evolution. However, a common phenomenon associated with lake eutrophication is that many phytoplankton, especially those with buoyancy or mobility, usually multiply in large numbers to form algal blooms, which can cause a series of serious water environment problems [1] [2]. Therefore, the cause, harm and control measures of algal blooms have become one of the important environmental issues concerned by the academic community.

There are many organisms in nature that can inhibit algal growth, mainly including: cyanobacteria virus (algaphage), alginolytic bacteria, protozoa, fungi and actinomycetes [3] [4] [5] [6]. At the same time, protozoa are an important link in aquatic food chain, many protozoa can eat algae, and some even take algae as their only food [7]. A large number of studies have shown that the decline of algal biomass is often accompanied by a sharp increase in the number of protozoa, considering the environmental adaptability, reproductive ability, algal control efficacy, host range, adaptability to host changes of various algal control biological factors, scholars believe that protozoa are a control factor with great application prospects [7] [8] [9]. The paper [10] mainly focused on the dispute and consensus of non classical biological manipulation technology dominated by silver carp and bighead carp. The paper [11] pointed out that silver and bighead carps were just suitable for controlling cyanobacteria bloom by comparison with the increasing of blue-green algae's proportion and the forming of microcystis bloom within the enclosures without fish. The paper [12] constructed a new aquatic ecological model to understand the dynamic relationship between *Microcystis aeruginosa* and filter-feeding fish, which could indirectly show the algae control effect of filter-feeding fish. The paper [13] proposed an aquatic amensalism model to explore the inhibition mechanism of algicidal bacteria on algae. In general, the use of protozoa to control algal blooms is a brand-new control idea, which is worth our in-depth exploration.

The Allee effect is an ecological concept with roots that go back at least to the 1920s, and fifty years have elapsed since the last edition of a book by W.C. Allee, the father of this process in the paper [14]. The paper [15] pointed out that Allee effect is divided into weak Allee effect and strong Allee effect, weak Allee effect refers to the unit individual growth rate at low densities, increasing with population density and always positive, with the population showing a positive growth trend, strong Allee effect refers to the unit individual growth rate at low density, increasing with population density but negative below a critical value, when population density below that becomes negative and tends to extinction. The paper [16] investigated sufficient conditions for the existence of coexisting solutions from the strong and weak Allee effects. The paper [17] established a predation-prey system with Allee effects to study the stability analysis of nonspatial systems and obtain the existence of Hopf branching at coexisting equilibrium points and the stability of branching periodic solutions. The paper [18] shown that the model with strong Allee effect has at most two positive equilibrium point in the first quadrant, while the model with weak Allee effect has at most three positive equilibrium point in the first quadrant. The paper [19] discussed the impacts of Allee effect on co-existence, stability, bistability and bifurcations, and pointed out that the introduction of Allee effect could induce more rich dynamics and compel the model to be more sensitive to initial population densities. The paper [20] pointed out that the model with strong Allee effect could exhibit multiple stability in the first quadrant, and the model with weak Allee effect could undergo saddle knot bifurcation, Hopf bifurcation and Bogdanov-Takes bifurcation in the first quadrant. In short, with more and more examples of Allee phenomenon in natural ecology, more and more researchers pay attention to Allee effect, then more and more excellent achievements will appear in the near future.

Other arrangements of this paper are as follows: In the second section, an aquatic ecological model with Allee effect is built to describe the ecological relationship between algae and protozoa. In the third section, the existence and stability of all possible equilibrium points are studied. In the fourth section, the possible bifurcation dynamic behavior of the model (2.2) is mainly explored. In the fifth section, relevant dynamic simulation tests are carried out to verify the feasibility of theoretical results and demonstrate the evolution trend of population coexistence mode. In the sixth section, we mainly give the main conclusions and make some explanations.

2. Ecological Mathematical Modeling

At present, it is of special significance to apply ecological models to study the problem of biological algae control, this is because that ecological model can form three basic of studying biological system, namely: trophic level analysis, system perspective and dynamic view [21] [22], which can improve the understanding of the ecological interactions between populations and their dependence on internal and external conditions [23] [24]. The paper [12] proposed a new aquatic ecological model to explore the aggregation behavior of algae population. According to the modeling framework of this new aquatic ecological model, we will propose an aquatic ecological model to characterize the dynamic relationship between algae and protozoa (protozoa can feed on algae), which can be described as follow:

$$\begin{cases} \dfrac{\mathrm{d}N}{\mathrm{d}T} = r_1 N\left(1-\dfrac{N}{K_1}\right)\dfrac{N}{N+a} - \dfrac{\alpha_1 (N-g)P}{c+N-g} - m_1 N, \\ \dfrac{\mathrm{d}P}{\mathrm{d}T} = r_2 P + \dfrac{\beta_1 \alpha_1 (N-g)P}{c+N-g} - m_2 P, \end{cases} \tag{2.1}$$

where $N(T)$ and $P(T)$ are density of algae population and protozoa respectively, r_1 is maximum growth rate of algae population, K_1 is maximum environmental capacity for algae population, m_1 is mortality rate of algae population, a is Allee effect coefficient, r_2 is intrinsic growth rate of protozoa, c is saturation coefficient, m_2 is mortality rate of protozoa, α_1 is capture rate of protozoa preying on algae, β_1 is energy conversion rate, and g is algal aggregation parameter.

For simplicity, we will replace the model (2.1) with the following variable:

$$N = cx, P = \frac{r_1 c y}{\alpha_1}, T = \frac{t}{r_1}, p = \frac{K_1}{c}, d = \frac{g}{c},$$

$$m = \frac{m_1}{r_1}, q = \frac{a}{c}, b = \frac{r_2}{r_1}, e = \frac{\beta_1 \alpha_1}{r_1}, n = \frac{m_2}{r_1},$$

then the model (2.2) is obtained:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{x^2}{x+q}\left(1-\dfrac{x}{p}\right) - \dfrac{(x-d)y}{1+x-d} - mx, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = by + \dfrac{ey(x-d)}{1+x-d} - ny. \end{cases} \tag{2.2}$$

For the model (2.2), the existence and stability of all possible equilibrium points will firstly be discussed. Then some critical conditions are given to demonstrate the occurrence of transcritical bifurcation and Hopf bifurcation. Finally, some numerical simulations were implemented to not only verify the feasibility of the theoretical results, but also dynamically evolve ecological dynamic relationship between algae and protozoa, which can abstract out ecological evolution significance represented by bifurcation dynamic evolution behavior.

3. Existence and Stability of All Possible Equilibrium Points

In this section, we will explore the existence and stability of all possible equilibrium points of the model (2.2), which represents the special dynamic relationship between populations.

To obtain all possible equilibrium points of the model (2.2), we list the following equations from the model (2.2):

$$\begin{cases} \dfrac{x^2}{x+q}\left(1-\dfrac{x}{p}\right) - \dfrac{(x-d)y}{1+x-d} - mx = 0, \\ by + \dfrac{ey(x-d)}{1+x-d} - ny = 0. \end{cases} \tag{2.3}$$

It is easy to find that the model (2.2) has five possible equilibrium points: $E_0(0,0)$, $E_1(x_1,0)$, $E_2(x_2,0)$, $E_3(x_3,0)$, $E_*(x_*,y_*)$, where

$$x_1 = \frac{(1-m)p+\sqrt{\Delta}}{2} > 0, x_2 = \frac{(1-m)p-\sqrt{\Delta}}{2} > 0,$$

$$x_3 = \frac{p(1-m)}{2} > 0, \Delta = (m-1)^2 p^2 - 4mpq > 0.$$

According to the equation(2.3), the model(2.2) has one internal equilibrium point if $n > b$, $d > \dfrac{b-n}{b+e-n}$ and $x_2 < x_* < x_2$, where

$$x_* = d - \frac{b-n}{b+e-n}, y_* = \frac{\left[x_*^2(p-x_*) - mpx_*(x_*+q)\right](1+x_*-d)}{(x_*+q)(x_*-d)p}.$$

Thus, we can give Theorem 1, which is mainly the critical condition for the existence of all possible equilibrium points.

Theorem 1 1) The boundary equilibrium point $E_0(0,0)$ always exists.

2) The boundary equilibrium point $E_1(x_1,0)$ and $E_2(x_2,0)$ exist if and only if $q < \dfrac{(1-m)^2 p}{4m}$ and $0 < m < 1$.

3) The boundary equilibrium point $E_3(x_3,0)$ exists if and only if

$q=\frac{(1-m)^2 p}{4m}$ and $0<m<1$.

4) The internal equilibrium point $E_*(x_*,y_*)$ exists if and only if $n>b$, $d>\frac{b-n}{b-n+e}$ and $x_2<x_*<x_1$.

Because the stability of the equilibrium point is determined by the properties of the eigenvalues of its Jacobian matrix, the stability of each equilibrium point is discussed, thus we can get that the Jacobi matrix of the model (2.2) is

$$J=\begin{bmatrix} \frac{-2x^3+(p-3q)x^2+2pqx}{p(x+q)^2}-\frac{y}{(1+x-d)^2}-m & -\frac{x-d}{1+x-d} \\ \frac{ey}{(1+x-d)^2} & b+\frac{e(x-d)}{1+x-d}-n \end{bmatrix}.$$

On the based of the Jacobi matrix, we can obtain Theorem 2-6, which mainly explore the types and stability of all equilibrium points.

Theorem 2 Under the premise of $n>b$, we have

1) If $b-n+e>0$ and $d>1$ or $e+b-n<0$ and $1<d<\frac{b-n}{e+b-n}$ hold, $E_0(0,0)$ is a saddle.

2) If $d<1$ or $e+b-n<0$ and $d>\frac{b-n}{e+b-n}$ hold $E_0(0,0)$ is a stable node.

Proof. The Jacobi matrix of the $E_0(0,0)$ is:

$$J_{E_0}=\begin{bmatrix} -m & \frac{d}{1-d} \\ 0 & b-\frac{ed}{1-d}-n \end{bmatrix}.$$

Apparently, the Jacobi matrix of J_{E_0} has two characteristic roots, $\lambda_1=-m<0$, $\lambda_2=b-\frac{ed}{1-d}-n$. Under the premise of $n>b$, it is easy to know that if $b-n+e>0$ and $d>1$ or $e+b-n<0$ and $1<d<\frac{b-n}{e+b-n}$ hold, the boundary equilibrium point $E_0(0,0)$ is a saddle; if $d<1$ or $e+b-n<0$ and $d>\frac{b-n}{e+b-n}$ hold, the boundary equilibrium point $E_0(0,0)$ is a stable node.

Theorem 3 Under the premise of $n>b$, we have

1) If $q<\frac{p(m-1)^2}{4m}$ and $\frac{e(x_1-d)}{1+x_1-d}>n-b$ hold, the boundary equilibrium point $E_1(x_1,0)$ is a saddle.

2) If $q<\frac{p(m-1)^2}{4m}$ and $\frac{e(x_1-d)}{1+x_1-d}<n-b$ hold, the boundary equilibrium

point $E_1(x_1,0)$ is a stable node.

Proof. The Jacobi matrix of the boundary equilibrium point $E_1(x_1,0)$ is:

$$J_{E_1(x_1,0)}=\begin{bmatrix}\dfrac{-2x_1^3+(p-3q)x_1^2+2pqx_1}{p(x_1+q)^2}-m & -\dfrac{x_1-d}{1+x_1-d}\\ 0 & b+\dfrac{e(x_1-d)}{1+x_1-d}-n\end{bmatrix}.$$

Then, the Jacobi matrix of $J_{E_1(x_1,0)}$ has two characteristic roots,

$$\lambda_1=\frac{-2x_1^3+(p-3q)x_1^2+2pqx_1}{p(x_1+q)^2}-m,\lambda_2=b+\frac{e(x_1-d)}{1+x_1-d}-n.$$

Firstly, we will analyze the positive and negativity of λ_1. It is easy to know that $p(x_1+q)^2>0$ and x_1 satisfies $x_1^2+p(m-1)x_1+mpq=0$, we put it in λ_1 and sort it out.

$$\lambda_1=\frac{\left[-(m-1)^2p^2+(3m-1)pq\right]x_1+2mpq^2+mp^2(1-m)q}{p(x_1+q)^2}.$$

According to $q<\dfrac{p(m-1)^2}{4m}$, and

$$x_1=\frac{(1-m)p+\sqrt{\Delta}}{2}>0,$$

therefore

$$\left[-(m-1)^2p^2+(3m-1)pq\right]x_1+2mpq^2+mp^2(1-m)q<\frac{p^2(1-m)^2(m+1)}{8m}\left(-\sqrt{\Delta}\right)<0,$$

so $\lambda_1<0$.

Secondly, we analyze the positive and negativity of λ_2. Through calculation, we can get that if $q<\dfrac{p(m-1)^2}{4m}$ and $\dfrac{e(x_1-d)}{1+x_1-d}>n-b$ hold, we have $\lambda_2>0$, then the boundary equilibrium point $E_1(x_1,0)$ is a saddle; If $q<\dfrac{p(m-1)^2}{4m}$ and $\dfrac{e(x_1-d)}{1+x_1-d}<n-b$ hold, we have $\lambda_2<0$, then the boundary equilibrium point $E_1(x_1,0)$ is a stable node.

Theorem 4 Under the premise of $n>b$, we have

1) If $q<\dfrac{p(m-1)^2}{4m}$, $m\neq\dfrac{1}{3}$ and $\dfrac{e(x_2-d)}{1+x_2-d}>n-b$ hold, the boundary equilibrium point $E_2(x_2,0)$ is a saddle.

2) If $q<\dfrac{p(m-1)^2}{4m}$, $m\neq\dfrac{1}{3}$ and $\dfrac{e(x_2-d)}{1+x_2-d}<n-b$ hold, the boundary equilibrium point $E_2(x_2,0)$ is a stable node.

Proof. The Jacobi matrix of the boundary equilibrium point $E_2(x_2,0)$ is

$$J_{E_2(x_2,0)} = \begin{bmatrix} \dfrac{-2x_2^3+(p-3q)x_2^2+2pqx_2}{p(x_2+q)^2}-m & -\dfrac{x_2-d}{1+x_2-d} \\ 0 & b+\dfrac{e(x_2-d)}{1+x_2-d}-n \end{bmatrix}.$$

Then, the Jacobi matrix of $J_{E_2(x_2,0)}$ has two characteristic roots,

$$\lambda_1 = \frac{-2x_2^3+(p-3q)x_2^2+2pqx_2}{p(x_2+q)^2}-m, \lambda_2 = b+\frac{e(x_2-d)}{1+x_2-d}-n.$$

Firstly, we will analyze the positive and negativity of λ_1, because $p(x_2+q)^2>0$, owing to x_2 satisfies $x_2^2+p(m-1)x_2+mpq=0$, we put it in λ_1 and sort it out, we can get

$$\lambda_1 = \frac{\left[-(m-1)^2 p^2+(3m-1)pq\right]x_2+2mpq^2+mp^2(1-m)q}{p(x_2+q)^2}.$$

1) If $\lambda_1>0$, then $\left[-(m-1)^2 p^2+(3m-1)pq\right]x_2>-\left(2mpq^2+mp^2(1-m)q\right)$.

a) If $\left[-(m-1)^2 p^2+(3m-1)pq\right]>0$, then $(3m-1)q>(m-1)^2 p$,

$$x_2 > \frac{-\left[2mpq^2+mp^2(1-m)q\right]}{-(m-1)^2 p^2+(3m-1)pq}.$$

i) When $0<m<\frac{1}{3}$, $q<\frac{p(m-1)^2}{3m-1}<0$ contradicts $q>0$.

ii) When $\frac{1}{3}<m<1$, $q>\frac{p(m-1)^2}{3m-1}<0$ has no intersection with $0<q<\frac{p(m-1)^2}{4m}$, so there is no solution.

b) If $\left[-(m-1)^2 p^2+(3m-1)pq\right]<0$, we have $(3m-1)q<(m-1)^2 p$, $x_2<\frac{-\left[2mpq^2+mp^2(1-m)q\right]}{-(m-1)^2 p^2+(3m-1)pq}$, also because $x_2=\frac{(1-m)p-\sqrt{\Delta}}{2}$, therefore, it must be satisfied that

$$\frac{(1-m)p-\sqrt{\Delta}}{2}+\frac{2mpq^2+mp^2(1-m)q}{-(m-1)^2 p^2+(3m-1)pq}<0.$$

It can be deformed

$$\frac{\left[(1-m)p-\sqrt{\Delta}\right]\cdot\left[-(m-1)^2 p+(3m-1)q\right]+4mq^2+2mpq(1-m)}{2\cdot\left[-(m-1)^2 p+(3m-1)q\right]}<0$$

for $\left[-(m-1)^2 p+(3m-1)q\right]<0$. So it needs to be proved that

$$\left[(1-m)p-\sqrt{\Delta}\right]\cdot\left[-(m-1)^2 p+(3m-1)q\right]+4mq^2+2mpq(1-m)>0,$$

simplified the upper type

$$\left(4mq-(m-1)^2 p\right)\cdot(p+q)>0,$$

because $(p+q)>0$, and $4mq-(1-m)^2 p>0$, $q>\frac{p(m-1)^2}{4m}$ contradicts $0<q<\frac{p(m-1)^2}{4m}$, so $\lambda_1<0$.

2) If $\lambda_1<0$, then $\left[-(m-1)^2 p^2+(3m-1)pq\right]x_2<-\left(2mpq^2+mp^2(1-m)q\right)$,

a) If $\left[-(m-1)^2 p^2+(3m-1)pq\right]>0$, then $x_2<\frac{-\left[2mpq^2+mp^2(1-m)q\right]}{-(m-1)^2 p^2+(3m-1)pq<0}$, it contradicts $x_2>0$.

b) If $\left[-(m-1)^2 p^2+(3m-1)pq\right]<0$, we have $(3m-1)q<(m-1)^2 p$, $x_2>\frac{-\left[2mpq^2+mp^2(1-m)q\right]}{-(m-1)^2 p^2+(3m-1)pq}$, also because $x_2=\frac{(1-m)p-\sqrt{\Delta}}{2}$, therefore, it must be satisfied that

$$\frac{(1-m)p-\sqrt{\Delta}}{2}+\frac{2mpq^2+mp^2(1-m)q}{-(m-1)^2 p^2+(3m-1)pq}>0.$$

It can be deformed

$$\frac{\left[(1-m)p-\sqrt{\Delta}\right]\cdot\left[-(m-1)^2 p+(3m-1)q\right]+4mq^2+2mpq(1-m)}{2\cdot\left[-(m-1)^2 p+(3m-1)q\right]}>0$$

for $\left[-(m-1)^2 p+(3m-1)q\right]<0$. So it needs to be proved that

$$\left[(1-m)p-\sqrt{\Delta}\right]\cdot\left[-(m-1)^2 p+(3m-1)q\right]+4mq^2+2mpq(1-m)<0,$$

simplified the upper type

$$\left(4mq-(m-1)^2 p\right)\cdot(p+q)<0,$$

because $(p+q)>0$, $4mq-(1-m)^2 p<0$, so $q<\frac{p(m-1)^2}{4m}$, therefore, this situation is true. On account of $(3m-1)q<(m-1)^2 p$,

i) When $0<m<\frac{1}{3}$, so $q>\frac{p(m-1)^2}{3m-1}<0$ and $0<q<\frac{p(m-1)^2}{4m}$, they take the intersection to get $0<q<\frac{p(m-1)^2}{4m}$.

ii) When $\frac{1}{3}<m<1$, $q<\frac{p(m-1)^2}{3m-1}<0$ and $0<q<\frac{p(m-1)^2}{4m}$, they take the intersection to get $0<q<\frac{p(m-1)^2}{4m}$.

In a word, when $0<q<\frac{p(m-1)^2}{4m}$ and $m\neq\frac{1}{3}$ hold, we have $\lambda_1<0$.

Secondly, we analyze the positive and negativity of λ_2. When $q<\frac{p(m-1)^2}{4m}$

and $\frac{e(x_2-d)}{1+x_2-d} > n-b$ hold, the boundary equilibrium point $E_2(x_2,0)$ is a saddle. when $q < \frac{p(m-1)^2}{4m}$ and $\frac{e(x_2-d)}{1+x_2-d} < n-b$ hold, the boundary equilibrium point $E_2(x_2,0)$ is a stable node.

Theorem 5 Under the premise of $n > b$, we have

1) If $q < \frac{p(m-1)^2}{4m}$ and $\frac{e(x_3-d)}{1+x_3-d} > n-b$ hold, the boundary equilibrium point $E_3(x_3,0)$ is a saddle.

2) If $q < \frac{p(m-1)^2}{4m}$ and $\frac{e(x_3-d)}{1+x_3-d} < n-b$ hold, the boundary equilibrium point $E_3(x_3,0)$ is a stable node.

Proof. The Jacobi matrix of the boundary equilibrium point $E_3(x_3,0)$ is

$$J_{E_3(x_3,0)} = \begin{bmatrix} \frac{-2x_3^3+(p-3q)x_3^2+2pqx_3}{p(x_3+q)^2}-m & -\frac{x_3-d}{1+x_3-d} \\ 0 & b+\frac{e(x_3-d)}{1+x_3-d}-n \end{bmatrix}.$$

It is easy to obtain that the Jacobi matrix of $J_{E_3(x_3,0)}$ has two characteristic roots,

$$\lambda_1 = \frac{-2x_3^3+(p-3q)x_3^2+2pqx_3}{p(x_2+q)^2}-m, \lambda_2 = b+\frac{e(x_3-d)}{1+x_3-d}-n.$$

Firstly, we will analyze the positive and negativity of λ_1, because x_3 satisfies $x_3^2+p(m-1)x_3+mpq=0$ and $x_3^2=-p(m-1)x_3-mpq$, we put it in λ_1 and sort it out, we can get

$$\lambda_1 = \frac{\left[-(m-1)^2 p^2+(3m-1)pq\right]x_3+2mpq^2+mp^2(1-m)q}{p(x_3+q)^2}.$$

Owing to $x_3 = \frac{(1-m)p}{2}$, simplified $\left[-(m-1)^2 p^2+(3m-1)pq\right]x_3+2mpq^2+mp^2(1-m)q$, we can get $\lambda_1 = \frac{-(1-m)^4 p^3}{p(x_3+q)^2} < 0$, so $\lambda_1 < 0$.

Secondly, we analyze the positive and negativity of λ_2. When $q = \frac{p(m-1)^2}{4m}$ and $\frac{e(x_3-d)}{1+x_3-d} > n-b$ hold, the boundary equilibrium point $E_3(x_3,0)$ is a saddle. When $q = \frac{p(m-1)^2}{4m}$, and $\frac{e(x_3-d)}{1+x_3-d} < n-b$ hold, the boundary equilibrium point $E_3(x_3,0)$ is a stable node.

Theorem 6 Under the condition of the internal equilibrium point $E_*(x_*, y_*)$,

1) If $Det(J_{E_*}) > 0$, $Tr(J_{E_*}) < 0$ hold, the internal equilibrium point $E_*(x_*, y_*)$ is a stable node(focus).

2) If $Det(J_{E_*}) > 0$, $Tr(J_{E_*}) > 0$ hold, the internal equilibrium point $E_*(x_*, y_*)$ is an unstable node(focus).

Proof. The Jacobi matrix of the internal equilibrium point $E_*(x_*, y_*)$ is,

$$J_{E_*(x_*,y_*)} = \begin{bmatrix} \dfrac{-2x_*^3 + (p-3q)x_*^2 + 2pqx_*}{p(x_*+q)^2} - \dfrac{y_*}{(1+x_*-d)^2} - m & -\dfrac{x_*-d}{1+x_*-d} \\ 0 & b + \dfrac{e(x_*-d)}{1+x_*-d} - n \end{bmatrix}$$

where $x_* = d - \dfrac{b-n}{b+e-n}$, $y_* = \dfrac{\left[x_*^2(p-x_*) - mpx_*(x_*+q)\right](1+x_*-d)}{(x_*+q)(x_*-d)p}$. The following characteristic equation is obtained as follow

$$\lambda^2 - \left[\frac{-2x_*^3 + (p-3q)x_*^2 + 2pqx_*}{p(x_*+q)^2} - \frac{y_*}{(1+x_*-d)^2} - m\right]\cdot\lambda + \frac{x_*-d}{1+x_*-d}\cdot\frac{ey_*}{(1+x_*-d)^2} = 0,$$

and

$$Tr(J_{E_*}) = \lambda_1 + \lambda_2, Det(J_{E_*}) = \lambda_1\lambda_2,$$

here

$$Det(J_{E_*}) = \frac{x_*-d}{1+x_*-d}\cdot\frac{ey_*}{(1+x_*-d)^2},$$

for $y_* > 0$. Then we can get

$$q < \frac{-x_*^2 - mpx_* + px_*}{mp},$$

for

$$T = \frac{-2x_*^3 + (p-3q)x_*^2 + 2pqx_*}{p(x_*+q)^2} - \frac{y_*}{(1+x_*-d)^2} - m.$$

Thus if $Det(J_{E_*}) > 0$, $Tr(J_{E_*}) < 0$ hold, the internal equilibrium point $E_*(x_*, y_*)$ is a stable node(focus); if $Det(J_{E_*}) > 0$, $Tr(J_{E_*}) > 0$ hold, the internal equilibrium point $E_*(x_*, y_*)$ is an unstable node(focus).

4. Local Bifurcation Analysis

In this section, we will choose parameter d as a bifurcation control parameter to investigate the bifurcation dynamics evolution characteristics of the model (2.2), and give the threshold conditions for transcritical bifurcation and Hopf bifurcation of the model (2.2).

4.1. Transcritical Bifurcation

Theorem 7 1) The model (2.2) undergoes a transcritical bifurcation at the equilibrium point $E_1(x_1,0)$ when $d = d_{TC1} = x_1 + \frac{b-n}{b+e-n}$.

2) The model (2.2) undergoes a transcritical bifurcation at the equilibrium point $E_2(x_2,0)$ when $d = d_{TC2} = x_2 + \frac{b-n}{b+e-n}$.

3) The model (2.2) undergoes a transcritical bifurcation at the equilibrium point $E_3(x_3,0)$ when $d = d_{TC3} = x_3 + \frac{b-n}{b+e-n}$.

Proof:

1) On the basis of the Theorem 3, when $d = d_{TC1} = x_1 + \frac{b-n}{b+e-n}$, the Jacobi matrix of the equilibrium point E_1 is

$$J_{E_{TC1}} = \begin{bmatrix} \frac{-2x_1^3+(p-3q)x_1^2+2pqx_1}{p(x_1+q)^2} - m & -\frac{x_1-d}{1+x_1-d} \\ 0 & 0 \end{bmatrix},$$

suppose V and W are eigenvectors of $J_{E_{TC1}}$ and $J_{E_{TC1}}^{\mathrm{T}}$, then

$$J_{E_{TC1}} V = 0 \cdot V,\ J_{E_{TC1}}^{\mathrm{T}} W = 0 \cdot W.$$

Then we can get

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1-d}{1+x_1-d} \\ \frac{-2x_1^3+(p-3q)x_1^2+2pqx_1}{p(x_1+q)^2} - m \end{bmatrix},$$

$$W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Due to

$$F_d(E_1;d_{TC1}) = \begin{bmatrix} F_{1d} \\ F_{2d} \end{bmatrix} = \begin{bmatrix} \frac{y}{(1+x-d)^2} \\ \frac{-ey}{(1+x-d)^2} \end{bmatrix}_{(E_1;d_{TC1})} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so

$$DF_d(E_1;d_{TC1})V = \begin{bmatrix} F_{1d_x} & F_{1d_y} \\ F_{2d_x} & F_{2d_y} \end{bmatrix}_{(E_1;d_{TC1})} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{(1+x_1-d)^2}\left(\frac{-2x_1^3+(p-3q)x_1^2+2pqx_1}{p(x_1+q)^2} - m\right) \\ \frac{-e}{(1+x_1-d)^2}\left(\frac{-2x_1^3+(p-3q)x_1^2+2pqx_1}{p(x_1+q)^2} - m\right) \end{bmatrix},$$

$$D^2F_d\left(E_1;d_{TC1}\right)(V,V)=\begin{bmatrix}\frac{\partial^2 F_1}{\partial x^2}v_1v_1+2\frac{\partial^2 F_1}{\partial x\partial y}v_1v_2+\frac{\partial^2 F_1}{\partial y^2}v_2v_2\\ \frac{\partial^2 F_2}{\partial x^2}v_1v_1+2\frac{\partial^2 F_2}{\partial x\partial y}v_1v_2+\frac{\partial^2 F_2}{\partial y^2}v_2v_2\end{bmatrix}_{(E_1;d_{TC1})}$$

$$=\begin{bmatrix}\frac{-2x_1^3-6qx_1^2-6q^2x_1+2pq^2}{p\left(x_1+q\right)^3}v_1v_1-\frac{2}{\left(1+x_1-d\right)^2}v_1v_2\\ \frac{2e}{\left(1+x_1-d\right)^2}v_1v_2\end{bmatrix}.$$

Thus, we can reach the following conclusions:

$$W^{\mathrm{T}}F_d\left(E_1;d_{TC1}\right)=[0,1]\begin{bmatrix}0\\0\end{bmatrix}=0,$$

$$W^{\mathrm{T}}\left[DF_d\left(E_1;d_{TC1}\right)V\right]$$
$$=\frac{-e}{\left(1+x_1-d\right)^2}\left(\frac{-2x_1^3+\left(p-3q\right)x_1^2+2pqx_1}{p\left(x_1+q\right)^2}-m\right)\neq 0,$$

$$W^{\mathrm{T}}\left[D^2F_d\left(E_1;d_{TC1}\right)(V,V)\right]$$
$$=\frac{2e\left(x_1-d\right)}{\left(1+x_1-d\right)^3}\left(\frac{-2x_1^3+\left(p-3q\right)x_1^2+2pqx_1}{p\left(x_1+q\right)^2}-m\right)\neq 0.$$

According to Sotomayors theorem, when $d=d_{TC1}=x_1+\frac{b-n}{b+e-n}$, then the model (2.2) undergoes a transcritical bifurcation at the equilibrium point $E_1\left(x_1,0\right)$.

2) On the basis of the Theorem 4, we next prove that the model (2.2) will undergo a transcritical bifurcation at the equilibrium point $E_2\left(x_2,0\right)$. When $d=d_{TC2}=x_2+\frac{b-n}{b+e-n}$, the Jacobi matrix of the equilibrium point E_2 is

$$J_{E_{TC2}}=\begin{bmatrix}\frac{-2x_2^3+\left(p-3q\right)x_2^2+2pqx_2}{p\left(x_2+q\right)^2}-m & -\frac{x_2-d}{1+x_2-d}\\ 0 & 0\end{bmatrix},$$

suppose V and W are eigenvectors of $J_{E_{TC2}}$ and $J_{E_{TC2}}^{\mathrm{T}}$, then

$$J_{E_{TC2}}V=0\cdot V,\ J_{E_{TC2}}^{\mathrm{T}}W=0\cdot W,$$

then

$$V=\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}\frac{x_2-d}{1+x_2-d}\\ \frac{-2x_2^3+\left(p-3q\right)x_2^2+2pqx_2}{p\left(x_2+q\right)^2}-m\end{bmatrix},W=\begin{bmatrix}w_1\\w_2\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}.$$

Owing to

$$F_d\left(E_2;d_{TC2}\right)=\begin{bmatrix}F_{1d}\\F_{2d}\end{bmatrix}=\begin{bmatrix}\dfrac{y}{\left(1+x-d\right)^2}\\\dfrac{-ey}{\left(1+x-d\right)^2}\end{bmatrix}_{\left(E_2;d_{TC2}\right)}=\begin{bmatrix}0\\0\end{bmatrix},$$

and

$$DF_d\left(E_2;d_{TC2}\right)V=\begin{bmatrix}F_{1d_x} & F_{1d_y}\\F_{2d_x} & F_{2d_y}\end{bmatrix}_{\left(E_2;d_{TC2}\right)}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$
$$=\begin{bmatrix}\dfrac{1}{\left(1+x_2-d\right)^2}\left(\dfrac{-2x_2^3+\left(p-3q\right)x_2^2+2pqx_2}{p\left(x_2+q\right)^2}-m\right)\\\dfrac{-e}{\left(1+x_2-d\right)^2}\left(\dfrac{-2x_2^3+\left(p-3q\right)x_2^2+2pqx_2}{p\left(x_2+q\right)^2}-m\right)\end{bmatrix},$$

$$D^2F_d\left(E_2;d_{TC2}\right)\left(V,V\right)=\begin{bmatrix}\dfrac{\partial^2F_1}{\partial x^2}v_1v_1+2\dfrac{\partial^2F_1}{\partial x\partial y}v_1v_2+\dfrac{\partial^2F_1}{\partial y^2}v_2v_2\\\dfrac{\partial^2F_2}{\partial x^2}v_1v_1+2\dfrac{\partial^2F_2}{\partial x\partial y}v_1v_2+\dfrac{\partial^2F_2}{\partial y^2}v_2v_2\end{bmatrix}_{\left(E_2;d_{TC2}\right)}$$
$$=\begin{bmatrix}\dfrac{-2x_2^3-6qx_2^2-6q^2x_2+2pq^2}{p\left(x_2+q\right)^3}v_1v_1-\dfrac{2}{\left(1+x_2-d\right)^2}v_1v_2\\\dfrac{2e}{\left(1+x_2-d\right)^2}v_1v_2\end{bmatrix},$$

we can obtain

$$W^{\mathrm{T}}F_d\left(E_2;d_{TC2}\right)=\left[0,1\right]\begin{bmatrix}0\\0\end{bmatrix}=0,$$

$$W^{\mathrm{T}}\left[DF_d\left(E_2;d_{TC2}\right)V\right]$$
$$=\frac{-e}{\left(1+x_2-d\right)^2}\left(\frac{-2x_2^3+\left(p-3q\right)x_2^2+2pqx_2}{p\left(x_2+q\right)^2}-m\right)\neq 0,$$

$$W^{\mathrm{T}}\left[D^2F_d\left(E_2;d_{TC2}\right)\left(V,V\right)\right]$$
$$=\frac{2e\left(x_2-d\right)}{\left(1+x_2-d\right)^3}\left(\frac{-2x_2^3+\left(p-3q\right)x_2^2+2pqx_2}{p\left(x_2+q\right)^2}-m\right)\neq 0.$$

According to Sotomayors theorem, when $d=d_{TC2}=x_2+\dfrac{b-n}{b+e-n}$, then the model (2.2) undergoes a transcritical bifurcation at the equilibrium point $E_2\left(x_2,0\right)$.

3) On the basis of the Theorem 5, we will prove that the model (2.2) will undergo a transcritical bifurcation at the equilibrium point $E_3\left(x_3,0\right)$. When $d=d_{TC3}=x_3+\dfrac{b-n}{b+e-n}$ holds, the equilibrium point $E_1\left(x_1,0\right)$ and $E_2\left(x_2,0\right)$ will coincide as an equilibrium point, here the Jacobian matrix at $E_3\left(x_3,0\right)$ is

$$J_{E_{TC3}}=\begin{bmatrix}\dfrac{-2x_3^3+(p-3q)x_3^2+2pqx_3}{p(x_3+q)^2}-m & -\dfrac{x_3-d}{1+x_3-d}\\ 0 & 0\end{bmatrix},$$

suppose V and W are eigenvectors of $J_{E_{TC3}}$ and $J_{E_{TC3}}^{\mathrm{T}}$, then

$$J_{E_{TC3}}V=0\cdot V,\ J_{E_{TC3}}^{\mathrm{T}}W=0\cdot W,$$

so

$$V=\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}\dfrac{x_3-d}{1+x_3-d}\\ \dfrac{-2x_3^3+(p-3q)x_3^2+2pqx_3}{p(x_3+q)^2}-m\end{bmatrix},W=\begin{bmatrix}w_1\\ w_2\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}.$$

Because of

$$F_d(E_3;d_{TC3})=\begin{bmatrix}F_{1d}\\ F_{2d}\end{bmatrix}=\begin{bmatrix}\dfrac{y}{(1+x-d)^2}\\ \dfrac{-ey}{(1+x-d)^2}\end{bmatrix}_{(E_3;d_{TC3})}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

then, we can obtain

$$DF_d(E_3;d_{TC3})V=\begin{bmatrix}F_{1d_x} & F_{1d_y}\\ F_{2d_x} & F_{2d_y}\end{bmatrix}_{(E_3;d_{TC3})}\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{1}{(1+x_3-d)^2}\left(\dfrac{-2x_3^3+(p-3q)x_3^2+2pqx_3}{p(x_3+q)^2}-m\right)\\ \dfrac{-e}{(1+x_3-d)^2}\left(\dfrac{-2x_3^3+(p-3q)x_3^2+2pqx_3}{p(x_3+q)^2}-m\right)\end{bmatrix},$$

$$D^2F_d(E_3;d_{TC3})(V,V)=\begin{bmatrix}\dfrac{\partial^2F_1}{\partial x^2}v_1v_1+2\dfrac{\partial^2F_1}{\partial x\partial y}v_1v_2+\dfrac{\partial^2F_1}{\partial y^2}v_2v_2\\ \dfrac{\partial^2F_2}{\partial x^2}v_1v_1+2\dfrac{\partial^2F_2}{\partial x\partial y}v_1v_2+\dfrac{\partial^2F_2}{\partial y^2}v_2v_2\end{bmatrix}_{(E_3;d_{TC3})}$$

$$=\begin{bmatrix}\dfrac{-2x_3^3-6qx_3^2-6q^2x_3+2pq^2}{p(x_3+q)^3}v_1v_1-\dfrac{2}{(1+x_3-d)^2}v_1v_2\\ \dfrac{2e}{(1+x_3-d)^2}v_1v_2\end{bmatrix}.$$

Thus, we can get the following conclusions:

$$W^{\mathrm{T}}F_d(E_3;d_{TC3})=[0,1]\begin{bmatrix}0\\ 0\end{bmatrix}=0,$$

$$W^{\mathrm{T}}\left[DF_d(E_3;d_{TC3})V\right]$$

$$=\frac{-e}{(1+x_3-d)^2}\left(\frac{-2x_3^3+(p-3q)x_3^2+2pqx_3}{p(x_3+q)^2}-m\right)\neq 0,$$

$$W^{\mathrm{T}}\left[D^2F_d\left(E_3;d_{TC3}\right)(V,V)\right]$$
$$=\frac{2e\left(x_3-d\right)}{\left(1+x_3-d\right)^3}\left(\frac{-2x_3^3+\left(p-3q\right)x_3^2+2pqx_3}{p\left(x_3+q\right)^2}-m\right)\neq 0.$$

According to Sotomayors theorem, when $d=d_{TC3}=x_3+\frac{b-n}{b+e-n}$, then the model (2.2) undergoes a transcritical bifurcation at the equilibrium point $E_3\left(x_3,0\right)$.

4.2. Hopf Bifurcation

According to the Theorem 6, the internal equilibrium point $E_*\left(x_*,y_*\right)$ can lose its stability, hence the model (2.2) may occur a Hopf bifurcation under certain conditions.

Theorem 8 Under the conditions of the Theorem 6, the internal equilibrium point E_* can change its stability when the controlling parameter d passes through a critical value $d=d_{Hp}$, then the model (2.2) will undergo a Hopf bifurcation, where $Tr\left(J_{E_*}\right)\Big|_{d=d_{Hp}}=0$.

Proof: To determine the internal equilibrium point $E_*\left(x_*,y_*\right)$ can change its stability by a Hopf bifurcation, we need to prove the cross-sectional condition of Hopf bifurcation:

$$Tr\left(J_{E_*}\right)=\frac{-2x_*^3+\left(p-3q\right)x_*^2+2pqx_*}{p\left(x_*+q\right)^2}-\frac{y_*}{\left(1+x_*-d\right)^2}-m=0,$$

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[Tr\left(J_{E_*}\right)\right]\Big|_{d=d_{Hp}}=\left(\frac{x_*^3-2pqx_*+3qx_*^2}{p\left(x_*+q\right)^2}+\frac{x_*^3-px_*^2}{p\left(1+x_*-d\right)\left(x_*+q\right)^2\left(x_*-d\right)}\right)\Bigg|_{d=d_{Hp}}\neq 0,$$

hence the model (2.2) can occur a Hopf bifurcation at $q=q_{Hp}$.

Next, we discuss the stability of the limit cycle by computing the first Lyapunov coefficient of the internal equilibrium point $E_*\left(x_*,y_*\right)$. Translating the origin of coordinates at this equilibrium point through the following transformation $x=x_d-x_*$, $y=y_d-y_*$,

we can obtain

$$\begin{cases}\dot{x}_d=\alpha_{10}x_d+\alpha_{01}y_d+\alpha_{20}x_d^2+\alpha_{11}x_dy_d+\alpha_{02}y_d^2+\alpha_{30}x_d^3+\alpha_{21}x_d^2y_d+\alpha_{12}x_dy_d^2+\alpha_{03}y_d^3+P\left(x_d,y_d\right)\\ \dot{y}_d=\beta_{10}x_d+\beta_{01}y_d+\beta_{20}x_d^2+\beta_{11}x_dy_d+\beta_{02}y_d^2+\beta_{30}x_d^3+\beta_{21}x_d^2y_d+\beta_{12}x_dy_d^2+\beta_{03}y_d^3+Q\left(x_d,y_d\right)\end{cases}$$

where

$$\alpha_{10}=\frac{-pqx_*-2qx_*^2+x_*^3}{p\left(q-x_*\right)}+\frac{y_*}{\left(1-x_*-d\right)^2}-m,\alpha_{01}=\frac{x_*+d}{1-x_*-d},$$
$$\alpha_{20}=\frac{q\left(p+2x_*\right)}{p\left(q-x_*\right)^2}+\frac{x_*^2}{\left(q-x_*\right)^3}-\frac{y_*}{\left(1-x_*-d\right)^3},\alpha_{11}=\frac{-1}{\left(1-x_*-d\right)^3},$$
$$\alpha_{30}=\frac{q^2-pq-2x_*^3}{p\left(q-x_*\right)^3}+\frac{x_*^2\left(q-p-2x_*\right)}{p\left(q-x_*\right)^4}+\frac{y_*}{\left(1-x_*-d\right)^4},\alpha_{21}=\frac{1}{\left(1-x_*-d\right)^3},$$
$$\alpha_{02}=\alpha_{12}=\alpha_{03}=0,$$

and

$$\beta_{10}=\frac{-ey_*}{(1-x_*-d)^2},\beta_{01}=b-n-\frac{e(x_*+d)}{1-x_*-d},\beta_{20}=\frac{ey_*}{(1-x_*-d)^3},$$
$$\beta_{11}=\frac{e}{(1-x_*-d)^2},\beta_{30}=\frac{-ey_*}{(1-x_*-d)^4},\beta_{21}=\frac{-e}{(1-x_*-d)^3},$$
$$\beta_{02}=\beta_{12}=\beta_{03}=0,$$

then $P(x_d,y_d),Q(x_d,y_d)$ are power series in (x_d,y_d) with terms $x_d^i y_d^j$ satisfying $i+j\geq 4$.

Thus, the first Lyapunov coefficient is

$$\begin{aligned}
l=&\frac{-3\pi}{2\alpha_{01}\Delta^{\frac{3}{2}}}\Big\{\Big[\alpha_{10}\beta_{10}\left(\alpha_{11}^2+\alpha_{11}\beta_{02}+\alpha_{02}\beta_{11}\right)+\alpha_{10}\alpha_{01}\left(\beta_{11}^2+\alpha_{20}\beta_{11}+\alpha_{11}\beta_{02}\right)\\
&+\beta_{10}^2\left(\alpha_{11}\alpha_{02}+2\alpha_{02}\beta_{02}\right)-2\alpha_{10}\beta_{10}\left(\beta_{02}^2-\alpha_{20}\alpha_{02}\right)-2\alpha_{10}\alpha_{01}\left(\alpha_{20}^2-\beta_{20}\beta_{02}\right)\\
&-\alpha_{01}^2\left(2\alpha_{20}\beta_{20}+\beta_{11}\beta_{20}\right)+\left(\alpha_{01}\beta_{10}-2\alpha_{10}^2\right)\left(\beta_{11}\beta_{02}-\alpha_{11}\alpha_{20}\right)\Big]\\
&-\left(\alpha_{10}^2+\alpha_{01}\beta_{10}\right)\Big[3\left(\beta_{10}\beta_{03}-\alpha_{01}\alpha_{30}\right)+2\alpha_{10}\left(\alpha_{21}+\beta_{12}\right)+\left(\alpha_{12}\beta_{10}-\alpha_{01}\beta_{21}\right)\Big]\Big\}\\
=&\frac{-3\pi}{2\alpha_{01}\Delta^{\frac{3}{2}}}\Big[\alpha_{10}\beta_{10}\alpha_{11}^2+\alpha_{10}\alpha_{01}\left(\beta_{11}^2+\alpha_{20}\beta_{11}\right)-2\alpha_{10}\alpha_{01}\alpha_{20}^2\\
&-\alpha_{01}^2\left(2\alpha_{20}\beta_{20}+\beta_{11}\beta_{20}\right)-\alpha_{11}\alpha_{20}\left(\alpha_{01}\beta_{10}-2\alpha_{10}^2\right)\\
&-\left(\alpha_{10}^2+\alpha_{01}\beta_{10}\right)\left(-3\alpha_{01}\alpha_{30}+2\alpha_{10}\alpha_{21}-\alpha_{01}\beta_{21}\right)\Big].
\end{aligned}$$

If $l<0$, the limit cycle is stable; if $l>0$, the limit cycle is unstable. However, the expression for Lyapunov number l is rather cumbersome, we cannot directly judge the sign of it, so we will give some numerical simulation results in section 5.

Based on the mathematical theory, the existence and stability threshold conditions of all possible equilibrium points of the model (2.2) are deduced, and some critical conditions for inducing transcritical bifurcation and Hopf bifurcation of the model (2.2) are explored, which can provide a theoretical basis for some numerical simulation works. Furthermore, it is also worth pointing out that the key parameter d has a serious effect on bifurcation dynamics of the model (2.2).

5. Simulation Analysis and Results

In order to verify the validity of theoretical results, find some key control parameters that can induce bifurcation dynamics of the model (2.2), and explore ecological interaction between algae and protozoa, some numerical simulations are given with parameter values $n=0.4$, $b=0.2$, $e=0.6$, $m=0.1$, $q=0.6$ and $p=2$. From the equations 2.3, we can obtain that the dynamic relationship between algae x and protozoa y is $y=\frac{\left[x^2(p-x)-mpx(x+q)\right](1+x-d)}{(x+q)(x-d)p}$. It is easy to find from **Figure 1(b)** that only when the algae density is greater than the value of *d*, the protozoa density can be positive, so the initial value of algae density in numerical simulation is larger than the value of *d*. At the same time, the density of protozoa *y* can reach a limit value and a maximum value within the range of algae *x* density, which implies that there may be an oscillatory coexistence mode between algae *x* and protozoa *y*. Furthermore, it is obvious to know from **Figure 1(a)** that the dynamic relationship between algae *x* and protozoa *y* is affected by the value of parameter *d*, thus, we can select parameter *d* as a control parameter of dynamic evolution process of the model (2.2).

The bifurcation dynamic evolution processes of the model (2.2) are shown in **Figures 2-5**. It is clearly visible from **Figure 2** that if the value of *d* is greater than a critical value $d_{TC} = 1.23$, the boundary equilibrium point E_1 is locally asymptotically stable, the model (2.2) has no internal equilibrium point. However, if the value of *d* is less than a critical value $d_{TC} = 1.23$, the boundary equilibrium point E_1 loses stability and a new internal equilibrium point appears, this process implies that a transcritical bifurcation occurs, the detailed dynamic results are shown in Fi.3. Therefore, if the value of *d* is within the interval $\left(d_{Hp}, d_{TC}\right)$, the model (2.2) has a stable internal equilibrium point, that is, algae and protozoa have a steady-state coexistence mode. As the value of *d* gradually decreases and is lower than a key value $d_{Hp} = 0.1882$, the internal equilibrium point loses stability and a limit cycle appears, which implies that the model (2.2) has a Hopf bifurcation dynamic behavior, the dynamic evolution process of Hopf bifurcation is shown in **Figure 4** and **Figure 5**, which shows that algae and protozoa coexist in a periodic oscillation mode. At the same time, because the first Lyapunov coefficient is $-0.0461268452701094560 77\Pi$, this limit cycle is stable. Furthermore, it is also worth emphasizing that when the value of parameter *d* gradually decreases from 1.5 to 0, the model (2.2) will undergo transcritical bifurcation and Hopf bifurcation successively, which means that the coexistence mode of algae and protozoa has changed fundamentally, from a protozoan extinction mode to a steady-state coexistence mode, and finally to a stable periodic oscillatory coexistence mode. Therefore, it is worth pointing out that the value of parameter *d* seriously affects the coexistence of algae and protozoa.

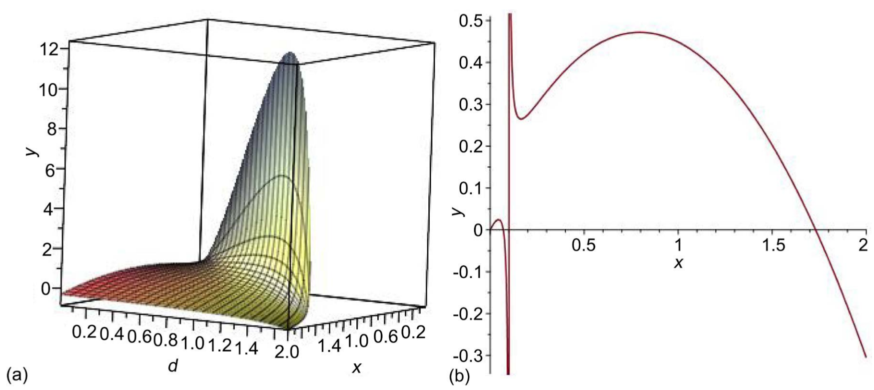


Figure 1. (a) Dynamic relationship between algae *x*, protozoa *y* and parameter *d* value; (b) Dynamic relationship between algae *x* and protozoa *y* with $d = 0.1$.

In order to investigate the influence mechanism of Allee effect on dynamic behavior of the model (2.2), we will select parameter q as a control parameter for relevant dynamic simulation experiments. It is relatively clear from Figure 6 and Figure 7 that the model (2.2) has a constant steady state and a stable periodic

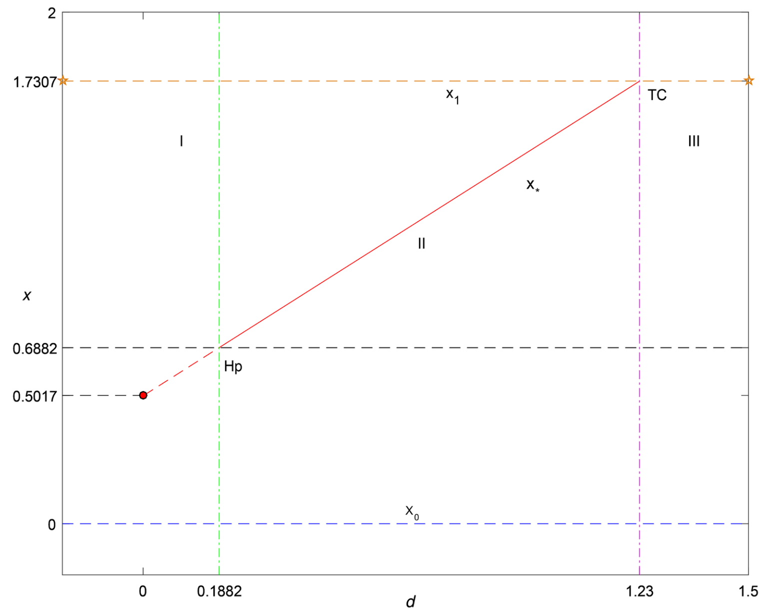
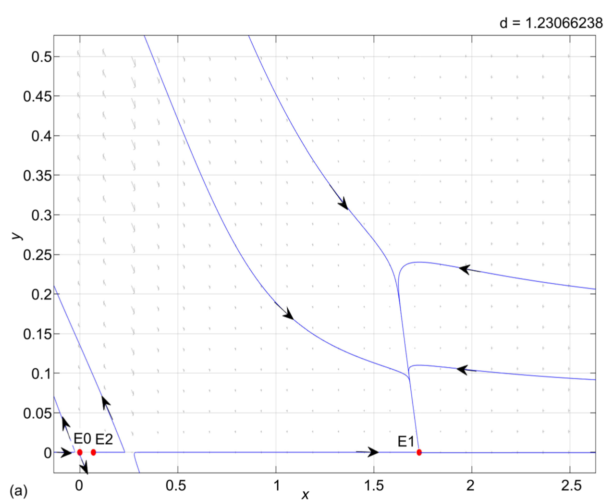


Figure 2. Bifurcation diagram of the model (2.2), here the red line indicates that the internal equilibrium point x_* changes with the parameter d, the blue line and yellow line stand for the boundary equilibrium point x_0 and x_1 with the parameter d value changing, respectively. Here x_0 and x_1 represent the boundary equilibrium point $E_0(0,0)$ and $E_1(x_1,0)$. The solid curve shows that the equilibrium point is stable, the cyan dotted curve shows that the equilibrium point is unstable, and the vertical dot dotted line indicates a critical value of the equilibrium point which can induce bifurcation. More detailed, H_P and TC are some critical values for the Hopf bifurcation and transcritical bifurcation. Besides, the red dots are solid points, where the boundary equilibrium point $(0,0)$ does not exist.



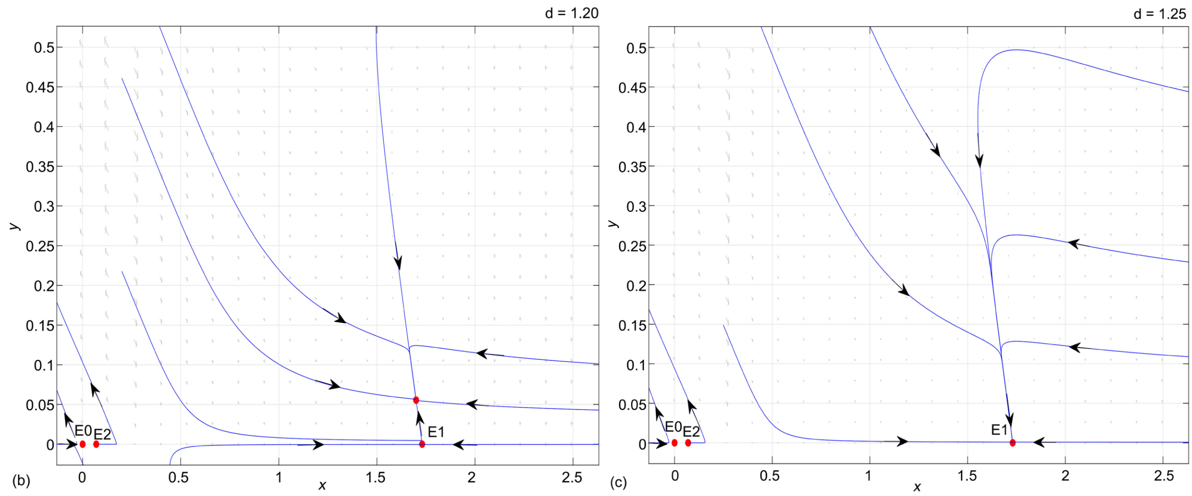


Figure 3. (a) If $d = d_{TC1} = 1.23066238$, then a transcritical bifurcation occurs, where the boundary equilibrium point $E_1(x_1, 0)$ and the equilibrium point $E_*(x_*, y_*)$ coincide; (b) $E_1(x_1, 0)$ is a saddle when $d = 1.2 < d_{TC1}$, which can separate an internal equilibrium point $E_*(x_*, y_*)$; (c) $E_1(x_1, 0)$ is a stable node when $d_{TC1} < d = 1.25$, and the internal equilibrium point $E_*(x_*, y_*)$ does not exist.

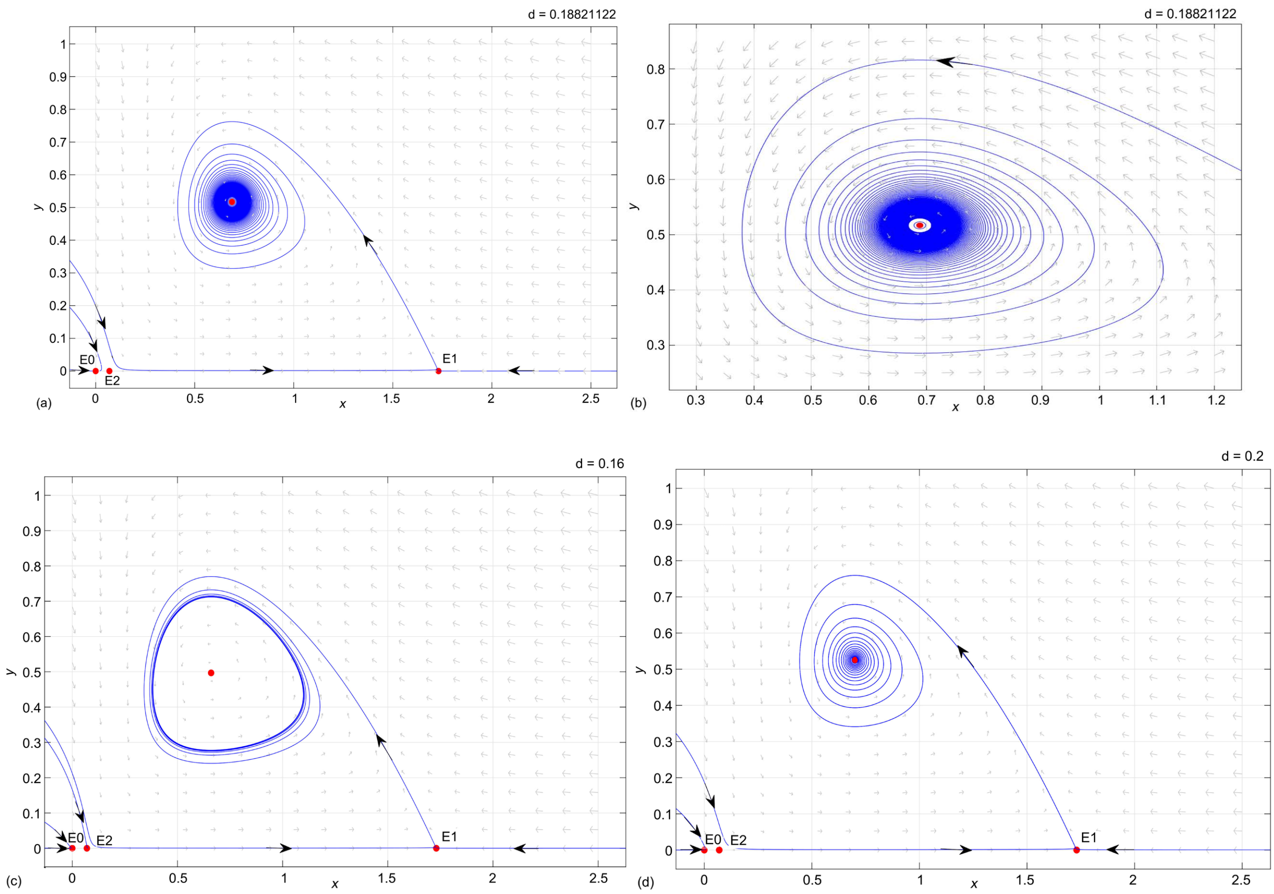


Figure 4. (a) If $d = d_{Hp} = 0.18821122$, then a Hopf bifurcation occurs at the internal equilibrium point $E_*(x_*, y_*)$ and can generate a periodic solution; (b) Local magnification plot of (a); (c) $E_*(x_*, y_*)$ is an unstable node (or spiral source)if $d = 0.16 < d_{Hp}$; (d) $E_*(x_*, y_*)$ is a stable node (or spiral source) if $d_{Hp} < d = 0.2$.

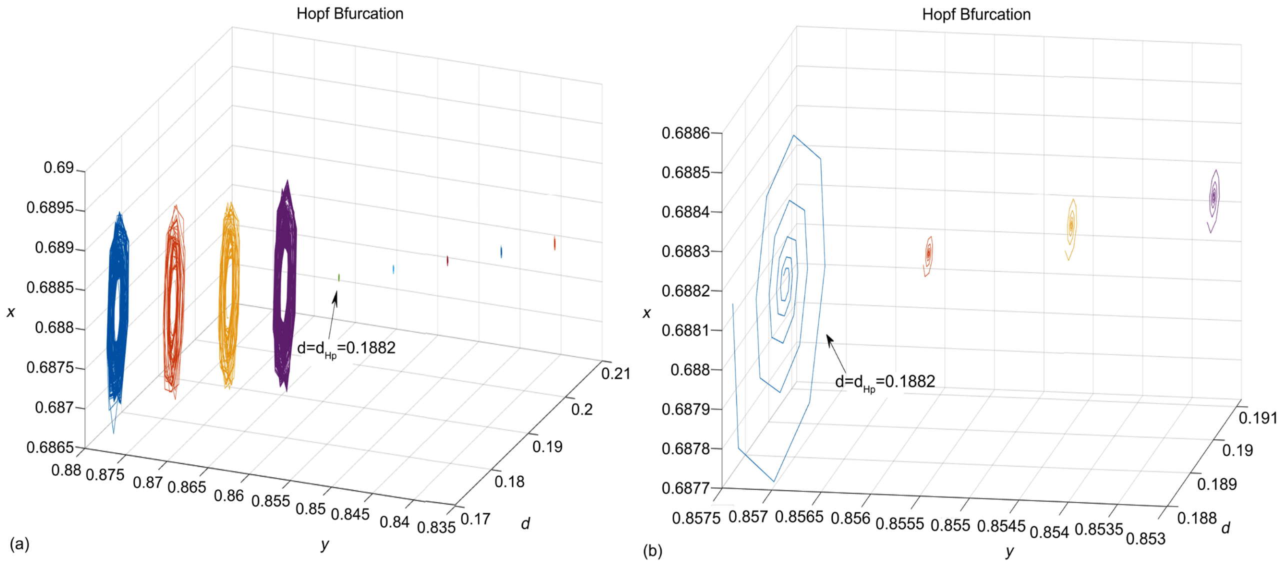


Figure 5. (a) Dynamic evolution diagram of Hopf bifurcation based on the change of parameter d value; (b) Local magnification plot of (a) when $d > d_{Hp} = 0.1882$.

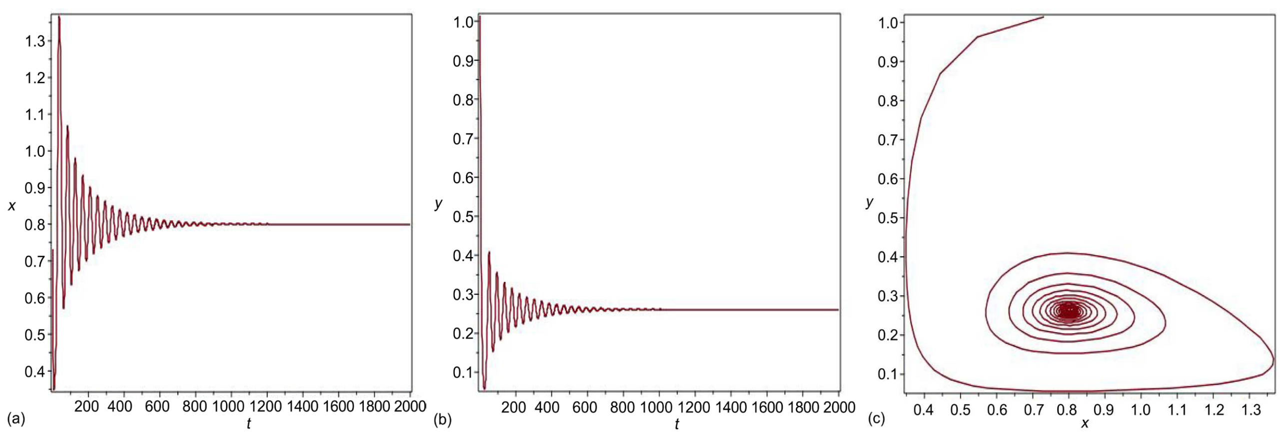


Figure 6. (a) Time series of algae x with $q = 1.5$; (b) Time series of protozoa y with $q = 1.5$; (c) Phase diagram of algae x and protozoa y with $q = 1.5$.

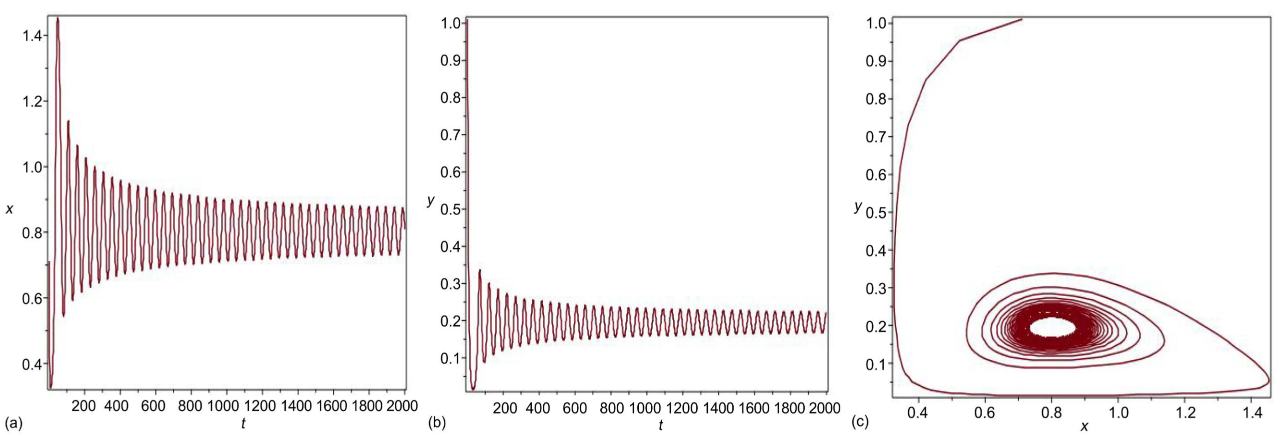


Figure 7. (a) Time series of algae x with $q = 1.85$; (b) Time series of protozoa y with $q = 1.85$; (c) Phase diagram of algae x and protozoa y with $q = 1.85$.

oscillation state, when the value of parameter q are 1.5 and 1.85 respectively, that is to say, the model (2.2) experiences a Hopf bifurcation dynamic behavior with the increase of parameter q value. Furthermore, this simulation result also indirectly shows that the size of Allee effect seriously affects the coexistence mode of algae and protozoa.

Based on the numerical simulation analysis, we first know that the model (2.2) has complex bifurcation dynamic behavior, mainly including transcritical bifurcation and Hopf bifurcation. Secondly, the algal population density gradually decreases with the transition from transcritical bifurcations to Hopf bifurcations. Finally, the value of key parameters of Allee effect seriously affects the coexistence mode of algae and protozoa.

6. Conclusions and Remarks

In this paper, based on the dynamic relationship between algae and protozoa, an aquatic ecological model with Allee effect was established to explore bifurcation behavior and investigate how Allee effect affects the coexistence mode of algae and protozoa. Some key conditions were given to ensure the existence and stability of all possible equilibrium points, and induce the model (2.2) to have transcritical bifurcation and Hopf bifurcation, which were theoretical basis for subsequent numerical simulation and the necessary conditions for parameter estimation value.

Through numerical simulation, we can see that the model (2.2) has complex bifurcation dynamics. It can be seen from **Figure 3** and **Figure 4** that transcritical bifurcation could make algae and protozoa to transform from a protozoa gradual extinction coexistence mode to a steady-state coexistence mode, and Hopf bifurcation could force algae and protozoa to transform from a constant steady-state coexistence mode to a stable periodic oscillation coexistence mode, which implied that the ecological relationship between algae and protozoa had changed substantially. Furthermore, it was easy to know from **Figure 6** and **Figure 7** that the ecological relationship between algae and protozoa could change from a constant steady state to a periodic oscillatory steady state with the increase of key parameters of Allee effect, which showed that Allee effect seriously affected coexistence mode of algae and protozoa.

Based on the theoretical analysis and numerical simulation results, it is worth pointing out that the algal aggregation behavior can change the coexistence mode of algae and protozoa, and the greater the algae population aggregation intensity, the more adverse to the permanent survival of protozoa, this research result is consistent with the fact that algae bloom is not conducive to the survival of protozoa. Furthermore, it should also be emphasized that algae population has Allee effect mechanism with small key value, which is conducive to form periodic oscillation coexistence mode of algae and protozoa.

Although some theoretical and numerical simulation results have been obtained in this study, there are still some deficiencies that need our follow-up research, such as: 1) the natural growth mode of protozoa is too simple, and the logistic growth function needs to be studied subsequently; 2) the influence of hydrodynamics on algae and protozoa should be considered in modeling dynamic process. However, it is hoped that the research results of this paper can play a theoretical supporting role in the study of aquatic ecological model.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Optimal Control Analysis of Influenza Epidemic Model

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Abstract

The implementation of optimal control strategies involving preventive measures and antiviral treatment can significantly reduce the number of clinical cases of influenza. In this paper, a model for the transmission dynamics of influenza is formulated and two control strategies involving preventive measures (awareness campaign, washing hand, using hand sanitizer, wearing mask) and treatment are considered and used to minimize the total number of infected individuals and associated cost of using these two controls. The resulting optimality system is solved numerically. Hamiltonian is formulated to investigate the existence of the optimal control, in the optimal control model. Pontryagin's Maximum Principle is applied to describe the control variables and the objective function is designed to reduce both the infection and the cost of interventions. From the numerical simulation, it is observed that in the case of high contact rate ($\beta = 3$), both the controls work for a longer period of time to reduce the disease burden. The optimal control analysis and numerical simulations reveal that the interventions reduce the number of exposed and infected individuals.

Keywords

Influenza, Optimal Control, Pontryagin's Maximum Principle, Transversality Condition, Hamiltonian

1. Introduction

Influenza viruses cause the infectious disease, influenza, commonly known as "the flu" and this infection primarily transmitted through respiratory droplets produced by sneezing and coughing by an infected person [1]. Symptoms range from mild to severe and often include fever, sore throat, runny nose, headache, muscle pain, coughing, and fatigue and these symptoms begin from one to four days after exposure to the virus (typically two days) and last for about 2 - 8 days. Particularly in children diarrhea and vomiting can occur. Some other complications caused by the infection include meningitis, acute respiratory distress syndrome, encephalitis and worsening of pre-existing health problems such as asthma and cardiovascular disease.

For the past few centuries, influenza remains a serious threat to public health globally [2]. During the past century, thousands of people lost their lives during three disastrous pandemics including the Spanish flu (1918), Asian flu (1958), and Hong Kong flu (1968) [2]. In 2009, the world experienced the H1N1 influenza, also known as the Swine Influenza, an epidemic that led to over 16,455 deaths globally.

In reducing the spread of influenza, frequent hand washing with soap and water, using hand sanitizers (alcohol-based) and not touching one's nose, eyes and mouth with one's hands, are highly effective. Covering nose and mouth when coughing or sneezing and staying home when sick, is important to limit influenza transmission [3]. Creating awareness among people about the aforementioned etiquette and hygiene by spreading health education through media is important. The disease can be treated with supportive measures and, in severe cases, with antiviral drugs such as oseltamivir.

In view of the serious consequences due to the H1N1 epidemic on the public health, various mathematical models have been proposed and analyzed in order to know the transmission dynamics of the H1N1 influenza [4]-[10].

Optimal control theory is another area of mathematics that is used extensively in controlling the spread of infectious diseases. It is a powerful mathematical tool that can be used to make decisions involving complex biological situation and is a decent strategy for deciding how to control a sickness best.

To overcome H1N1 influenza, mitigation strategies are proposed in [11], an H1N1 influenza model was analyzed in [12] that accounts for the role of an imperfect vaccine and antiviral drugs that administered to infected individuals, the evolutionary model of influenza A with drift and shift was discussed in [13]. Authors in [14] discussed two strain influenza model with vaccination for strain 1 and transmission dynamics of H1N1 influenza was rigorously analyzed with optimal control in [5]. All these studies reveal the complex feature of transmission dynamics of influenza and to the author's knowledge no such model for transmission of influenza in a population has been developed in which optimal control strategies have been designed on the basis of considering all possible preventive measures and treatment and this is the novelty of this research work.

The task of identifying optimal control strategies with a simple SEIR model that minimize the impact of influenza epidemics through the use of antiviral drug in combination with aforementioned preventive measures (which is highly prioritized) like covering nose and mouth, washing hand, using hand sanitizer, creating awareness through health education are the focus of this manuscript.

Optimal control theory [15] [16] [17], is the primary tool used in the analysis. To complement the aforementioned studies by formulating a simple model and considering all possible preventive measures and treatment as control parameters to reduce the disease burden, is the main objective of this study.

This paper is organized as follows: Section 2 (Model formulation), Section 3 (Existence of an optimal control), Section 4 (Necessary conditions of the optimal control), Section 5 (The optimality system), Section 6 (Numerical simulations and discussion) and Section 7 (Conclusion).

2. Model Formulation

The total population is divided into four mutually exclusive compartments, namely susceptible ($S(t)$), exposed ($E(t)$), infected ($I(t)$), and recovered ($R(t)$) at any time t. Thus, the total population can be written as $N(t)=S(t)+E(t)+I(t)+R(t)$.

The corresponding system of nonlinear ODEs is,

$$\begin{aligned} S'(t) &= bN(t)-\beta S(t)I(t)-\mu_1 S(t), \\ E'(t) &= \beta S(t)I(t)-(\mu_1+\gamma)E(t), \\ I'(t) &= \gamma E(t)-(\mu_1+\mu_2+r)I(t), \\ R'(t) &= rI(t)-\mu_1 R(t), \\ N'(t) &= (b-\mu_1)N(t)-\mu_2 I(t). \end{aligned} \tag{1}$$

with initial conditions

$$S(0)=S_0\ge 0, E(0)=E_0\ge 0, I(0)=I_0\ge 0, R(0)=R_0\ge 0, N(0)=N_0\ge 0 \tag{2}$$

Here b is the recruitment rate, γ is the transmission rate from exposed class to infected class, r is the recovery rate and μ_1, μ_2 are natural death rate and disease induced death rate respectively. Susceptible individuals acquire infection at a per capita rate $\beta I(t)$, where β is the transmission coefficients.

To control various types of diseases, optimal control techniques are of great use in developing optimal strategies. In this model two control strategies are introduced namely $v_1(t)$, which represents the preventive measures like covering nose and mouth, washing hand, using hand sanitizer, awareness campaign among the community and $v_2(t)$, which represents the treatment of infectious people. The modified model to estimate the effect of controlling strategies, is given below,

$$\begin{aligned} S'(t) &= bN(t)-\beta S(t)I(t)-(v_1+\mu_1)S(t), \\ E'(t) &= \beta S(t)I(t)-(\mu_1+\gamma)E(t), \\ I'(t) &= \gamma E(t)-(\mu_1+\mu_2+r+v_2)I(t), \\ R'(t) &= v_1 S(t)+v_2 I(t)+rI(t)-\mu_1 R(t), \end{aligned} \tag{3}$$

with initial conditions

$$S(0)=S_0\ge 0, E(0)=E_0\ge 0, I(0)=I_0\ge 0, R(0)=R_0\ge 0 \tag{4}$$

To limit the number of infectious individuals and minimize the cost of applied controls v_1, v_2, the required objective functional J is defined as follows,

$$J(v_1, v_2) = \int_0^T \left(YI(t) + \frac{1}{2}\left(w_1 v_1^2 + w_2 v_2^2\right)\right) dt \tag{5}$$

and the control set is, $V = \left\{ \left(v_1(t), v_2(t)\right) : 0 \le v_1(t) \le 1, 0 \le v_2(t) \le 1, t \in [0, T] \right\}$.

A linear combination of quadratic terms $\left(v_i^2, i = 1, 2\right)$ are used to model the control efforts, and the constants Y, w_1, w_2 are a measure of the relative cost of the interventions over [0, T]. Here the problem is to find optimal controls, $\left(v_1^*, v_2^*\right)$ such that

$$J\left(v_1^*(t), v_2^*(t)\right) = \min_V J\left(v_1(t), v_2(t)\right) \tag{6}$$

3. Existence of an Optimal Control

From the model (3), $N'(t) \le (b - \mu_1) N(t)$. Then there exists $M \in \mathbb{R}^+$ such that

$$N(t) \le N_0 e^{(b-\mu_1)T} = M, t \in [0, T]$$

Since, $N(t) = S(t) + E(t) + I(t) + R(t)$ and the state variables, $S(t), E(t), I(t)$, and $R(t)$ are bounded above, then there exists solution for the system (3).

To prove the existence of the optimal control, it's required to check the following hypotheses [15].

(M_1) The set consisting of controls and corresponding state variables is nonempty.

(M_2) The admissible control set V is convex and closed.

(M_3) R.H.S of state system (3) is bounded by a linear function in the state and control variables.

(M_4) J, the objective functional, has a convex integrand on V and the integrand is bounded below by $-b_1 + b_2 \left|\left(v_1, v_2\right)\right|^\eta$ with $b_1 > 0, b_2 > 0$ and $\eta > 1$.

To prove the above statements the following theorem is required,

Theorem 1. If each of the functions $\tilde{F}_i$, for $i = 1, \cdots n$ and the partial derivatives $\frac{\partial \tilde{F}_i}{\partial x_j}$ for $i, j = 1, \cdots n$, are continuous in $\mathcal{R}^{n+1}$ space, then there exists a unique solution $\left(x_1 = \phi_1(t), \cdots, x_n = \phi_n(t)\right)$ of the system of differential equations, $x_i' = \tilde{F}_i\left(t, x_1, \cdots, x_n\right)$ for $i = 1, \cdots n$, with initial conditions $x_i(t_0) = x_i^0$ for $i = 1, \cdots n$, and the solution also satisfies the initial conditions [15].

To prove the hypotheses (M_1-M_4), let us consider the system,

$$\begin{aligned} \frac{dS}{dt} &= \tilde{F}_1(t, S, E, I, R), \\ \frac{dE}{dt} &= \tilde{F}_2(t, S, E, I, R), \\ \frac{dI}{dt} &= \tilde{F}_3(t, S, E, I, R), \\ \frac{dR}{dt} &= \tilde{F}_4(t, S, E, I, R), \end{aligned} \tag{7}$$

where $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$ and $\tilde{F}_4$ represent the right side of the system (3) and for some constants c_1 and c_2, let $v_1(t) = c_1$ and $v_2(t) = c_2$. The functions $\tilde{F}_i$ for $i = 1, \cdots 4$, must be linear and their partial derivatives with respect to all state variables are constants. Hence the functions and their partial derivatives are continuous everywhere. So, according to the theorem 1 we can say that, there exists a unique solution $S(t) = \phi_1(t), E(t) = \phi_2(t), I(t) = \phi_3(t), R(t) = \phi_4(t)$, which satisfies the initial conditions. Therefore, the consisting set of controls and corresponding state variables is nonempty.

Now for any two controls $v_1, v_2 \in V$ and $\theta \in [0,1]$, $0 \le \theta v_1 + (1-\theta) v_2 \le 1$. Therefore, the set V is convex and closed (by definition).

Now comparing (7) with (3),

$$\begin{aligned} \tilde{F}_1 &\le bN - v_1 S \\ \tilde{F}_2 &\le KI \\ \tilde{F}_3 &\le \gamma E - v_2 I \\ \tilde{F}_4 &\le v_1 S + v_2 I + rI \end{aligned}$$

in matrix form,

$$\bar{F}(t, \bar{X}, V) \le \bar{m}_1 \left(t, \begin{bmatrix} S \\ E \\ I \\ R \end{bmatrix} \right) \bar{X}(t) + \bar{m}_2 \left(t, \begin{bmatrix} S \\ E \\ I \\ R \end{bmatrix} \right) V \tag{8}$$

where,

$$\bar{m}_1 \left(t, \begin{bmatrix} S \\ E \\ I \\ R \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & r & 0 \end{bmatrix} \text{ and } \bar{m}_2 \left(t, \begin{bmatrix} S \\ E \\ I \\ R \end{bmatrix} \right) = \begin{bmatrix} -S \\ 0 \\ -I \\ S + I \end{bmatrix}$$

Here all the parameters are constant and nonnegative. Therefore from (8),

$$\left| \bar{F}(t, \bar{X}, V) \right| \le \|\bar{m}_1\| \left| \bar{X} \right| + |S + I| \left| (v_1(t), v_2(t)) \right| \le q \left(\left| \bar{X} \right| + \left| (v_1(t), v_2(t)) \right| \right)$$

Therefore, the right side of the state system (3) is bounded by a linear function in the state and control variables.

Moreover, the integrand, $YI(t) + \frac{1}{2}\left(w_1 v_1^2 + w_2 v_2^2 \right)$ of the objective functional J, is convex and satisfies $J(v_1, v_2) = -b_1 + b_2 \left| (v_1, v_2) \right|^2$ where $b_1 > 0, b_2 > 0$ and $\eta = 2 > 1$, according to [18] [19] [20].

Hence, we have the following theorem.

Theorem 2 For $V = \left\{ (v_1(t), v_2(t)) : 0 \le v_1(t) \le 1, 0 \le v_2(t) \le 1, t \in [0, T] \right\}$ subject to Equation (3) having the initial conditions and $J(v_1, v_2) = \int_0^T \left(YI(t) + \frac{1}{2}\left(w_1 v_1^2 + w_2 v_2^2 \right) \right) dt$, there is an optimal control (v_1^*, v_2^*) such that $J(v_1^*(t), v_2^*(t)) = \min_V J(v_1(t), v_2(t))$ [7].

For the solution of the system (3), it's Hamiltonian has to be defined.

4. Necessary Conditions of the Optimal Control

Let $\tilde{X}=(S,E,I,R), V=(v_1,v_2)$ and $\lambda=(\lambda'_S,\lambda'_E,\lambda'_I,\lambda'_R)$. Then the Hamiltonian H for the optimal control problem is,

$$\begin{aligned} H(\tilde{X},V,\lambda) &= YI(t)+\frac{1}{2}(w_1v_1^2+w_2v_2^2)+\lambda_S(bN-\beta SI-(v_1+\mu_1)S) \\ &+\lambda_E(\beta SI-(\mu_1+\gamma)E)+\lambda_I(\gamma E-(\mu_1+\mu_2+r+v_2)I) \\ &+\lambda_R(v_1S+v_2I+rI-\mu_1R) \end{aligned} \tag{9}$$

Pontryagin's maximum principle [17] is used to derive the necessary conditions for the optimal control, which (Pontryagin's maximum principle) converts the problem (6) into the problem of minimizing the Hamiltonian.

Hamiltonian H is used for determining the adjoint equations and transversality conditions.

The following can be derived from the differentiation of H, with respect to each state variables

$$\lambda'_S=-\frac{\partial H}{\partial S}=\lambda_S(\beta I+(v_1+\mu_1))-\beta I\lambda_E-v_1\lambda_R,$$

$$\lambda'_E=-\frac{\partial H}{\partial E}=(\mu_1+\gamma)\lambda_E-\gamma\lambda_I,$$

$$\lambda'_I=-\frac{\partial H}{\partial I}=\beta S\lambda_S-\beta S\lambda_E+(\mu_1+\mu_2+r+v_2)\lambda_I-(v_2+r)\lambda_R-Y,$$

$$\lambda'_R=-\frac{\partial H}{\partial R}=\mu_1\lambda_R$$

with transversality conditions, $\lambda_S(T)=\lambda_E(T)=\lambda_I(T)=\lambda_R(T)=0$.

With the help of controls and conditions of optimality,

$$\left.\frac{\partial H}{\partial v_1}\right|_{v_1=v_1^*}=0$$

$$\Rightarrow v_1^*w_1-S\lambda_S+S\lambda_R=0$$

$$\Rightarrow v_1^*=\frac{S(\lambda_S-\lambda_R)}{w_1}$$

and

$$\left.\frac{\partial H}{\partial v_2}\right|_{v_2=v_2^*}=0$$

$$\Rightarrow v_2^*w_2-I\lambda_I+I\lambda_R=0$$

$$\Rightarrow v_2^*=\frac{I(\lambda_I-\lambda_R)}{w_2}$$

5. The Optimality System

The resulting optimality system is given as follows,

State equations with initial conditions,

$$\begin{aligned} S'(t) &= bN(t) - \beta S(t)I(t) - (v_1 + \mu_1)S(t), \\ E'(t) &= \beta S(t)I(t) - (\mu_1 + \gamma)E(t), \\ I'(t) &= \gamma E(t) - (\mu_1 + \mu_2 + r + v_2)I(t), \\ R'(t) &= v_1 S(t) + v_2 I(t) + rI(t) - \mu_1 R(t), \end{aligned} \tag{10}$$

with initial conditions

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0 \tag{11}$$

Adjoint system with transversality conditions,

$$\begin{aligned} \lambda'_S &= -\frac{\partial H}{\partial S} = \lambda_S\left(\beta I + (v_1 + \mu_1)\right) - \beta I \lambda_E - v_1 \lambda_R, \\ \lambda'_E &= -\frac{\partial H}{\partial E} = (\mu_1 + \gamma)\lambda_E - \gamma \lambda_I, \\ \lambda'_I &= -\frac{\partial H}{\partial I} = \beta S \lambda_S - \beta S \lambda_E + (\mu_1 + \mu_2 + r + v_2)\lambda_I - (v_2 + r)\lambda_R - Y, \\ \lambda'_R &= -\frac{\partial H}{\partial R} = \mu_1 \lambda_R \end{aligned} \tag{12}$$

and,

$$\lambda_S(T) = \lambda_E(T) = \lambda_I(T) = \lambda_R(T) = 0 . \tag{13}$$

Controls v_1^* and v_2^* are given by,

$$v_1^* = \begin{cases} 0, & \text{if } \dfrac{S(\lambda_S - \lambda_R)}{w_1} < 0 \\ \dfrac{S(\lambda_S - \lambda_R)}{w_1}, & \text{if } 0 \leq \dfrac{S(\lambda_S - \lambda_R)}{w_1} \leq 1 \\ 1, & \text{if } \dfrac{S(\lambda_S - \lambda_R)}{w_1} > 1 \end{cases} \tag{14}$$

and

$$v_2^* = \begin{cases} 0, & \text{if } \dfrac{I(\lambda_I - \lambda_R)}{w_2} < 0 \\ \dfrac{I(\lambda_I - \lambda_R)}{w_2}, & \text{if } 0 \leq \dfrac{I(\lambda_I - \lambda_R)}{w_2} \leq 1 \\ 1, & \text{if } \dfrac{I(\lambda_I - \lambda_R)}{w_2} > 1 \end{cases} \tag{15}$$

6. Numerical Simulations and Discussion

For the numerical solution of the system (10), the Runge-Kutta method is used. The simulation of the model is done with different scenarios. For this, the considered initial population size for susceptible class, exposed class, infected class and recovered class are $S_0 = 0.8, E_0 = 0.06, I_0 = 0.05$ and $R_0 = 0.05$ respectively and total number of years, $T = 8$.

Here intead of whole numbers the proportions are used. Description of the variables of the model and other parameter values are given in **Table 1** and **Table 2** respectively.

Figure 1 shows the density of the susceptible, exposed, infected and recovered population with and without control. It is noticed that, the extinction of infected and exposed class is possible if the control parameters are kept. Otherwise the infection reaches to the maximum level.

Figure 2 and **Figure 3** show a comparative situation with varying effective contact rate. For low contact rate ($\beta = 0.4$) there is a significant increase in the recovered compartment, compared to the high contact rate ($\beta = 3$). In the case of high contact rate, both the controls v_1 and v_2 work for a longer period of time to reduce the disease burden. **Figure 4** and **Figure 5** portray the solution of the optimal control problem with different control weights ($w_1 = 0.2, w_2 = 0.5$) and ($w_1 = 0.5, w_2 = 0.2$) respectively and there are no significant changes in the infected and exposed class. Applying more awareness control does not significantly bring down the number of exposed and infected individuals as compared to the case when applying more of the treatment control. In both cases there are increase in the infected individuals after the time $t = 7.4$ years.

Table 1. Description of the model variables.

Variable	Description
$S(t)$	Susceptible Population at time t
$E(t)$	Exposed Population at time t
$I(t)$	Infected Population at time t
$R(t)$	Recovered Population at time t

Table 2. Description and nominal value of the model parameter.

Parameter	Description	Value
b	Birth rate	0.03 [assumed]
μ_1	Natural death rate	0.02 [5]
μ_2	Disease induced death rate	0.01 [5]
β	Effective contact rate	0.9 [5]
γ	Transmission rate from $E(t)$ to $I(t)$ class	0.53 [5]
r	Recovery rate	0.2 [5]
Y	Weight parameter	10 [assumed]
w_1	Weight parameter	0.2 [assumed]
w_2	Weight parameter	0.3 [assumed]

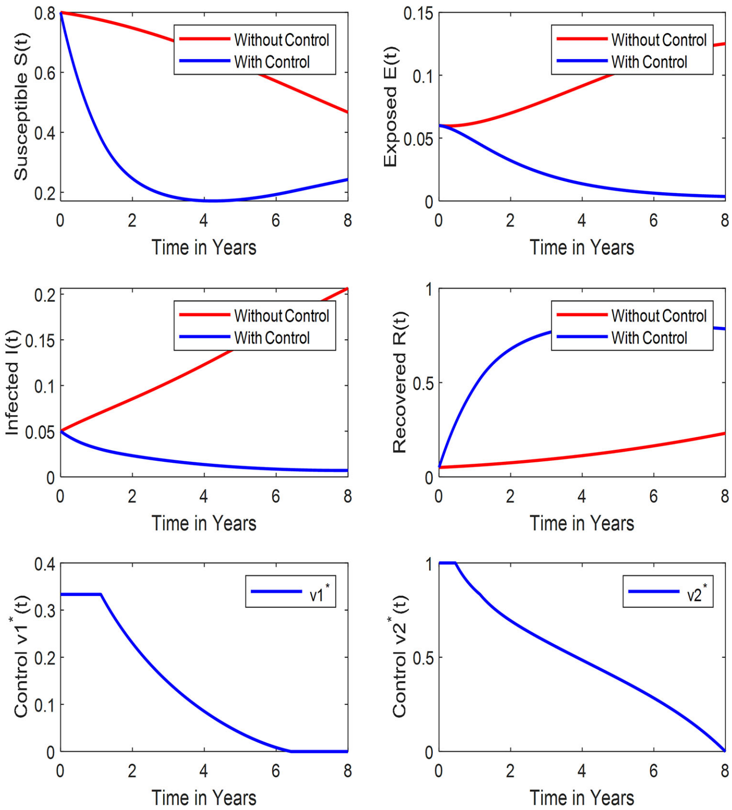

Figure 1. The graph shows the comparison of changes in population with and without control.

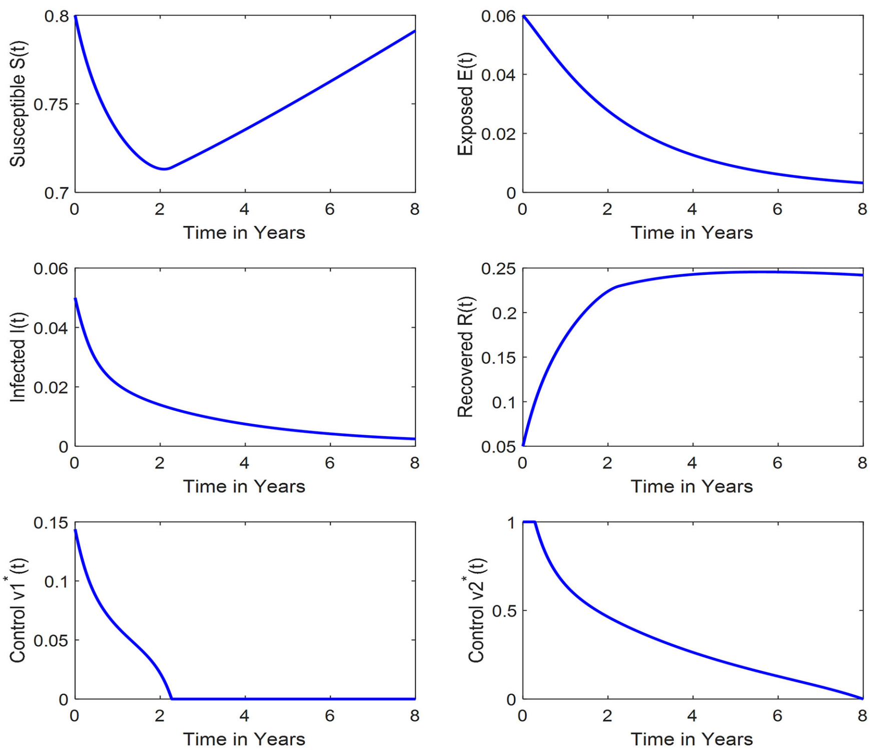

Figure 2. The graph shows the effect of low contact rate ($\beta = 0.4$).

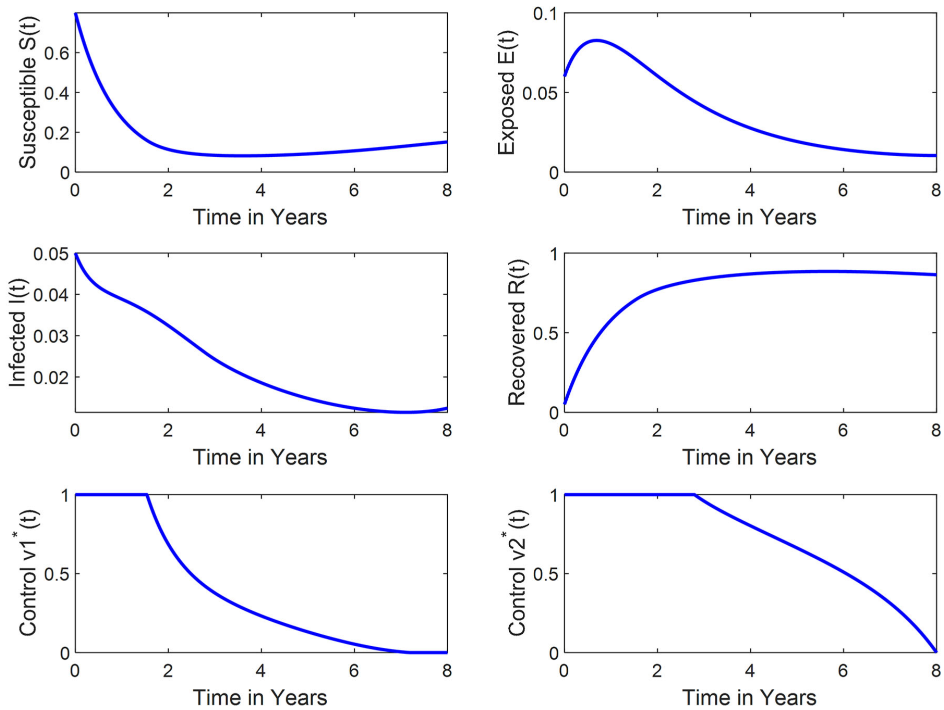

Figure 3. The graph shows the effect of high contact rate ($\beta = 3$).

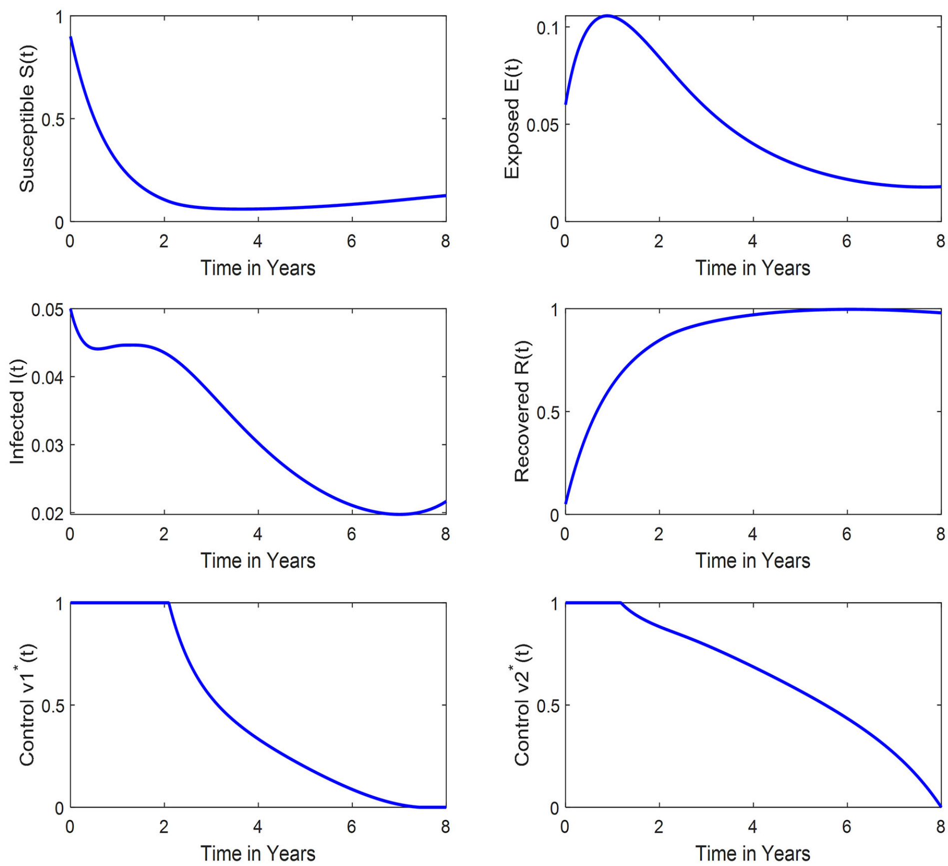

Figure 4. The graph shows the effect of weight parameters ($w_1 = 0.2, w_2 = 0.5$).

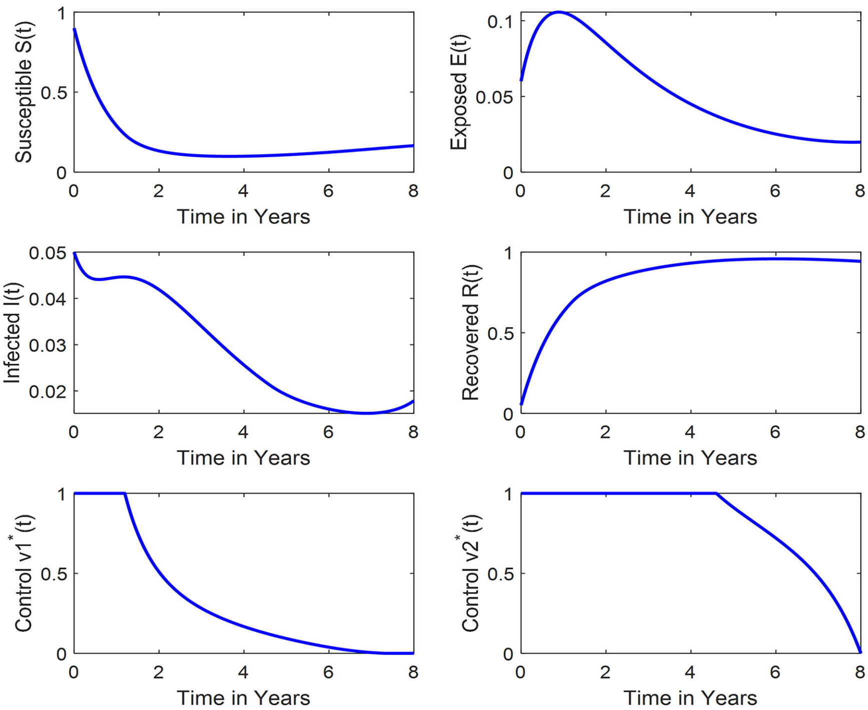

Figure 5. The graph shows the effect of weight parameters ($w_1 = 0.5, w_2 = 0.2$).

7. Conclusions

In this paper, a deterministic mathematical model of Influenza was formulated with preventive measures (awareness campaign, washing hand, using hand sanitizer, wearing mask) and treatment as interventions. It is monitored that there is a significant effect of using control strategies in reducing the exposed and infected individuals. In case of high effective contact rate, the effectiveness of the controls should last for longer period of time because of increasing disease burden. Moreover, the combination of both the controls has positive impact on reducing the disease burden and minimizing the corresponding cost. The main findings are:

For high contact rate ($\beta = 3$), to reduce disease burden both the controls, preventive measures and treatment should work for long period and in the case of low contact rate ($\beta = 0.4$), exposed and infected individuals decrease rapidly and for this, control v_1 needs to work for more than 2 years.

For different control weights ($w_1 = 0.2, w_2 = 0.5$) and ($w_1 = 0.5, w_2 = 0.2$), it is monitored that the number of recovered individuals increases more rapidly and reaches it maximum level faster whenever preventive measures get more priority and this is more economical than treatment cost.

In this study, the optimal control problem does not include vaccination, which is important and for further study, this problem extends by considering the vaccination as intervention.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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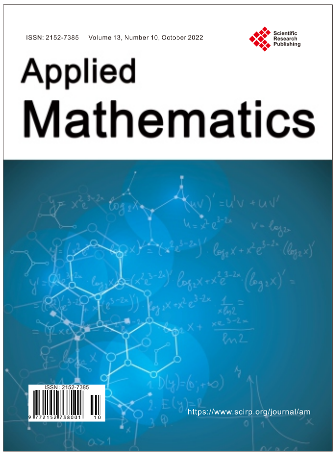

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