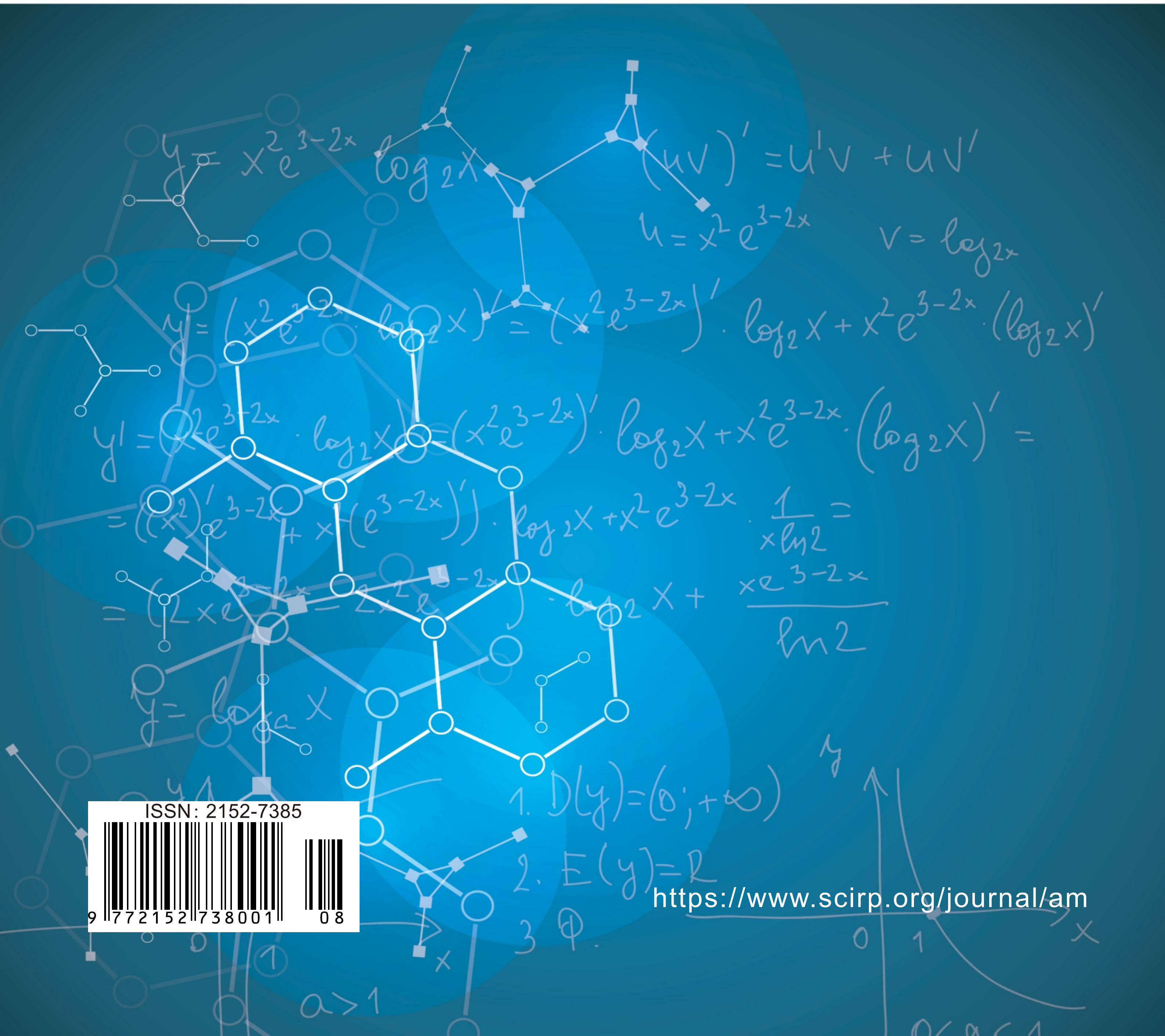
ISSN: 2152-7385 Volume 13, Number 8, August 2022

Scientific Research Publishing

Applied Mathematics



ISSN: 2152-7385

https://www.scirp.org/journal/am

Journal Editorial Board

ISSN Print: 2152-7385 ISSN Online: 2152-7393
https://www.scirp.org/journal/am

Editorial Board

Applied Mathematics, 2022, 13, 651-703
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Table of Contents

Volume 13 Number 8 August 2022

Applied Mathematics (AM)
Journal Information

SUBSCRIPTIONS

The *Applied Mathematics* (Online at Scientific Research Publishing, https://www.scirp.org/) is published monthly by Scientific Research Publishing, Inc., USA.

Subscription rates:
Print: $89 per copy.
To subscribe, please contact Journals Subscriptions Department, E-mail: sub@scirp.org

SERVICES

Advertisements
Advertisement Sales Department, E-mail: service@scirp.org

Reprints (minimum quantity 100 copies)
Reprints Co-ordinator, Scientific Research Publishing, Inc., USA.
E-mail: sub@scirp.org

COPYRIGHT

PRODUCTION INFORMATION

For manuscripts that have been accepted for publication, please contact:
E-mail: am@scirp.org

Applied Mathematics, 2022, 13, 651-657
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

On Invertibility of Some Functional Operators with Shift

Aleksandr Karelin*, **Anna Tarasenko**, **Manuel Gonzalez-Hernandez**

Institute of Basic Sciences and Engineering, Hidalgo State University, Pachuca, Mexico
Email: *karelin@uaeh.edu.mx, anataras@uaeh.edu.mx, mghdez@uaeh.edu.mx

How to cite this paper: Karelin, A., Tarasenko, A. and Gonzalez-Hernandez, M. (2022) On Invertibility of Some Functional Operators with Shift. *Applied Mathematics*, **13**, 651-657.
https://doi.org/10.4236/am.2022.138040

Received: July 13, 2022
Accepted: August 16, 2022
Published: August 19, 2022

Abstract

In this paper, we consider operators arising in the modeling of renewable systems with elements that can be in different states. These operators are functional operators with non-Carlemann shifts and they act in Holder spaces with weight. The main attention was paid to non-linear equations relating coefficients to operators with a shift. The solutions of these equations were used to reduce the operators under consideration to operators with shift, the invertibility conditions for which were found in previous articles of the authors. To construct the solution of the non-linear equation, we consider the coefficient factorization problem (the homogeneous equation with a zero right-hand side) and the jump problem (the non-homogeneous equation with a unit coefficient). The solution of the general equation is represented as a composition of the solutions to these two problems.

Keywords

Operator with a Non-Carlemann Shift, Inverse Operator, Non-Linear Equation, Factorization of Coefficient, Equation with Unit Coefficient

1. Introduction

In Hölder's space with weight $H_\mu^0(J,p)$, we consider operators

$$A = a(x)I + b(x)B_\alpha, \quad F = (\delta(x)I + \mu(x)B_\alpha)(\rho(x)I + \eta(x)B_\alpha),$$

$$V = k(x)I + (a_r(x)I + b_r(x)B_\alpha)(a_u(x)I + b_u(x)B_\alpha),$$

$$(\mathbb{W}v)(x) = l(x)I + s(x)B_\alpha + t(x)B_\alpha^2$$

and equations with the operators $A\varphi(x) = q(x)$, $F\varphi(x) = q(x)$, $V\varphi(x) = q(x)$, $W\varphi(x) = q(x)$.

Carrying out the modeling of renewable systems with elements that are in different states, such functional equations with shift appears in balance relations [1]

[2] [3] [4] [5]. The main method for studying these balance relations is their reduction to a system of integral equations with degenerate kernels of the second kind [6] [7]. This reduction uses inverse operators A^{-1}, V^{-1}, W^{-1}. The interest and motivation for studying such operators are growing. Since the conditions for the invertibility of operator A in the space $H_\mu^0(J,p)$ have already been calculated [8], it follows that the conditions for the invertibility of operator F are found as the intersection of the conditions for the invertibility of the factors that make up F. The question arises: when can equations $(A\varphi)(x)=q(x)$ and $(W\varphi)(x)=q(x)$ be represented as an operator F? Studying the possibility of reducing the equation $(W\varphi)(x)=q(x)$ to $(F\varphi)(x)=q(x)$, a nonlinear system of equations that describes links between coefficients arises: $\delta(x)\rho(x)=l(x)$, $\delta(x)\eta(x)+\mu(x)(B_\alpha\rho)(x)=s(x)$, $\mu(x)(B_\alpha\eta)(x)=t(x)$. Substituting $\delta(x)$ from the first equation of the system and $\mu(x)$ from the third equation into the second equation, we obtain $\varphi(x)-\dfrac{G(x)}{(B_\alpha\varphi)(x)}=g(x)$, where $G(x)=-\dfrac{t(x)}{l(x)}$, $g(x)=\dfrac{s(x)}{l(x)}$, $\varphi(x)=\dfrac{\eta(x)}{\rho(x)}$. Substituting $\rho(x)$ from the first equation of the system and $(B_\alpha\eta)(x)$ from the third equation into the second equation, we obtain a non-homogeneous and non-linear equation with an unknown function $\varphi(x)$. An analogous situation takes place when the equation $(V\varphi)(x)=q(x)$ is reduced to $(F\varphi)(x)=q(x)$.

Let $l(x)$ not vanish on the interval J. When getting connections between the coefficients of the operator W and the coefficients of the operator F, we can assume that $\delta(x)=1$. Non-linear system describing the relationship between the coefficients of operators F, W will be as follows: $\rho(x)=l(x)$, $\eta(x)+\mu(x)(B_\alpha l(x))(x)=s(x)$, $\mu(x)(B_\alpha\eta)(x)=t(x)$ and the unknown function of the non-linear equation will be the same as $\eta(x)$, $\varphi(x)=\eta(x)$. From here we find the remaining coefficients of the operator F. The conditions for the invertibility of the operator W are written out as the intersection of the conditions for the invertibility of operators $I+\mu(x)B_\alpha$ and $\rho(x)I+\eta(x)B_\alpha$.

As we can see, the solvability of non-linear equations plays an especially important role.

We will devote the third section to the solution of such non-linear equations. And now, let us recall the definition of a Hölder space with weight and formulate, obtained by us, the conditions for the invertibility of the operator A in this space.

2. On the Invertibility of a Functional Operator with Shift $A=aI+bB_\alpha$ in the Hölder Space with Weight

Let us recall the definition of a Hölder space with weight and formulate, obtained by us in [8], the conditions for the invertibility of operator A in this space.

A function $\varphi(x)$ that satisfies the following condition on $J=[0,1]$,

$$\left|\varphi(x_1)-\varphi(x_2)\right| \le C\left|x_1-x_2\right|^{\mu}, \quad x_1 \in J, x_2 \in J, \mu \in (0,1),$$

is called a Hölder function with exponent μ and constant C on *J*.

Let ρ be a power function which has zeros at the endpoints $x=0$, $x=1$:

$$\rho(x)=(x-0)^{\mu_0}(1-x)^{\mu_1}, \quad \mu<\mu_0<1+\mu, \mu<\mu_1<1+\mu.$$

The functions that become Hölder functions and have zero values at the points $x=0$, $x=1$, after being multiplied by $\rho(x)$, form a Banach space:

$$H_\mu^0(J,\rho), \quad J=[0,1].$$

The norm in the space $H_\mu^0(J,\rho)$ is defined by

$$\left\|f(x)\right\|_{H_\mu^0(J,\rho)}=\left\|\rho(x)f(x)\right\|_{H_\mu(J)},$$

where

$$\left\|\rho(x)f(x)\right\|_{H_\mu(J)}=\left\|\rho(x)f(x)\right\|_C+\left\|\rho(x)f(x)\right\|_\mu,$$

and

$$\left\|\rho(x)f(x)\right\|_C=\max_{x\in J}\left|\rho(x)f(x)\right|,$$

$$\left\|\rho(x)f(x)\right\|_\mu=\max_{x_1,x_2\in J,x_1\ne x_2}\left|\rho(x)f(x)\right|_\mu,$$

$$\left|\rho(x)f(x)\right|_\mu=\frac{\left|\rho(x_1)f(x_1)-\rho(x_2)f(x_2)\right|}{\left|x_1-x_2\right|^\mu}.$$

Let $\alpha(x)$ be a bijective orientation-preserving shift on J:
if $x_1<x_2$, then $\alpha(x_1)<\alpha(x_2)$ for any $x_1\in J, x_2\in J$; and let $\alpha(x)$ have only two fixed points:

$$\alpha(0)=0, \quad \alpha(1)=1, \quad \alpha(x)\ne x, \quad \text{when } x\ne 0, x\ne 1.$$

In addition, let $\alpha(x)$ be a differentiable function with $\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x)\ne 0$ and $\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x)\in H_\mu(J)$.

The shift operator is defined by the formula $(B_\alpha\varphi)(x)=\varphi[\alpha(x)]$.

Consider the operator $A=aI-bB_\alpha$ with coefficients from the Hölder space $a(x)\in H_\mu(J)$, $b(x)\in H_\mu(J)$. Operator A acts on the Hölder space with weight described above, $H_\mu^0(J,\rho)$.

We will now formulate conditions of ivertibility for operator A in the space of Hölder class functions with weight [8].

Operator A, acting in Banach space $H_\mu^0(J,\rho)$, is invertible if the following condition is fulfilled: $\theta_\alpha\left[a(x),b(x),H_\mu^0(J,\rho)\right]\ne 0, x\in J$, where the function σ_α is defined by

$$\theta_\alpha\left[a(x),b(x),H_\mu^0(J,\rho)\right]$$
$$=\begin{cases} a(x), \text{ when } |a(0)|>\left[\alpha'(0)\right]^{-\mu_0+\mu}|b(0)| \text{ and } |a(1)|>\left[\alpha'(1)\right]^{-\mu_1+\mu}|b(1)|; \\ b(x), \text{ when } |a(0)|<\left[\alpha'(0)\right]^{-\mu_0+\mu}|b(0)| \text{ and } |a(1)|<\left[\alpha'(1)\right]^{-\mu_1+\mu}|b(1)|; \\ 0 \qquad \text{in other cases.} \end{cases} \tag{1}$$

Note that the condition for the invertibility of the operator A in Hölder space with weight $H_\mu^0(J,p)$ can be obtained in terms of convergent series based on the recurrence relation. We represent the solution of the linear operator equation $I\varphi(x)-G(x)B_\alpha\varphi(x)=g(x)$ using the recurrence relation $\varphi(x)=G(x)(B_\alpha\varphi)(x)+g(x)$:

$$\begin{aligned}\varphi(x)&=G(x)B_\alpha\left[G(x)(B_\alpha\varphi)(x)+g(x)\right]+g(x)\\&=G(x)B_\alpha G(x)B_\alpha^2\left[\varphi(x)\right]+G(x)B_\alpha g(x)+g(x)\\&=G(x)(B_\alpha G)(x)B_\alpha^2\left[G(x)(B_\alpha\varphi)(x)+g(x)\right]+G(x)B_\alpha g(x)+g(x)\\&=G(x)(B_\alpha G)(x)\left(B_\alpha^2G\right)(x)\left(B_\alpha^3\varphi\right)(x)+G(x)(B_\alpha G)(x)\left(B_\alpha^2\right)g(x)\\&\quad+G(x)B_\alpha g(x)+g(x)\end{aligned}$$

and

$$\begin{aligned}\varphi(x)&=G(x)(B_\alpha G)(x)\left(B_\alpha^2G\right)(x)\left(B_\alpha^3[\ldots]\right)+G(x)(B_\alpha G)(x)\left(B_\alpha^2\right)g(x)\\&\quad+G(x)B_\alpha g(x)+g(x).\end{aligned}\tag{2}$$

Operator A is invertible in $H_\mu^0(J,\rho)$ if the series (2) converges in this space.

Note that condition (1) is a condition for the convergence of the series (2), which is a solution to equation $I\varphi(x)-G(x)B_\alpha\varphi(x)=g(x)$.

3. Solution of the Obtained Non-Linear Equations

In Hölder's space $H_\mu(J)$, we consider the non-linear non-homogeneous equation

$$\varphi(x)-\frac{G(x)}{(B_\alpha\varphi)(x)}=g(x)\ \text{ or }\ \varphi(x)(B_\alpha\varphi)(x)-g(x)(B_\alpha\varphi)(x)=G(x),\tag{3}$$

where $G(x)\in H_\mu(J)$, $g(x)\in H_\mu(J)$ and the function $\varphi(x)$ is sought in space $H_\mu(J)$.

We will assume that all considered functions are positive.

Consider factorization problem, $G(x)\in H_\mu(J)$, $g(x)=0, x\in J$, $\nu(x)\in H_\mu(J)$,

$$\nu(x)-\frac{G(x)}{(B_\alpha\nu)(x)}=0\ \text{ or }\ G(x)=\nu(x)(B_\alpha\nu)(x).\tag{4}$$

We write the recurrent relation $\nu(x)=\dfrac{G(x)}{(B_\alpha\nu)(x)}$, from here:

$$\begin{aligned}\nu(x)&=\frac{G(x)}{B_\alpha\left[\dfrac{G(x)}{(B_\alpha\nu)(x)}\right]}=\frac{G(x)}{(B_\alpha G)(x)}\left(B_\alpha^2\nu\right)(x)=\frac{G(x)}{(B_\alpha G)(x)}\left(B_\alpha^2\left[\frac{G}{B_\alpha\nu}\right]\right)(x)\\&=\frac{G(x)}{(B_\alpha G)(x)}\frac{\left(B_\alpha^2G\right)(x)}{\left(B_\alpha^3[\nu]\right)(x)}=\frac{G(x)}{(B_\alpha G)(x)}\frac{\left(B_\alpha^2G\right)(x)}{\left(B_\alpha^3\left[\dfrac{G}{B_\alpha\nu}\right]\right)(x)}\\&=\frac{G(x)}{(B_\alpha G)(x)}\frac{\left(B_\alpha^2G\right)(x)}{\left(B_\alpha^3G\right)(x)}\left(B_\alpha^4[\nu]\right)(x)=\cdots\end{aligned}$$

The condition for the solvability of Equation (4) is the condition for the convergence of the infinite product according to Hölder norm $\|f(x)\|_{H_\mu(J)} = \max_{x\in J}|f(x)| + \max_{x_1,x_2\in J, x_1\neq x_2}\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^\mu}$ to a function from $H_\mu(J)$.

Theorem 1 *If the infinite product converges in Hölder norm to a function* $v(x)\in H_\mu(J)$, *then the only solution of the non-linear non-homogeneous* Equation (4) *will be*.

$$v(x)=\sqrt{G(1)}\frac{G(x)}{(B_\alpha G)(x)}\frac{(B_\alpha^2 G)(x)}{(B_\alpha^3 G)(x)}\frac{(B_\alpha^4 G)(x)}{(B_\alpha^5 G)(x)}\cdots \tag{5}$$

Consider jump problem: $G(x)=1, x\in J$, $h(x)\in H_\mu(J)$, $f(x)\in H_\mu(J)$,

$$f(x)-\frac{1}{(B_\alpha f)(x)}=h(x). \tag{6}$$

We write the recurrent relation $f(x)=\frac{1}{(B_\alpha f)(x)}+h(x)$, from here:

$$\begin{aligned} f(x) &= \frac{1}{\left(B_\alpha\left[\frac{1}{(B_\alpha f)(x)}+h(x)\right]\right)(x)}+h(x) \\ &= \frac{1}{\frac{1}{(B_\alpha^2[f])(x)}+(B_\alpha h)(x)}+h(x) \\ &= \frac{1}{\frac{1}{\left(B_\alpha^2\left[\frac{1}{(B_\alpha f)(x)}+h(x)\right]\right)(x)}+(B_\alpha h)(x)}+h(x)=\cdots \end{aligned}$$

The condition for the solvability of Equation (6) is the condition for the convergence of the continued infinite fraction [9] in Hölder norm to a function from $H_\mu(J)$.

Theorem 2 *If the continued infinite fraction converges in Hölder norm to a function* $f(x)\in H_\mu(J)$, *then the only solution of the non-linear homogeneous* Equation (6) *will be the function*.

$$f(x)=\frac{1}{\frac{1}{\left(B_\alpha^2\left[\frac{1}{(B_\alpha[\ldots])}+h(x)\right]\right)(x)}+(B_\alpha h)(x)}+h(x) \tag{7}$$

Let us turn to the solution of the general nonlinear equation based on the solution of the factorization problem and the jump problem. If the factorization problem is solvable and its solution is $v(x)$, then the coefficient $G(x)$ is represented by a product $v(x)(B_\alpha v)(x)$. The general non-linear equation takes on the form of a jump problem $f(x)-\frac{1}{(B_\alpha f)(x)}=h(x)$, where $h(x)=\frac{g(x)}{\nu(x)}$, $h(x)=\frac{\varphi(x)}{\nu(x)}$. If the jump problem is solvable, then its solution $f(x)$ is represented by an infinite continued fraction (7). The solution of the general nonlinear equation is determined by the formula $\varphi(x)=f(x)\nu(x)$.

Now, let's go back to Equation (3) and write down, without solving, factorization and jump problems, only on the basis of the reduction relation $\varphi(x)=\frac{G(x)}{(B_\alpha\varphi)(x)}+g(x)$, the solution of nonlinear Equation (3) in the form of an infinite continued fraction.

$$\begin{aligned}\varphi(x)&=\frac{G(x)}{B_\alpha\left[\frac{G(x)}{(B_\alpha\varphi)(x)}+g(x)\right](x)}+g(x)\\&=\frac{G(x)}{\frac{B_\alpha G(x)}{\left(\left[\frac{G(x)}{\left(B_\alpha^2[\ldots]\right)}+g(x)\right]\right)}+(B_\alpha g)(x)}+g(x).\end{aligned}\tag{8}$$

To conclude, we note that because the continued fraction (8) has a more complex structure than (7), then the conditions for its convergence do not look "transparent" as the condition for the solvability of the factorization problem, that is, the condition for the convergence of the infinite product (5) and the condition for the solvability of the jump problem, that is, the condition for the convergence of the continuous function (7). On the other hand, the problem of factorization of the coefficient may turn out to be unsolvable; it remains to investigate the convergence of the continued fraction (8).

In this section, we will not touch on problems related to the description of the classes of solutions of the considered non-linear equations. We will look for solutions among the Hölder class functions $H_\mu(J)$. We will not deal with finding conditions for the convergence of the resulting infinite products and infinite continuous functions. Coefficients and free terms of non-linear equations arising in the modeling of renewable systems are real non-negative functions. Of course, this particularity and other specifics will be taken into account when applying the proposed mathematical apparatus in the analysis of the balance relations of the models.

4. Conclusion

The authors intend to generalize the method for solving nonlinear equations proposed in Section 3, which is based on the separation of the coefficient factorization problem and the jump problem. Then, we apply it to solve equations with an abstract operator defined through the description of its non-linear properties. We propose to develop a model for renewable systems with elements that are in different states, to study balance relations and find the equilibrium state, and use inverse operators constructed on the basis of solving equations with abstract nonlinear operators.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Tarasenko, A., Karelin, O., Perez-Lechuga, P. and Gonzalez-Hernandez, M. (2010) Modelling Systems with Renewable Resources Based on Functional Operators with Shift. *Applied Mathematics and Computation*, **216**, 1938-1944. https://doi.org/10.1016/j.amc.2010.03.023

[2] Karelin, O., Tarasenko, A. and Gonzalez-Hernandez, M. (2013) Application of Functional Operators with Shift to the Study of Renewable Systems when the reproductive Processed is Describedby Integrals with Degenerate Kernels. *Applied Mathematics*, **4**, 1376-1380. https://doi.org/10.4236/am.2013.410186

[3] Karelin, O., Gonzalez-Hernandez, M., Hernandez-Romero, N. and Seck-Tuoh-Mora, J. (2015) Study of Renewable Systems Based on Functional Operators with Shift. *Applied Mathematical Sciences*, **9**, 7167-7173. https://doi.org/10.12988/ams.2015.510622

[4] Karelin, O., Tarasenko, A., Zolotov, V. and Gonzalez-Hernandez, M. (2018) Study of the Equilibrium State of Systems with Two Renewable Resources Based on Functional Operators with Shift. *IAENG International Journal of Applied Mathematics*, **48**, 98-104.

[5] Karelin, O., Tarasenko, A., Barabash, O., Gonzalez-Hernandez, M. and Medina-Marin, J. (2019) Study of Changes of Individual Parameter of Resources in the Modelling of Renewable Systems. In: Ao, S., Gelman, L. and Kim, H., Eds., *Transactions on Engineering Technologies. WCE* 2018, Springer, Singapore, 19-28. https://doi.org/10.1007/978-981-32-9531-5_2

[6] Atkinson, K.E. (1997) The Numerical Solution of Integral Equations of the Second Kind. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511626340

[7] Jerry, A.J. (1999) Introduction to Integral Equations with Application. Wiley-Interscience, New York.

[8] Tarasenko, A. and Karelin, A. (2016) Condition of Invertibility for Functional Operators with Shift in Weighted Hölder Spaces. *Ukrainian Mathematical Journal*, **67**, 1758-1769. https://doi.org/10.1007/s11253-016-1187-4

[9] Khinchin, A.Y. (1964) Continued Fraction. University of Chicago Press, Chicago.

Applied Mathematics, 2022, 13, 658-671
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Global Stability of a Three-Species System with Attractive Prey-Taxis

Mengxin Chen[1], **Qianqian Zheng**[2]

[1]College of Mathematics and Information Science, Henan Normal University, Xinxiang, China
[2]School of Science, Xuchang University, Xuchang, China
Email: chmxdc@163.com

How to cite this paper: Chen, M.X. and Zheng, Q.Q. (2022) Global Stability of a Three-Species System with Attractive Prey-Taxis. *Applied Mathematics*, **13**, 658-671.
https://doi.org/10.4236/am.2022.138041

Received: July 30, 2022
Accepted: August 16, 2022
Published: August 19, 2022

Abstract

This paper reports the global asymptotic stability of a three-species predator-prey system involving the prey-taxis. With the assumptions, we establish the global asymptotic stability results of its equilibria, respectively. Our results illustrate that 1) the global asymptotic stability of the semi-trivial equilibrium does not involve the prey-taxis coefficients χ,ξ; 2) the global asymptotic stability of two boundary equilibria relies on a single prey-taxis coefficient χ and ξ, respectively; 3) the global asymptotic stability of the unique positive equilibrium depends on two prey-taxis coefficients χ and ξ.

Keywords

Predator-Prey Model, Global Asymptotic Stability, Prey-Taxis, Lyapunov Function

1. Introduction

In the past few decades, predator-prey systems involving the prey-taxis have attracted more and more scholars to investigate them. Chen *et al.* [1] reported stationary patterns of a predator-prey model with prey-taxis and investigated the stability of the nonconstant steady states by employing the Crandall-Rabinowitz bifurcation theory. Tu *et al.* [2] considered the asymptotic behaviors of a parabolic-elliptic chemotaxis system with competitive kinetics and loop of a predator-prey model. Bell and Haskell [3] established the global existence of positive classical solutions and the existence of nontrivial steady states via the bifurcation theory of a predator-prey system. The global existence and uniform boundedness of solutions to a predator-prey system with prey-taxis for general functional responses in any spatial dimensions have been investigated by Ahn and Yoon [4]. The existence of the unique global bounded classical solution is proven, and the steady-state bifurcation, the Hopf bifurcation, and Hopf/steady-state mode interaction are studied via the Lyapunov-Schmidt procedure by Qiu *et al.* [5]. We recommend more existing results about the predator-prey systems with directed prey-taxis, see Refs. [6] [7] [8] [9] [10], etc.

In this present paper, we focus on a predator-prey model with two predators and one prey as well as the prey-taxis as follows.

$$
\begin{cases}
\partial_t u = \partial_{xx} u - \chi \partial_x \cdot \left(\dfrac{u}{(1+\delta w)^2} \partial_x w \right) + \dfrac{\beta_1 uw}{\alpha_1 + w + su} - \dfrac{\rho_1 uw}{\alpha_1 + w + su} - \delta_1 u, & x \in \Omega,\ t > 0, \\
\partial_t v = \partial_{xx} v - \xi \partial_x \cdot \left(\dfrac{v}{(1+\delta w)^2} \partial_x w \right) + \dfrac{\beta_2 vw}{\alpha_2 + w} - \dfrac{\rho_2 vw}{\alpha_2 + w} - \delta_2 v, & x \in \Omega,\ t > 0, \\
\partial_t w = d \partial_{xx} w + rw \left(1 - \dfrac{w}{K} \right) - \dfrac{\mu_1 uw}{\alpha_1 + w + su} - \dfrac{\mu_2 vw}{\alpha_2 + w}, & x \in \Omega,\ t > 0, \\
\partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega,\ t \ge 0, \\
u(x,0) = u_0(x) \ge 0,\ v(x,0) = v_0(x) \ge 0,\ w(x,0) = w_0(x) \ge 0, & x \in \Omega,
\end{cases}
\tag{1}
$$

where $u = u(x,t), v = v(x,t)$ and $w = w(x,t)$ are predator and prey densities at position x and time t, respectively. $\Omega \subset \mathbb{R}^N$ is a bounded domain with its smooth boundary $\partial\Omega$; constant d describes the diffusive rate of prey; For $j = 1, 2$, β_j are the ratios of biomass conversion of predators species; ρ_j represent the rates of toxic substances produced by per unit biomass about predators due to prey species are toxic corresponding to the predators; δ_j are the natural mortality of the predators u and v; α_j describe the half-saturation constant of the predators; s describes the measure of mutual interference among the predator u; two constants r and K in the third equation are the intrinsic growth rate and the maximum environmental capacity of prey species, respectively. Moreover, $-\chi \partial_x \cdot \left(\dfrac{u}{(1+\delta w)^2} \partial_x w \right)$ and $-\xi \partial_x \cdot \left(\dfrac{v}{(1+\delta w)^2} \partial_x w \right)$ are prey-taxis terms. They imply the tendency of predators moving toward the positive direction of the increasing gradient of prey population as $\chi > 0$ and $\xi > 0$. If $\chi < 0$ and $\xi < 0$, we say that predators move toward the opposite direction of the increasing gradient of prey population to avoid group defense by a large number of prey species or volume-filling effect in predator species [11]. Consequently, $\chi, \xi > 0$ and $\chi, \xi < 0$ corresponding to attractive and repulsive prey-taxis, respectively. Moreover, $\dfrac{u}{(1+\delta w)^2}$ and $\dfrac{v}{(1+\delta w)^2}$ represent the distribution variations of the directed species dispersals [12]. Obviously, they depend on the density of the prey population. All parameters exhibited in the system (1) are set to be positive.

For system (1), define

$$
f(u,v,w) = \frac{\beta_1 uw}{\alpha_1 + w + su} - \frac{\rho_1 uw}{\alpha_1 + w + su} - \delta_1 u,
$$

$$g(u,v,w)=\frac{\beta_2 vw}{\alpha_2+w}-\frac{\rho_2 vw}{\alpha_2+w}-\delta_2 v,$$

and

$$h(u,v,w)=rw\left(1-\frac{w}{K}\right)-\frac{\mu_1 uw}{\alpha_1+w+su}-\frac{\mu_2 vw}{\alpha_2+w},$$

as well as some assumptions

(H1) $\beta_1-\rho_1-\delta_1>0,\ \hat{w}>\dfrac{\alpha_1\delta_1}{\beta_1-\rho_1-\delta_1}$ with $\hat{w}=\dfrac{-\gamma_2+\sqrt{\gamma_2^2-4\gamma_1\gamma_3}}{2\gamma_1}$, $\gamma_1=rs(\beta_1-\rho_1),\gamma_2=\mu_1(\beta_1-\rho_1-\delta_1)K-rs(\beta_1-\rho_1)K,\gamma_3=-\mu_1\alpha_1\delta_1K.$

(H2) $\beta_2-\rho_2-\delta_2>0,(\beta_2-\rho_2)K-(\alpha_2+K)\delta_2>0.$

(H3) $\beta_2-\rho_2-\delta_2>0,\alpha_2\delta_2(\beta_1-\rho_1-\delta_1)-\alpha_1\delta_1(\beta_2-\rho_2-\delta_2)>0,$ $(r-\mu_1)(\beta_2-\rho_2-\delta_2)K-r\alpha_2\delta_2>0.$

As a result, we can conclude the classifications of the equilibria of system (1). 1) system (1) has a trivial equilibrium $E_0=(0,0,0)$ and a semi-trivial equilibrium $E_1=(0,0,K)$; 2) if (H1) holds, system (1) has a boundary equilibrium $E_2=(\hat{u},0,\hat{w})$, where

$$\hat{w}=\frac{-\gamma_2+\sqrt{\gamma_2^2-4\gamma_1\gamma_3}}{2\gamma_1},\ \hat{u}=\frac{(\beta_1-\rho_1-\delta_1)\hat{w}-\alpha_1\delta_1}{s\delta_1};$$

3) if (H2) is valid, system (1) has a boundary equilibrium $E_3=(0,\tilde{v},\tilde{w})$, where

$$\tilde{w}=\frac{\alpha_2\delta_2}{\beta_2-\rho_2-\delta_2},\ \tilde{v}=\frac{r\alpha_2(\beta_2-\rho_2)\left[(\beta_2-\rho_2)K-(\alpha_2+K)\delta_2\right]}{\mu_2(\beta_2-\rho_2-\delta_2)K};$$

4) if (H3) is satisfied, system (1) has a unique positive equilibrium $E_*=(u^*,v^*,w^*)$, where

$$u^*=\frac{\alpha_2\delta_2(\beta_1-\rho_1-\delta_1)-\alpha_1\delta_1(\beta_2-\rho_2-\delta_2)}{s\delta_1(\beta_2-\rho_2-\delta_2)},$$

$$v^*=\frac{(\alpha_1+w^*)\gamma_4}{\delta_2\mu_2\alpha_2K(\beta_1-\rho_1)(\beta_2-\rho_2-\delta_2)},$$

and

$$w^*=\frac{\alpha_2\delta_2}{\beta_2-\rho_2-\delta_2},$$

with $\begin{aligned}\gamma_4=&\alpha_2\delta_2(\beta_1-\rho_1)\left[(r-\mu_1)(\beta_2-\rho_2-\delta_2)K-r\alpha_2\delta_2\right]\\&+\mu_1\delta_1K(\beta_2-\rho_2-\delta_2)\left[\alpha_2\delta_2+\alpha_1(\beta_2-\rho_2-\delta_2)\right]\end{aligned}.$

In this present paper, we will establish the global asymptotic stabilities of semi-trivial equilibrium $E_1=(0,0,K)$, boundary equilibria $E_2=(\hat{u},0,\hat{w}),E_3=(0,\tilde{v},\tilde{w})$ and the unique positive equilibrium $E_*=(u^*,v^*,w^*)$ by constructing some suitable time evolution Lyapunov functions, respectively.

This paper is structured as follows. In Section 2, we perform the main results of the present paper. In Section 3, the local-in-time existence of the classical solution of the model is given. In Section 4, the proofs of the main results are displayed. Finally, some conclusions are made in Section 5.

2. Main Results

Theorem 2.1 *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with the smooth boundary* $\partial\Omega$. *Suppose* (u,v,w) *is a classical solution of system* (1) *with the initial conditions* $(u_0(x), v_0(x), w_0(x)) \in \left[W^{1,p}(\Omega)\right]^3$ *and* $u_0(x) \geq 0, v_0(x), w_0(x) \geq 0$ *for* $x \in \bar{\Omega}$. *We have the following global asymptotic stability results.*

1) *For any* $\chi, \xi > 0$ *and*

$$0 < \beta_1 \leq \rho_1 + \mu_1,\ 0 < \beta_2 \leq \rho_2 + \mu_2,\ 0 < K \leq \min\left\{\frac{\alpha_1\delta_1}{\mu_1}, \frac{\alpha_2\delta_2}{\mu_2}\right\}, \quad (2)$$

then $E_1 = (0,0,K)$ *is globally asymptotically stable.*

2) *If the condition* $(H1)$ *holds and*

$$\beta_2 \leq \mu_2 \leq \frac{\delta_2\alpha_2}{\hat{w}},\ \mu_1 = \frac{(\beta_1 - \rho_1)(\alpha_1 + s\hat{u})}{\alpha_1 + \hat{w}},\ 0 < K \leq \frac{r\alpha_1^2}{\mu_1\hat{u}},\ 0 < \chi^2 \leq \frac{4d\hat{w}}{C^2\hat{u}}, \quad (3)$$

then boundary equilibrium $E_2 = (\hat{u}, 0, \hat{w})$ *is globally asymptotically stable for any* $\xi > 0$.

3) *If the condition* $(H2)$ *is valid and*

$$\beta_1 \leq \mu_1 \leq \frac{\delta_1\alpha_1}{\tilde{w}},\ \mu_2 = \frac{\alpha_2(\beta_2 - \rho_2)}{\alpha_2 + \tilde{w}},\ 0 < K \leq \frac{r\alpha_2^2}{\mu_2\tilde{v}},\ 0 < \xi^2 \leq \frac{4d\tilde{w}}{C^2\tilde{v}}, \quad (4)$$

then $E_3 = (0, \tilde{v}, \tilde{w})$ *is globally asymptotically stable for any* $\chi > 0$.

4) *If the condition* $(H3)$ *holds and*

$$\mu_1 = \frac{(\beta_1 - \rho_1)(\alpha_1 + su^*)}{\alpha_1 + w^*},\ \mu_2 = \frac{\alpha_2(\beta_2 - \rho_2)}{\alpha_2 + w^*},\ 0 < K \leq \frac{r\alpha_1^2}{\mu_1 u^*} + \frac{r\alpha_2^2}{\mu_2 v^*}, \quad (5)$$

as well as

$$0 < \chi^2 + \xi^2 \leq \frac{4dw^*}{\max\{u^*, v^*\}C^2}, \quad (6)$$

then $E_* = (u^*, v^*, w^*)$ *is globally asymptotically stable, where* $C = \max\left\{\|w_0(x)\|_{L^\infty(\Omega)}, K\right\}$.

Remark 2.1 *From Theorem* 2.1, *we can find that the global asymptotic stability of the semi-trivial equilibrium* $E_1 = (0,0,K)$ *does not involve the prey-taxis coefficients* χ *and* ξ. *The global asymptotic stabilities of the boundary equilibria* $E_2 = (\hat{u}, 0, \hat{w})$ *and* $E_3 = (0, \tilde{v}, \tilde{w})$ *only depend on prey-taxis coefficient* χ *and* ξ, *respectively. However, the global asymptotic stability of the unique positive equilibrium* $E_* = (u^*, v^*, w^*)$ *depends on prey-taxis coefficients* χ *and* ξ.

Remark 2.2 *The control conditions* (3), (4) *and* (6) *of the global asymptotic* *stabilities of the equilibria only involve the initial spatial density* $w_0(x)$ *of prey species but are independent of the initial spatial densities* $u_0(x)$ *and* $v_0(x)$ *of the predators.*

The following conclusion is helpful to obtain the desired results.

3. Existence

Lemma 1 *Suppose that* $\Omega \subset \mathbb{R}^N$ *with the smooth boundary* $\partial\Omega$. *For any initial conditions* $(u_0(x), v_0(x), w_0(x)) \in [W^{1,p}(\Omega)]^3$ *satisfies* $u_0(x) \ge 0, v_0(x), w_0(x) \ge 0$ *for* $x \in \bar{\Omega}$. *Then there is a maximal existence time* $T_{\max} > 0$ *such that system* (1) *has a unique local non-negative classical solution* $(u(x,t), v(x,t), w(x,t)) \in \left[C([0,T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max}))\right]^3$. *Moreover, we have* $u(x,t), v(x,t) > 0, w(x,t) \le C$ *for* $x \in \bar{\Omega}$, $t \in [0, T_{\max})$, *where* $C = \max\left\{\|w_0(x)\|_{L^\infty(\Omega)}, K\right\}$.

Proof. Denote by $\psi(x,t) = (u(x,t), v(x,t), w(x,t))$. Then system (1) takes the form

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = \partial_x \cdot (z(\psi)\partial_x \psi) + \Psi(\psi), & x \in \Omega,\ t > 0, \\ \dfrac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega,\ t \ge 0, \\ \psi(\cdot, 0) = (u_0(x), v_0(x), w_0(x)), & x \in \Omega. \end{cases}$$

where

$$z(\psi) = \begin{pmatrix} 1 & 0 & -\chi \dfrac{u}{(1+\delta w)^2} \\ 0 & 1 & -\xi \dfrac{u}{(1+\delta w)^2} \\ 0 & 0 & d \end{pmatrix}, \quad \Psi(\psi) = \begin{pmatrix} \dfrac{\beta_1 uw}{\alpha_1 + w + su} - \dfrac{\rho_1 uw}{\alpha_1 + w + su} - \delta_1 u \\ \dfrac{\beta_2 vw}{\alpha_2 + w} - \dfrac{\rho_2 vw}{\alpha_2 + w} - \delta_2 v \\ rw\left(1 - \dfrac{w}{K}\right) - \dfrac{\mu_1 uw}{\alpha_1 + w + su} - \dfrac{\mu_2 vw}{\alpha_2 + w} \end{pmatrix}.$$

Obviously, $z(\psi)$ is an upper-triangular matrix and is positive definite since $d > 0$ is valid. Therefore, the local existence can be checked by Amman's fixed point argument [13]. Now rewrite the first equation of system (1) as follows.

$$\begin{cases} \partial_t u = \partial_{xx} u - \chi \dfrac{\partial_x u \cdot \partial_x w}{(1+\delta w)^2} + \dfrac{2\chi\delta u}{(1+\delta w)^3} \partial_{xx} w + u f_1(u,v,w), & x \in \Omega,\ t \in (0, T_{\max}), \\ \partial_\nu u = 0, & x \in \partial\Omega,\ t \in (0, T_{\max}), \\ u(x,0) = u_0(x) \ge 0, & x \in \Omega, \end{cases} \tag{7}$$

where $f_1(u,v,w) = \dfrac{\beta_1 w}{\alpha_1 + w + su} - \dfrac{\rho_1 w}{\alpha_1 + w + su} - \delta_1$. Obviously, 0 is a lower solution of (7). Therefore, the maximum principle shows $u(x,t) \ge 0$ for all $(x,t) \in \Omega \times (0, T_{\max})$. Combine $u_0(x) \ge 0 (\not\equiv 0)$ with the strong maximum principle, $u(x,t) > 0$ is valid. By the same way, we have $v(x,t), w(x,t) > 0$ for all $(x,t) \in \Omega \times (0, T_{\max})$. Finally, the maximum principle ensures that $w(x,t) \le C$

for $(x,t)\in\Omega\times(0,T_{\max})$. This ends the proof.

In the sequel, we shall give proof of Theorem 2.1 by constructing some suitable time evolution Lyapunov functions.

4. Proof of Theorem 2.1

1) For $E_1=(0,0,K)$, define the following Lyapunov function

$$V_1(t)=\int_\Omega u(\cdot,t)\mathrm{d}x+\int_\Omega v(\cdot,t)\mathrm{d}x+\int_\Omega\int_K^w\frac{w(\cdot,t)-K}{w(\cdot,t)}\mathrm{d}w\mathrm{d}x. \tag{8}$$

Then we deduce

$$\begin{aligned}
\dot{V}_1(t)=&\int_\Omega\left(\frac{\beta_1 uw}{\alpha_1+w+su}-\frac{\rho_1 uw}{\alpha_1+w+su}-\delta_1 u\right)\mathrm{d}x\\
&+\int_\Omega\left(\frac{\beta_2 vw}{\alpha_2+w}-\frac{\rho_2 vw}{\alpha_2+w}-\delta_2 v\right)\mathrm{d}x-\int_\Omega\frac{dK|\partial_x w|^2}{w^2}\mathrm{d}x\\
&+\int_\Omega(w-K)\left[r\left(1-\frac{w}{K}\right)-\frac{\mu_1 u}{\alpha_1+w+su}-\frac{\mu_2 v}{\alpha_2+w}\right]\\
\le&\int_\Omega\left(\frac{(\beta_1-\rho_1-\mu_1)uw}{\alpha_1+w+su}-\delta_1 u\right)\mathrm{d}x+\int_\Omega\frac{\mu_1 Ku}{\alpha_1+w+su}\mathrm{d}x\\
&+\int_\Omega\frac{\mu_2 Kv}{\alpha_2+w}\mathrm{d}x+\int_\Omega\left(\frac{(\beta_2-\rho_2-\mu_2)vw}{\alpha_2+w}-\delta_2 v\right)\mathrm{d}x\\
&-\int_\Omega\frac{dK|\partial_x w|^2}{w^2}\mathrm{d}x+\int_\Omega r(w-K)\left(1-\frac{w}{K}\right)\mathrm{d}x\\
\le&\int_\Omega\frac{(\beta_1-\rho_1-\mu_1)uw}{\alpha_1+w+su}\mathrm{d}x+\int_\Omega\left(\frac{\mu_1 K}{\alpha_1}-\delta_1\right)u\mathrm{d}x+\int_\Omega\frac{(\beta_2-\rho_2-\mu_2)vw}{\alpha_2+w}\mathrm{d}x\\
&+\int_\Omega\left(\frac{\mu_2 K}{\alpha_2}-\delta_2\right)v\mathrm{d}x-\int_\Omega\frac{dK|\partial_x w|^2}{w^2}\mathrm{d}x-\frac{r}{K}\int_\Omega(w-K)^2\mathrm{d}x.
\end{aligned}$$

Consequently, $\dot{V}_1(t)\le 0$ and $E_1=(0,0,K)$ is globally asymptotically stable if (2) holds.

2) Define

$$V_2(t)=\int_\Omega\int_{\hat{u}}^u\frac{u(\cdot,t)-\hat{u}}{u(\cdot,t)}\mathrm{d}u\mathrm{d}x+\int_\Omega v(\cdot,t)\mathrm{d}x+\int_\Omega\int_{\hat{w}}^w\frac{w(\cdot,t)-\hat{w}}{w(\cdot,t)}\mathrm{d}w\mathrm{d}x. \tag{9}$$

As a result, one deduces

$$\begin{aligned}
\dot{V}_2(t)=&\int_\Omega\left(1-\frac{\hat{u}}{u}\right)\partial_t u\mathrm{d}x+\int_\Omega\partial_t v\mathrm{d}x+\int_\Omega\left(1-\frac{\hat{w}}{w}\right)\partial_t w\mathrm{d}x\\
=&\int_\Omega(u-\hat{u})\left(\frac{\beta_1 w}{\alpha_1+w+su}-\frac{\rho_1 w}{\alpha_1+w+su}-\delta_1\right)\mathrm{d}x-\int_\Omega\frac{\hat{u}|\partial_x u|^2}{u^2}\mathrm{d}x\\
&+\int_\Omega\frac{\chi\hat{u}\partial_x w\cdot\partial_x u}{u(1+\delta w)^2}\mathrm{d}x+\int_\Omega\left(\frac{\beta_2 vw}{\alpha_2+w}-\frac{\rho_2 vw}{\alpha_2+w}-\delta_2 v\right)\mathrm{d}x\\
&-\int_\Omega\frac{d\hat{w}|\partial_x w|^2}{w^2}\mathrm{d}x+\int_\Omega(w-\hat{w})\left[r\left(1-\frac{w}{K}\right)-\frac{\mu_1 u}{\alpha_1+w+su}-\frac{\mu_2 v}{\alpha_2+w}\right]\\
=&\hat{J}_1(t)+\hat{J}_2(t),
\end{aligned}$$

where

$$\hat{J}_1(t)=\int_\Omega (u-\hat{u})\left(\frac{\beta_1 w}{\alpha_1+w+su}-\frac{\rho_1 w}{\alpha_1+w+su}-\delta_1\right)\mathrm{d}x$$
$$+\int_\Omega \left(\frac{\beta_2 vw}{\alpha_2+w}-\frac{\rho_2 vw}{\alpha_2+w}-\delta_2 v\right)\mathrm{d}x$$
$$+\int_\Omega (w-\hat{w})\left[r\left(1-\frac{w}{K}\right)-\frac{\mu_1 u}{\alpha_1+w+su}-\frac{\mu_2 v}{\alpha_2+w}\right],$$

and

$$\hat{J}_2(t)=-\int_\Omega \frac{\hat{u}|\partial_x u|^2}{u^2}\mathrm{d}x+\int_\Omega \frac{\chi\hat{u}\partial_x w\cdot\partial_x u}{u(1+\delta w)^2}\mathrm{d}x-\int_\Omega \frac{d\hat{w}|\partial_x w|^2}{w^2}\mathrm{d}x.$$

By using

$$\delta_1=\frac{\beta_1\hat{w}}{\alpha_1+\hat{w}+s\hat{u}}+\frac{\rho_1\hat{w}}{\alpha_1+\hat{w}+s\hat{u}},\quad r=\frac{r\hat{w}}{K}+\frac{\mu_1\hat{u}}{\alpha_1+\hat{w}+s\hat{u}},$$

one yields

$$\begin{aligned}
\hat{J}_1(t)&=\int_\Omega (u-\hat{u})\left(\frac{\beta_1 w}{\alpha_1+w+su}-\frac{\rho_1 w}{\alpha_1+w+su}-\delta_1\right)\mathrm{d}x\\
&\quad+\int_\Omega \left(\frac{\beta_2 vw}{\alpha_2+w}-\frac{\rho_2 vw}{\alpha_2+w}-\delta_2 v\right)\mathrm{d}x\\
&\quad+\int_\Omega (w-\hat{w})\left[r\left(1-\frac{w}{K}\right)-\frac{\mu_1 u}{\alpha_1+w+su}-\frac{\mu_2 v}{\alpha_2+w}\right]\\
&=\int_\Omega \frac{(\beta_1-\rho_1)(\alpha_1+s\hat{u})(u-\hat{u})(w-\hat{w})}{(\alpha_1+w+su)(\alpha_1+\hat{w}+s\hat{u})}\mathrm{d}x\\
&\quad-\int_\Omega \frac{s\hat{w}(\beta_1-\rho_1)(u-\hat{u})^2}{(\alpha_1+w+su)(\alpha_1+\hat{w}+s\hat{u})}\mathrm{d}x+\int_\Omega \left(\frac{\beta_2 vw}{\alpha_2+w}-\frac{\rho_2 vw}{\alpha_2+w}-\delta_2 v\right)\mathrm{d}x\\
&\quad+\int_\Omega \frac{\mu_1\hat{u}(w-\hat{w})^2}{(\alpha_1+w+su)(\alpha_1+\hat{w}+s\hat{u})}\mathrm{d}x-\int_\Omega \frac{\mu_1(\alpha_1+\hat{w})(u-\hat{u})(w-\hat{w})}{(\alpha_1+w+su)(\alpha_1+\hat{w}+s\hat{u})}\mathrm{d}x\\
&\quad-\frac{r}{K}\int_\Omega (w-\hat{w})^2\,\mathrm{d}x-\int_\Omega \frac{\mu_2 v(w-\hat{w})}{\alpha_2+w}\mathrm{d}x\\
&\le\int_\Omega \frac{\left[(\beta_1-\rho_1)(\alpha_1+s\hat{u})-\mu_1(\alpha_1+\hat{w})\right](u-\hat{u})(w-\hat{w})}{(\alpha_1+w+su)(\alpha_1+\hat{w}+s\hat{u})}\mathrm{d}x-\int_\Omega \frac{\rho_2 vw}{\alpha_2+w}\mathrm{d}x\\
&\quad+\int_\Omega \frac{\mu_1\hat{u}(w-\hat{w})^2}{(\alpha_1+w+su)(\alpha_1+\hat{w}+s\hat{u})}\mathrm{d}x-\frac{r}{K}\int_\Omega (w-\hat{w})^2\,\mathrm{d}x+\int_\Omega \frac{(\beta_2-\mu_2)vw}{\alpha_2+w}\mathrm{d}x\\
&\quad+\int_\Omega \left(\frac{\mu_2\hat{w}}{\alpha_2+w}-\delta_2\right)v\mathrm{d}x-\int_\Omega \frac{s\hat{w}(\beta_1-\rho_1)(u-\hat{u})^2}{(\alpha_1+w+su)(\alpha_1+\hat{w}+s\hat{u})}\mathrm{d}x\\
&\le\int_\Omega \frac{\left[(\beta_1-\rho_1)(\alpha_1+s\hat{u})-\mu_1(\alpha_1+\hat{w})\right](u-\hat{u})(w-\hat{w})}{(\alpha_1+w+su)(\alpha_1+\hat{w}+s\hat{u})}\mathrm{d}x\\
&\quad+\int_\Omega \left(\frac{\mu_1 s\hat{u}}{\alpha_1^2}-\frac{r}{K}\right)(w-\hat{w})^2\,\mathrm{d}x\\
&\quad+\int_\Omega \frac{(\beta_2-\mu_2)vw}{\alpha_2+w}\mathrm{d}x+\int_\Omega \left(\frac{\mu_2\hat{w}}{\alpha_2}-\delta_2\right)v\mathrm{d}x\\
&\le 0,
\end{aligned}$$

due to (3) is valid. For $\hat{J}_2(t)$, we have

$$\begin{aligned}\hat{J}_2(t) &= -\int_\Omega \frac{\hat{u}|\partial_x u|^2}{u^2}\mathrm{d}x + \int_\Omega \frac{\chi\hat{u}\partial_x w\cdot\partial_x u}{u(1+\delta w)^2}\mathrm{d}x - \int_\Omega \frac{d\hat{w}|\partial_x w|^2}{w^2}\mathrm{d}x \\ &\le -\int_\Omega \frac{\hat{u}|\partial_x u|^2}{u^2}\mathrm{d}x + \int_\Omega \frac{\chi\hat{u}|\partial_x w|\cdot|\partial_x u|}{u}\mathrm{d}x - \int_\Omega \frac{d\hat{w}|\partial_x w|^2}{w^2}\mathrm{d}x \\ &= -\int_\Omega X_1 Q_1 X_1^{\mathrm{T}}\mathrm{d}x,\end{aligned}$$

where we define $X_1(x,t) = (|\partial_x u(x,t)|, |\partial_x w(x,t)|)$ in $\Omega\times(0,\infty)$, and the matrix Q_1 is

$$Q_1 = \begin{pmatrix} \frac{\hat{u}}{u^2} & -\frac{\chi\hat{u}}{2u} \\ -\frac{\chi\hat{u}}{2u} & \frac{d\hat{w}}{w^2}\end{pmatrix}.$$

Accordingly, we have $\frac{\hat{u}}{u^2} > 0$ and $\frac{\hat{u}}{u^2}\left(\frac{d\hat{w}}{w^2} - \frac{\chi^2\hat{u}}{4}\right) \ge 0$ as (3) holds. These imply $\dot{V}_2(t) = \hat{J}_1(t) + \hat{J}_2(t) \le 0$ and $E_2 = (\hat{u}, 0, \hat{w})$ is globally asymptotically stable.

3) Consider the following function

$$V_3(t) = \int_\Omega u(\cdot,t)\mathrm{d}x + \int_\Omega \int_{\tilde{v}}^{v} \frac{v(\cdot,t)-\tilde{v}}{v(\cdot,t)}\mathrm{d}v\mathrm{d}x + \int_\Omega \int_{\tilde{w}}^{w} \frac{w(\cdot,t)-\tilde{w}}{w(\cdot,t)}\mathrm{d}w\mathrm{d}x. \tag{10}$$

Straightforward computation showing

$$\begin{aligned}\dot{V}_3(t) &= \int_\Omega \partial_t u\mathrm{d}x + \int_\Omega \left(1-\frac{\tilde{v}}{v}\right)\partial_t v\mathrm{d}x + \int_\Omega \left(1-\frac{\tilde{w}}{w}\right)\partial_t w\mathrm{d}x \\ &= \int_\Omega \left(\frac{\beta_1 uw}{\alpha_1+w+su} - \frac{\rho_1 uw}{\alpha_1+w+su} - \delta_1 u\right)\mathrm{d}x - \int_\Omega \frac{\tilde{v}|\partial_x v|^2}{v^2}\mathrm{d}x \\ &\quad + \int_\Omega \frac{\xi\tilde{v}\partial_x w\cdot\partial_x v}{v(1+\delta w)^2}\mathrm{d}x + \int_\Omega (v-\tilde{v})\left(\frac{\beta_2 w}{\alpha_2+w} - \frac{\rho_2 w}{\alpha_2+w} - \delta_2\right)\mathrm{d}x \\ &\quad - \int_\Omega \frac{d\tilde{w}|\partial_x w|^2}{w^2}\mathrm{d}x + \int_\Omega (w-\tilde{w})\left[r\left(1-\frac{w}{K}\right) - \frac{\mu_1 u}{\alpha_1+w+su} - \frac{\mu_2 v}{\alpha_2+w}\right] \\ &= \tilde{J}_1(t) + \tilde{J}_2(t),\end{aligned}$$

where

$$\begin{aligned}\tilde{J}_1(t) &= \int_\Omega \left(\frac{\beta_1 uw}{\alpha_1+w+su} - \frac{\rho_1 uw}{\alpha_1+w+su} - \delta_1 u\right)\mathrm{d}x \\ &\quad + \int_\Omega (v-\tilde{v})\left(\frac{\beta_2 w}{\alpha_2+w} - \frac{\rho_2 w}{\alpha_2+w} - \delta_2\right)\mathrm{d}x \\ &\quad + \int_\Omega (w-\tilde{w})\left[r\left(1-\frac{w}{K}\right) - \frac{\mu_1 u}{\alpha_1+w+su} - \frac{\mu_2 v}{\alpha_2+w}\right],\end{aligned}$$

and

$$\tilde{J}_2(t) = -\int_\Omega \frac{\tilde{v}|\partial_x v|^2}{v^2}\mathrm{d}x + \int_\Omega \frac{\xi\tilde{v}\partial_x w\cdot\partial_x v}{v(1+\delta w)^2}\mathrm{d}x - \int_\Omega \frac{d\tilde{w}|\partial_x w|^2}{w^2}\mathrm{d}x.$$

Note that

$$\delta_2=\frac{\beta_2\tilde{w}}{\alpha_2+\tilde{w}}+\frac{\rho_2\tilde{w}}{\alpha_2+\tilde{w}},\ r=\frac{r\tilde{w}}{K}+\frac{\mu_2 v}{\alpha_2+\tilde{w}},$$

we get

$$\begin{aligned}
\tilde{J}_1(t)&=\int_\Omega\left(\frac{\beta_1 uw}{\alpha_1+w+su}-\frac{\rho_1 uw}{\alpha_1+w+su}-\delta_1 u\right)\mathrm{d}x\\
&\quad+\int_\Omega(v-\tilde{v})\left(\frac{\beta_2 w}{\alpha_2+w}-\frac{\rho_2 w}{\alpha_2+w}-\delta_2\right)\mathrm{d}x\\
&\quad+\int_\Omega(w-\tilde{w})\left[r\left(1-\frac{w}{K}\right)-\frac{\mu_1 u}{\alpha_1+w+su}-\frac{\mu_2 v}{\alpha_2+w}\right]\\
&=\int_\Omega\left(\frac{\beta_1 uw}{\alpha_1+w+su}-\frac{\rho_1 uw}{\alpha_1+w+su}-\delta_1 u\right)\mathrm{d}x\\
&\quad+\int_\Omega\frac{\alpha_2(\beta_2-\rho_2)(v-\tilde{v})(w-\tilde{w})}{(\alpha_2+w)(\alpha_2+\tilde{w})}\mathrm{d}x\\
&\quad+\int_\Omega\frac{\mu_2\tilde{v}(w-\tilde{w})^2}{(\alpha_2+w)(\alpha_2+\tilde{w})}\mathrm{d}x-\int_\Omega\frac{\mu_2(\alpha_2+\tilde{w})(v-\tilde{v})(w-\tilde{w})}{(\alpha_2+w)(\alpha_2+\tilde{w})}\mathrm{d}x\\
&\quad-\frac{r}{K}\int_\Omega(w-\tilde{w})^2\,\mathrm{d}x-\int_\Omega\frac{\mu_1 u(w-\tilde{w})}{\alpha_1+w+su}\mathrm{d}x\\
&=-\int_\Omega\frac{\rho_1 uw}{\alpha_1+w+su}\mathrm{d}x+\int_\Omega\frac{\left[\alpha_2(\beta_2-\rho_2)-\mu_2(\alpha_2+\tilde{w})\right](v-\tilde{v})(w-\tilde{w})}{(\alpha_2+w)(\alpha_2+\tilde{w})}\mathrm{d}x\\
&\quad+\int_\Omega\left(\frac{\mu_2\tilde{v}}{(\alpha_2+w)(\alpha_2+\tilde{w})}-\frac{r}{K}\right)(w-\tilde{w})^2\,\mathrm{d}x\\
&\quad+\int_\Omega\frac{(\beta_1-\mu_1)uw}{\alpha_1+w+su}\mathrm{d}x+\int_\Omega\left(\frac{\mu_1\tilde{w}}{\alpha_1+w+su}-\delta_1\right)u\mathrm{d}x\\
&\le\int_\Omega\frac{\left[\alpha_2(\beta_2-\rho_2)-\mu_2(\alpha_2+\tilde{w})\right](v-\tilde{v})(w-\tilde{w})}{(\alpha_2+w)(\alpha_2+\tilde{w})}\mathrm{d}x\\
&\quad+\int_\Omega\left(\frac{\mu_2\tilde{v}}{\alpha_2^2}-\frac{r}{K}\right)(w-\tilde{w})^2\,\mathrm{d}x+\int_\Omega\frac{(\beta_1-\mu_1)uw}{\alpha_1+w+su}\mathrm{d}x\\
&\quad+\int_\Omega\left(\frac{\mu_1\tilde{w}}{\alpha_1}-\delta_1\right)u\mathrm{d}x\\
&\le 0,
\end{aligned}$$

if (4) is satisfied. For $\tilde{J}_2(t)$, we have

$$\begin{aligned}
\tilde{J}_2(t)&=-\int_\Omega\frac{\tilde{v}|\partial_x v|^2}{v^2}\mathrm{d}x+\int_\Omega\frac{\xi\tilde{v}\partial_x w\cdot\partial_x v}{v(1+\delta w)^2}\mathrm{d}x-\int_\Omega\frac{d\tilde{w}|\partial_x w|^2}{w^2}\mathrm{d}x\\
&\le-\int_\Omega\frac{\tilde{v}|\partial_x v|^2}{v^2}\mathrm{d}x+\int_\Omega\frac{\xi\tilde{v}|\partial_x w|\cdot|\partial_x v|}{v}\mathrm{d}x-\int_\Omega\frac{d\tilde{w}|\partial_x w|^2}{w^2}\mathrm{d}x\\
&=-\int_\Omega X_2 Q_2 X_2^{\mathrm{T}}\mathrm{d}x,
\end{aligned}$$

where denote by $X_2(x,t)=\left(|\partial_x v(x,t)|,|\partial_x w(x,t)|\right)$ in $\Omega\times(0,\infty)$ and

$$Q_2 = \begin{pmatrix} \dfrac{\tilde{v}}{v^2} & -\dfrac{\xi \tilde{v}}{2v} \\ -\dfrac{\xi \tilde{v}}{2v} & \dfrac{d\tilde{w}}{w^2} \end{pmatrix}.$$

It is clear that $\frac{\tilde{v}}{v^2} > 0$ and $|Q_2| = \frac{\tilde{v}}{v^2}\left(\frac{d\tilde{w}}{w^2} - \frac{\xi^2 \tilde{v}}{4}\right) \geq 0$ since $0 < \xi^2 \leq \frac{4d\tilde{w}}{C^2 \tilde{v}}$. Consequently, $\dot{V}_3(t) = \tilde{J}_1(t) + \tilde{J}_2(t) \leq 0$ and thus $E_3 = (0, \tilde{v}, \tilde{w})$ is globally asymptotically stable.

4) Introducing the following time evolution Lyapunov function

$$\begin{aligned} V_4(t) = &\int_\Omega \int_{u^*}^{u} \frac{u(\cdot,t) - u^*}{u(\cdot,t)} \mathrm{d}u\mathrm{d}x + \int_\Omega \int_{v^*}^{v} \frac{v(\cdot,t) - v^*}{v(\cdot,t)} \mathrm{d}v\mathrm{d}x \\ &+ \int_\Omega \int_{w^*}^{w} \frac{w(\cdot,t) - w^*}{w(\cdot,t)} \mathrm{d}w\mathrm{d}x. \end{aligned} \tag{11}$$

Direct computations illustrate that

$$\begin{aligned} \dot{V}_4(t) &= \int_\Omega \left(1 - \frac{u^*}{u}\right) \partial_t u \mathrm{d}x + \int_\Omega \left(1 - \frac{v^*}{v}\right) \partial_t v \mathrm{d}x + \int_\Omega \left(1 - \frac{w^*}{w}\right) \partial_t w \mathrm{d}x \\ &= \int_\Omega (u - u^*) \left(\frac{\beta_1 w}{\alpha_1 + w + su} - \frac{\rho_1 w}{\alpha_1 + w + su} - \delta_1\right) \mathrm{d}x \\ &\quad + \int_\Omega (v - v^*) \left(\frac{\beta_2 w}{\alpha_2 + w} - \frac{\rho_2 w}{\alpha_2 + w} - \delta_2\right) \mathrm{d}x \\ &\quad + \int_\Omega (w - w^*) \left[r\left(1 - \frac{w}{K}\right) - \frac{\mu_1 u}{\alpha_1 + w + su} - \frac{\mu_2 v}{\alpha_2 + w}\right] \\ &\quad - \int_\Omega \frac{u^* |\partial_x u|^2}{u^2} \mathrm{d}x - \int_\Omega \frac{v^* |\partial_x v|^2}{v^2} \mathrm{d}x - \int_\Omega \frac{dw^* |\partial_x w|^2}{w^2} \mathrm{d}x \\ &\quad + \int_\Omega \frac{\chi u^* \partial_x w \cdot \partial_x u}{u(1 + \delta w)^2} \mathrm{d}x + \int_\Omega \frac{\xi v^* \partial_x w \cdot \partial_x v}{v(1 + \delta w)^2} \mathrm{d}x \\ &= J_1^*(t) + J_2^*(t), \end{aligned}$$

where

$$\begin{aligned} J_1^*(t) = &\int_\Omega (u - u^*) \left(\frac{\beta_1 w}{\alpha_1 + w + su} - \frac{\rho_1 w}{\alpha_1 + w + su} - \delta_1\right) \mathrm{d}x \\ &+ \int_\Omega (v - v^*) \left(\frac{\beta_2 w}{\alpha_2 + w} - \frac{\rho_2 w}{\alpha_2 + w} - \delta_2\right) \mathrm{d}x \\ &+ \int_\Omega (w - w^*) \left[r\left(1 - \frac{w}{K}\right) - \frac{\mu_1 u}{\alpha_1 + w + su} - \frac{\mu_2 v}{\alpha_2 + w}\right], \end{aligned}$$

and

$$\begin{aligned} J_2^*(t) = &-\int_\Omega \frac{u^* |\partial_x u|^2}{u^2} \mathrm{d}x - \int_\Omega \frac{v^* |\partial_x v|^2}{v^2} \mathrm{d}x - \int_\Omega \frac{dw^* |\partial_x w|^2}{w^2} \mathrm{d}x \\ &+ \int_\Omega \frac{\chi u^* \partial_x w \cdot \partial_x u}{u(1 + \delta w)^2} \mathrm{d}x + \int_\Omega \frac{\xi v^* \partial_x w \cdot \partial_x v}{v(1 + \delta w)^2} \mathrm{d}x. \end{aligned}$$

By employing these facts

$$\delta_1=\frac{\beta_1 w^*}{\alpha_1+w^*+su^*}+\frac{\rho_1 w^*}{\alpha_1+w^*+su^*},\ \delta_2=\frac{\beta_2 w^*}{\alpha_2+w^*}+\frac{\rho_2 w^*}{\alpha_2+w^*},$$

$$r=\frac{w^*}{K}+\frac{\mu_1 u^*}{\alpha_1+w^*+su^*}+\frac{\mu_2 v^*}{\alpha_2+w^*},$$

we can obtain

$$\begin{aligned}
J_1^*(t)=&\int_\Omega \frac{(\beta_1-\rho_1)(\alpha_1+su^*)(u-u^*)(w-w^*)}{(\alpha_1+w+su)(\alpha_1+w^*+su^*)}\mathrm{d}x\\
&-\int_\Omega \frac{sw^*(\beta_1-\rho_1)(u-u^*)^2}{(\alpha_1+w+su)(\alpha_1+w^*+su^*)}\mathrm{d}x\\
&+\int_\Omega \frac{\alpha_2(\beta_2-\rho_2)(v-v^*)(w-w^*)}{(\alpha_2+w)(\alpha_2+w^*)}\mathrm{d}x\\
&-\int_\Omega \frac{\mu_2(\alpha_2+w^*)(v-v^*)(w-w^*)}{(\alpha_2+w)(\alpha_2+w^*)}\mathrm{d}x\\
&+\int_\Omega \frac{\mu_2 v^*(w-w^*)^2}{(\alpha_2+w)(\alpha_2+w^*)}\mathrm{d}x-\frac{r}{K}\int_\Omega (w-w^*)^2\,\mathrm{d}x\\
&+\int_\Omega \frac{\mu_1 u^*(w-w^*)^2}{(\alpha_1+w+su)(\alpha_1+w^*+su^*)}\mathrm{d}x\\
&-\int_\Omega \frac{\mu_1(\alpha_1+w^*)(u-u^*)(w-w^*)}{(\alpha_1+w+su)(\alpha_1+w^*+su^*)}\mathrm{d}x\\
\le&\int_\Omega \frac{\left[(\beta_1-\rho_1)(\alpha_1+su^*)-\mu_1(\alpha_1+w^*)\right](u-u^*)(w-w^*)}{(\alpha_1+w+su)(\alpha_1+w^*+su^*)}\mathrm{d}x\\
&+\int_\Omega \frac{\left[\alpha_2(\beta_2-\rho_2)-\mu_2(\alpha_2+w^*)\right](v-v^*)(w-w^*)}{(\alpha_2+w)(\alpha_2+w^*)}\mathrm{d}x\\
&+\int_\Omega \left(\frac{\mu_1 u^*}{\alpha_1^2}+\frac{\mu_2 v^*}{\alpha_2^2}-\frac{r}{K}\right)(w-w^*)^2\,\mathrm{d}x\\
\le&\,0,
\end{aligned}$$

here we use the Condition (5). Moreover

$$\begin{aligned}
J_2^*(t)=&-\int_\Omega \frac{u^*|\partial_x u|^2}{u^2}\mathrm{d}x-\int_\Omega \frac{v^*|\partial_x v|^2}{v^2}\mathrm{d}x-\int_\Omega \frac{dw^*|\partial_x w|^2}{w^2}\mathrm{d}x\\
&+\int_\Omega \frac{\chi u^*\partial_x w\cdot\partial_x u}{u(1+\delta w)^2}\mathrm{d}x+\int_\Omega \frac{\xi v^*\partial_x w\cdot\partial_x v}{v(1+\delta w)^2}\mathrm{d}x.\\
\le&-\int_\Omega \frac{u^*|\partial_x u|^2}{u^2}\mathrm{d}x-\int_\Omega \frac{v^*|\partial_x v|^2}{v^2}\mathrm{d}x-\int_\Omega \frac{dw^*|\partial_x w|^2}{w^2}\mathrm{d}x\\
&+\int_\Omega \frac{\chi u^*|\partial_x w|\cdot|\partial_x u|}{u}\mathrm{d}x+\int_\Omega \frac{\xi v^*|\partial_x w|\cdot|\partial_x v|}{v}\mathrm{d}x\\
=&-\int_\Omega X_3 Q_3 X_3^{\mathrm{T}}\mathrm{d}x,
\end{aligned}$$

where the vector function $X_3(x,t)$ is given by $X_3(x,t)=\left(\left|\partial_x u(x,t)\right|,\left|\partial_x v(x,t)\right|,\left|\partial_x w(x,t)\right|\right)$ in $\Omega\times(0,\infty)$ and

$$Q_3=\begin{pmatrix} \frac{u^*}{u^2} & 0 & -\frac{\chi u^*}{2u} \\ 0 & \frac{v^*}{v^2} & -\frac{\xi v^*}{2v} \\ -\frac{\chi u^*}{2u} & -\frac{\xi v^*}{2v} & \frac{dw^*}{w^2} \end{pmatrix}.$$

We can obtain

$$\left|\frac{u^*}{u^2}\right|>0,\quad \begin{vmatrix} \frac{u^*}{u^2} & 0 \\ 0 & \frac{v^*}{v^2} \end{vmatrix}=\frac{u^*v^*}{u^2v^2}>0.$$

as well as

$$\begin{aligned}|Q_3|&=\frac{u^*}{u^2}\begin{vmatrix} \frac{v^*}{v^2} & -\frac{\xi v^*}{2v} \\ -\frac{\xi v^*}{2v} & \frac{dw^*}{w^2} \end{vmatrix}-\frac{\chi u^*}{2u}\begin{vmatrix} 0 & \frac{v^*}{v^2} \\ -\frac{\chi u^*}{2u} & -\frac{\xi v^*}{2v} \end{vmatrix}\\ &=\frac{u^*v^*}{u^2v^2}\left(\frac{dw^*}{w^2}-\frac{\xi^2 v^*}{4}-\frac{\chi^2 u^*}{4}\right)\\ &\geq 0\end{aligned}$$

if

$$0<\chi^2+\xi^2\leq\frac{4dw^*}{\max\left\{u^*,v^*\right\}C^2}.$$

Thence A_3 is a nonnegative definite matrix, which gives $J_2^*(t)=-\int_\Omega X_3Q_3X_3^{\mathrm{T}}\mathrm{d}x\leq 0$. We conclude that $E_*=\left(u^*,v^*,w^*\right)$ is globally asymptotically stable. These end the proof.

5. Conclusions

This present paper deals with the global asymptotic stability of a three-species predator-prey model with prey-taxis. This system possesses a semi-trivial equilibrium $E_1=(0,0,K)$, two boundary equilibria $E_2=(\hat{u},0,\hat{w})$ and $E_3=(0,\tilde{v},\tilde{w})$, and a unique positive equilibrium $E_*=\left(u^*,v^*,w^*\right)$. By constructing some suitable Lyapunov functions, we establish their global asymptotic stability, respectively. It is concluded that the prey-taxis coefficients χ,ξ can not influence the global asymptotic stability of the semi-trivial equilibrium $E_1=(0,0,K)$. Also, the global asymptotic stability of two boundary equilibria $E_2=(\hat{u},0,\hat{w})$ and $E_3=(0,\tilde{v},\tilde{w})$ rely on the single prey-taxis coefficient χ and ξ, respectively. However, the global asymptotic stability of the unique positive equilibrium $E_*=\left(u^*,v^*,w^*\right)$ is determined by prey-taxis coefficients χ and ξ. These phenomena suggest that the prey-taxis has an influence on the global asymptotic stability of the equilibria of the System (1). Consequently, we will continuously explore the complicated dynamics of the System (1) with prey-taxis effect in the future.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 12002297) and the China Postdoctoral Science Foundation (No. 2021M701118).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Chen, M.J., Cao, H.H. and Fu, S.M. (2021) Stationary Patterns of a Predator-Prey Model with Prey-Stage Structure and Prey-Taxis. *International Journal of Bifurcation and Chaos*, **31**, Article ID: 2150038. https://doi.org/10.1142/S0218127421500383

[2] Tu, X.Y., Mu, C.L. and Qiu, S.Y. (2022) Global Asymptotic Stability in a Parabolic-Elliptic Chemotaxis System with Competitive Kinetics and Loop. *Applicable Analysis*, **101**, 1532-1551. https://doi.org/10.1080/00036811.2020.1783536

[3] Bell, J. and Haskell, E.C. (2021) Attraction-Repulsion Taxis Mechanisms in a Predator-Prey Model. *Partial Differential Equations and Applications*, Article No. 234. https://doi.org/10.1007/s42985-021-00080-0

[4] Ahn, I. and Yoon, C. (2020) Global Well-Posedness and Stability Analysis of Prey-Predator Model with Indirect Prey-Taxis. *Journal of Differential Equations*, **268**, 4222-4255. https://doi.org/10.1016/j.jde.2019.10.019

[5] Qiu, H.H., Guo, S.J. and Li, S.Z. (2020) Stability and Bifurcation in a Bredator-Prey System with Prey-Taxis. *International Journal of Bifurcation and Chaos*, **30**, Article ID: 2050022. https://doi.org/10.1142/S0218127420500224

[6] Haskell, E.C. and Bell, J. (2020) Pattern Formation in a Predator-Mediated Coexistence Model with Prey-Taxis. *American Institute of Mathematical Sciences*, **25**, 2895-2921. https://doi.org/10.3934/dcdsb.2020045

[7] Miao, L.Y., Yang, H. and Fu, S.M. (2021) Global Boundedness in a Two-Species Predator-Prey Chemotaxis Model. *Applied Mathematics Letters*, **111**, Article ID: 106639. https://doi.org/10.1016/j.aml.2020.106639

[8] Choi, W. and Ahn, I. (2019) Effect of Prey-Taxis on Predator's Invasion in a Spatially Heterogeneous Environment. *Applied Mathematics Letters*, **98**, 256-262. https://doi.org/10.1016/j.aml.2019.06.021

[9] Xu, X., Wang, Y.B. and Wang, Y.W. (2019) Local Bifurcation of a Ronsenzwing-MacArthur Predator Prey Model with Two Prey-Taxis. *Mathematical Biosciences and Engineering*, **16**, 1786-1797. https://doi.org/10.3934/mbe.2019086

[10] Xing, J., Zheng, P. and Pan, X. (2021) A Quasilinear Predator-Prey Model with Indirect Prey-Taxis. *Qualitative Theory of Dynamical Systems*, **20**, Article No. 70. https://doi.org/10.1007/s12346-021-00508-3

[11] Wang, Q., Song, Y. and Shao, L. (2017) Nonconstant Positive Steady States and Pattern Formation of 1D Prey-Taxis Systems. *Journal of Nonlinear Science*, **27**, 71-97. https://doi.org/10.1007/s00332-016-9326-5

[12] Winkler, M. (2010) Absence of Collapse in a Parabolic Chemotaxis System with Signal-Dependent Sensitivity. *Mathematische Nachrichten*, **283**, 1664-1673. https://doi.org/10.1002/mana.200810838

[13] Amann, H. (1990) Dynamic Theory of Quasilinear Parabolic Equations II. *Differential Integral Equations*, **3**, 13-75.

Applied Mathematics, 2022, 13, 672-682
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Goodness of Fit Test Based on BLUS Residuals for Error Distribution of Regression Model

Jianxin Zhao[1], Xinmin Li[2]

[1]Navy Submarine Academy, Qingdao, China
[2]School of Mathematics and Statistics, Qingdao University, Qingdao, China
Email: bit_zjx@163.com

How to cite this paper: Zhao, J.X. and Li, X.M. (2022) Goodness of Fit Test Based on BLUS Residuals for Error Distribution of Regression Model. *Applied Mathematics*, **13**, 672-682.
https://doi.org/10.4236/am.2022.138042

Received: July 14, 2022
Accepted: August 20, 2022
Published: August 23, 2022

Abstract

The error distribution testing plays an important role in linear regression as distribution misspecification seriously affects the validity and efficiency of regression analysis. The least squares (OLS) residuals are often used to construct test statistics; in order to overcome the non-independent and identically residuals, the best linear unbiased scale (BLUS) residuals are applied in this paper, which, unlike OLS residuals, the residuals vector is identically and independently distributed. Based on the BLUS residuals, a new test statistic is constructed by using the sample random distance between sample quantile and quasi sample quantile derived from the null distribution, and the goodness-of-fit test of error distribution in the linear model is studied. The powers of the new tests under certain alternatives are examined. They are more powerful tests for the hypotheses concerned.

Keywords

Goodness of Fit, Cramér-Von Mises Statistic, Kolmogorov-Smirnov Statistic, Anderson-Darling Statistic, Sample Quantiles, Stochastic Sample Quantiles

1. Introduction

Parametric and nonparametric regression models are widely applied to the fields such as biology, chemistry and economics. The general form is often written as

$$Y = m(x) + \varepsilon, \tag{1}$$

where m is the regression function, the error ε satisfies $\mathrm{E}(\varepsilon) = 0$ and $\mathrm{E}(\varepsilon^2) > 0$, x are explanatory variables. The following general assumption is that the errors corresponding to different observations are independent and identically distributed. The most popular model is the linear model, that is, $m(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{\beta}$, $\boldsymbol{\beta}$ assumed as unknown parameters, and the error terms of the model are distributed according to the normal distribution. Under the assumption, the linear model has been attracting practical and theoretical workers because of its simplicity and validity.

Obviously, it is important to test whether the error is distributed as the assumed distribution before using the linear model to analyze the data under a certain assumption of the error distribution. On the other hand, even in general nonparametric models, additional knowledge of the distribution of errors can also improve the effectiveness of statistical analysis. For example, under the assumption of normal error, accurate or optimal tests can be obtained in many cases. A typical example is the goodness-of-fit test of regression function (See ref. [1] [2] [3]).

From the above analysis and the existing literature, it is easy to see that there are two kinds of important goodness of fit tests to the model. One is the goodness-of-fit test of the error distribution; the other is about the goodness of fit test of the regression function. That is to test the hypothesis:

$$H : \varepsilon \sim N\left(0, \sigma^2\right) \tag{2}$$

and

$$H : m(x) \in \mathcal{M}, \tag{3}$$

where $\mathcal{M}$ is a class of regression functions with certain properties.

There are many kinds of literatures about the two kinds of tests mentioned above, especially about the second kind of tests. For example, Of the first kind of test are ref. [4] [5] [6], etc., the second kind of test are ref. [7]-[17], etc.

For Linear regression models, there are usually two kinds of residual construction tests. One is based on the ordinary least square (OLS) residuals, under the normal assumption, the residual vector of OLS has a singular normal distribution, and the components of a vector are no longer independently identically distributed. The other is best linear unbiased scale residuals (BLUS) (See ref. [18]). It is a kind of residuum given by Theil in consideration of such a fact where the residuals are neither independent nor identical, even if the error distribution is independent and identical. Using OLS residuals directly to test the sequence independence or the same variance is impossible (See ref. [19]). However, the BLUS residual vector, under normal assumptions, is different from the OLS residual. There is a non-singular normal distribution, and the components of the vector are identically and independently distributed. It is a fact that most of the existing literature uses the OLS residuals to construct the test. But as you can see from the above analysis, using residuals as a new sample, it is not natural to construct a test with the existing test, such as the Shapiro-Wilk test, because the test of the original structure is based on independent samples from the same distribution, at the same time, it was proved that the skewness and Kurtosis of OLS residuals cannot exceed the skewness and Kurtosis of the error term (See ref. [20]). So, if the distribution of error terms is not normal, the distribution of OLS residuals is always close to the normal form, not the probability distribution of error terms. This shows that when the null hypothesis is not true, any normal test using the OLS residuals directly seems to have a tendency not to reject the null hypothesis. In view of the above analysis, this paper uses BLUS residuals to construct test statistics.

This paper is organized as follows. Section 2 constructs the new tests. Power comparisons are given in Section 3. Simulated powers for the new statistics are tabulated in this section. Section 4 gives some comments.

2. Test Statistics Based on BLUS Residuals

Let's have a linear model as follows:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \mathrm{E}\boldsymbol{\varepsilon} = \boldsymbol{0}, \mathrm{E}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\mathrm{T}} = \sigma^2 \boldsymbol{I}, \tag{4}$$

where $\boldsymbol{y}$ is a $n \times 1$ random vector, $\boldsymbol{X}$ is a non-random $n \times k$ matrix with a known rank of *k*, $\boldsymbol{\beta}$ is a $k \times 1$ unknown parametric vector. For the above model, the Normality test of the error term distribution is important. Many scholars have studied this problem. The early literature includes ref. [20] [21] and so on. For the past twenty years, the two papers (See ref. [22] [23]) are based on Shapiro & Wilk test statistics (See ref. [24]) to construct tests using OLS residuals. Furthermore, ref. [25] constructs test statistics by the normalized residuals based on Shapiro & Wilk test statistics. While ref. [26] using Bootstrap method and based on the process of empirical residuals, constructs a new test statistic by using KS and AD statistics. Recently, ref. [27] proposed several tests based on partial sums of residuals where the test statistics are based on sums of a subset of the (ordered and standardized) residuals (Also see ref. [28] [29]).

For a linear model (4), the least square estimate of the Regression Coefficient is $\hat{\boldsymbol{\beta}} = \left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{y}$, using $\boldsymbol{e}$ to represent the OLS residuals associated with the error vector $\boldsymbol{\varepsilon}$, we have

$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}} = \left(\boldsymbol{I} - \boldsymbol{X}\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\mathrm{T}}\right)\boldsymbol{y} = \boldsymbol{M}\boldsymbol{y} = \boldsymbol{M}\boldsymbol{\varepsilon}. \tag{5}$$

And from that, $\mathrm{E}\boldsymbol{\varepsilon} = \boldsymbol{0}, \mathrm{E}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\mathrm{T}} = \sigma^2\boldsymbol{M}$.

According to Theil (See ref. [18]), the BLUS residuals are obtained by the following steps:

First, select the smallest K elements in the main diagonal elements of the matrix $\boldsymbol{M}$ and rearrange the observed value Y according to the position of the row in which the K elements are located. Might as well be, place them in the first K position (See ref. [30]). The original model is then divided into blocks:

$$\begin{pmatrix} \boldsymbol{y}_0 \\ \boldsymbol{y}_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_0 \\ \boldsymbol{X}_1 \end{pmatrix}\boldsymbol{\beta} + \begin{pmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\varepsilon}_1 \end{pmatrix} \tag{6}$$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{X}_0\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}\right)^{-1}\boldsymbol{X}_0^{\mathrm{T}} & -\boldsymbol{X}_0\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}\right)^{-1}\boldsymbol{X}_1^{\mathrm{T}} \\ -\boldsymbol{X}_1\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}\right)^{-1}\boldsymbol{X}_0^{\mathrm{T}} & \boldsymbol{I} - \boldsymbol{X}_1\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}\right)^{-1}\boldsymbol{X}_1^{\mathrm{T}} \end{bmatrix}, \tag{7}$$

where the $\boldsymbol{I}$ in the upper left-hand corner in (7) is the $k\times k$ unit matrix, and the $\boldsymbol{I}$ in the lower right-hand corner is the $(n-k)\times(n-k)$ unit matrix.

Secondly, compute the eigenvalues, denoted by $d_1^2,\cdots,d_k^2$, of the matrix $\boldsymbol{X}_0\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}\right)^{-1}\boldsymbol{X}_0^{\mathrm{T}}$, and the corresponding eigenvectors by $q_1,\cdots,q_k$.

Finally, calculate the BLUS residuals $\hat{\boldsymbol{\varepsilon}}=\boldsymbol{e}_1-\boldsymbol{X}_1\boldsymbol{X}_0^{-1}\left[\sum_{i=1}^{k}\frac{d_i}{1+d_i}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathrm{T}}\right]\boldsymbol{e}_0$, here, $\boldsymbol{e}_0,\boldsymbol{e}_1$ is the block residuals vector corresponding to the error term in (6). At this point, under the original assumption (2), the BLUS residual vector is

$$\hat{\boldsymbol{\varepsilon}}\sim N\left(0,\sigma^2\boldsymbol{I}_{n-k}\right). \tag{8}$$

The tests in this section and the next are constructed from this new sample. Denote $\hat{\boldsymbol{\varepsilon}}_{(1)},\cdots,\hat{\boldsymbol{\varepsilon}}_{(m)}$ be an order statistic of BLUS residuals, $m=n-k$, $F_0(x)$ be the normal distribution.

BLUS residuals were mainly applied to multivariate models (See ref. [31]), the cusum and the cusum-of-squares tests (See ref. [19]) which have higher power than those based on the more popular recursive residuals for structural break, and the Dickey-Fuller unit root test which is based on BLUS residuals (See ref. [32]), etc.

The test statistics constructed in this paper and the competition test statistics available in the literature are described below.

The competition test statistics based on the empirical distribution function are KS statistics

$$\hat{D}_m=\sqrt{m}\max_{1\le i\le m}\left\{\left|\frac{i}{m}-F_0(\tilde{\varepsilon})\right|,\left|\frac{i-1}{m}-F_0(\tilde{\varepsilon})\right|\right\} \tag{9}$$

and AD statistics

$$\hat{A}_m^2=-\frac{1}{m}\sum_{i=1}^{m}\left[(2i-1)\log F_0(\tilde{\varepsilon})+(2m+1-2i)\log\left(1-F_0(\tilde{\varepsilon})\right)\right]-m, \tag{10}$$

here $\tilde{\varepsilon}=\hat{\varepsilon}_{(i)}\big/\sqrt{S_m^2}$, since the mean of the original hypothesis is 0, the parameter estimation of the variance of the error is $S_m^2=1/m\sum_{i=1}^{m}\hat{\varepsilon}_i^2$.

The test statistic based on sample order statistics or sample quantiles is Shapiro & Wilk test statistic (See ref. [33]):

$$W=\frac{\left(\sum_{i=1}^{m}a_i\hat{\varepsilon}_{(i)}\right)^2}{m\cdot S_m^2} \tag{11}$$

where $\boldsymbol{a}=(a_1,\cdots,a_m)^{\mathrm{T}}=\frac{\boldsymbol{V}^{-1}\mathrm{E}\tilde{\boldsymbol{W}}}{\sqrt{\mathrm{E}\tilde{\boldsymbol{W}}^{\mathrm{T}}\boldsymbol{V}^{-1}\boldsymbol{V}^{-1}\mathrm{E}\tilde{\boldsymbol{W}}}}$, $V=(v_{ij})=Var\left(\tilde{\boldsymbol{W}}-\mathrm{E}\tilde{\boldsymbol{W}}\right)$, $\tilde{\boldsymbol{W}}=\left(\tilde{W}_{(1)},\cdots,\tilde{W}_{(m)}\right)$ be the sample order statistics from $F_0(\cdot)$.

The test statistic constructed based on De Wet and Venter' idea (See ref. [34]) is

$$\hat{T}_2 = \sum_{i=1}^{m}\left(\frac{\hat{\varepsilon}_{(i)}}{\hat{\theta}} - F_0^{-1}\left(\frac{i}{m+1}\right)\right)^2, \tag{12}$$

where $F_0^{-1}(u) = \inf\{x : F_0(x) \geq u\}$, $\hat{\theta} = \sqrt{S_m^2}$ be the estimation of variance of standard error.

Del Barrio, Couesta-Albertos, Matrán, Rodríguez-Rodríguez (See ref. [35]) proposed that the statistic constructed based on the distance of L_2-Wasserstein is

$$\widehat{BCMR} = 1 - \frac{\left(\sum_{i=1}^{m}\hat{\varepsilon}_{(i)}\int_{(i-1)/m}^{i/m}\Phi^{-1}(t)\,\mathrm{d}t\right)^2}{S_m^2}, \tag{13}$$

Let $X_{(1)},\cdots,X_{(m)}$ be the order statistics of $X_1,\cdots,X_m$ and $U_{(1)},\cdots,U_{(m)}$ are the order statistics from $U(0,1)$. If $X_1,\cdots,X_m$ is an iid sample from the continuous cumulative distribution function $F(x)$, the following equation holds (See ref. [36]).

$$\left(X_{(1)},\cdots,X_{(m)}\right) \stackrel{d}{=} \left(F^{-1}\left(U_{(1)}\right),\cdots,F^{-1}\left(U_{(m)}\right)\right), \tag{14}$$

where $\stackrel{d}{=}$ stands for equality in distribution.

Using this conclusion, Zhao (See ref. [37] [38]) thinks that if $X_1,\cdots,X_m$ doesn't come from the null distribution $F(x)$, (14) doesn't hold and then there are differences between $\left(X_{(1)},\cdots,X_{(m)}\right)$ and $\left(F^{-1}\left(U_{(1)}\right),\cdots,F^{-1}\left(U_{(m)}\right)\right)$. The larger the differences are, the greater the evidence against the null hypothesis is. Using this idea, Zhao (See ref. [37]) constructed criteria to describe the differences based on random distance.

$$ZR = \sum_{i=1}^{m}\left(X_{(i)} - F^{-1}\left(U_{(i)}\right)\right)^2 \Big/ \sigma_i^2 \tag{15}$$

In this paper, the null hypothesis is that the distribution function is a normal distribution function with only scale parameters. So (14) is reduced to

$$X_{(i)} \stackrel{d}{=} \sigma Z_{(i)}, i = 1,\cdots,m. \tag{16}$$

And then, with respect to the parameter σ^2, take the smallest value of (15), it can be obtained that:

$$ZR = \sum_{i=1}^{m} Z_{(i)}^2 + \frac{\left[\sum_{i=1}^{m} X_{(i)}Z_{(i)}\right]^2}{\sum_{i=1}^{m} X_{(i)}^2} \tag{17}$$

where $Z_{(1)},\cdots,Z_{(m)}$ is the order statistics for a sample of m size from the normal distribution. Note that (8) and above analysis, using $\hat{\varepsilon}_{(i)}$ instead of $X_{(i)}$, we obtain the following test statistics in this paper.

$$\widehat{ZR} = \sum_{i=1}^{m} Z_{(i)}^2 + \frac{\left[\sum_{i=1}^{m} \hat{\varepsilon}_{(i)}Z_{(i)}\right]^2}{\sum_{i=1}^{m} \hat{\varepsilon}_{(i)}^2}, \tag{18}$$

where $\widehat{ZR}$ is scale-invariant. For the same reason as in Zhao (See ref. [37]), we can take the q quantile and expectation of $\widehat{ZR}$ as the test statistics. Let $\widehat{ZR}_q$

and $\widehat{ZR}_\mu$ be the qth quantile and the expectation of $\widehat{ZR}$. Here we select only $q = 0.05$, $q = 0.50$, and $q = 0.95$ as quantile statistics. Using BLUS residuals to solve the critical value of each test statistic is the same as Zhao's (See ref. [37]) algorithm, which is omitted here.

3. Power Comparisons

In order to compare the power of the above tests, the following linear models are considered.

$$y_i = 1 + 2x_{1i} - 3x_{2i} + \varepsilon_i, \quad i = 1, 2, \cdots, n. \tag{19}$$

The variable x is independent of ε_i, which comes from a uniform distribution, these values are constant for a given sample size. The error variable ε follows an alternative distribution.

Many continuous alternative distributions are chosen which were used in the power studying by Shapiro *et al.* (See ref. [33]), Gan and Koehler (See ref. [39]) and Eva Krauczi (See ref. [40]).

The first group: alternative distributions are the non-normal distribution, such as the chi-square distribution with a degree of freedom of 3, denoted by $\chi^2(3)$; the exponential distribution with a mean of 1, denoted by *Exp* (1); the gamma distribution functions with shape parameters of 0.8 and 1.5, *Gamma* (0.8), *Gamma* (1.5), respectively; the double exponential distribution *laplace* (0,1); the log-normal distribution *lognormal* (0,1); the Cauchy distribution *Cauchy* (0,1); the student distribution with degree of freedom of 3, *t* (3); the unbounded Johnson's distribution of the random variable sinh (*Z*), denoted as *SU* (0,1), where $Z \sim N(0,1)$.

The second group: in heteroscedasticity case, the he model under consideration at this point is

$$y_i = 1 + 2x_{1i} - 3x_{2i} + \sigma_i \bar{\varepsilon}_i, \quad i = 1, 2, \cdots, n, \tag{20}$$

where σ_i be distributed as uniform distribution $U(0,50)$, $i = 1, 2, \cdots, n$. At the same time, the error variable $\bar{\varepsilon}$ follows *logistic* (0,1), *Cauchy* (0,1), *N* (0,1) and *Laplace* (0,1), respectively.

The third group: the case of outlier with non-zero mean of regression error, such as

$$\varepsilon_1 \sim N(2,1), \varepsilon_i \sim N(0,1), i = 2, \cdots, n,$$

and

$$\varepsilon_i \sim N(2,1), i = 1, \cdots, 5,\ \varepsilon_i \sim N(0,1), i = 6, \cdots, n,$$

and also in high Leverage outlier case, like that

$$\varepsilon_1 \sim N(8,1), \varepsilon_i \sim N(0,1), i = 2, \cdots, n,\ x_{11} = x_{21} = 2.$$

In this section, the selected test level is 5%, the sample size is $n = 20, 50$, and the empirical power of the test is based on 10,000 simulations. The results are placed in **Table 1** and **Table 2**. From these two tables, you can see that:

Table 1. Powers for testing the error distribution is normal, against different non-normal alternative distributions, at significance level 5% and $n = 20, n = 50$.

Alternatives	$\chi^2(3)$	$exp(1)$	$G(0.8)^a$	$G(1.5)^a$	$laplace$	$F_1(x)^b$	$Cauchy$	$t_3(x)$	$F_2(x)^c$
					$n = 20$				
$\widehat{ZR}_{q=0.05}$	0.2875	0.3968	0.4744	0.2811	0.1917	0.6167	0.7496	0.2533	0.3166
$\widehat{ZR}_{q=0.50}$	0.1623	0.2237	0.2881	0.1580	0.1034	0.4444	0.6277	0.1619	0.1994
$\widehat{ZR}_{\mu}$	0.1349	0.1869	0.2421	0.1320	0.0923	0.3912	0.5880	0.1431	0.1746
$\widehat{ZR}_{q=0.95}$	0.0710	0.0858	0.1073	0.0709	0.0707	0.2026	0.4134	0.0915	0.1014
D_n	0.0972	0.1160	0.1335	0.1029	0.0756	0.1872	0.3686	0.0792	0.1047
A_n^2	0.0851	0.0997	0.1195	0.0898	0.0666	0.1892	0.3942	0.0790	0.0926
$BCMR$	0.1233	0.1722	0.2235	0.1209	0.0836	0.3698	0.5553	0.1273	0.1532
T_2	0.1415	0.1959	0.2511	0.1393	0.0990	0.4031	0.6099	0.1507	0.1856
W	0.1201	0.1677	0.2161	0.1173	0.0811	0.3609	0.5438	0.1214	0.1463
					$n = 50$				
$\widehat{ZR}_{q=0.05}$	0.7992	0.9136	0.9473	0.7902	0.4586	0.9754	0.9837	0.5928	0.7087
$\widehat{ZR}_{q=0.50}$	0.6621	0.8316	0.8963	0.6489	0.2760	0.9593	0.9709	0.4426	0.5480
$\widehat{ZR}_{\mu}$	0.6072	0.7903	0.8692	0.5970	0.2407	0.9478	0.9648	0.4058	0.5080
$\widehat{ZR}_{q=0.95}$	0.3692	0.5743	0.6866	0.3605	0.1329	0.8736	0.9166	0.2698	0.3318
D_n	0.2440	0.3877	0.4918	0.2503	0.1390	0.7121	0.8728	0.1892	0.2433
A_n^2	0.2518	0.4270	0.5470	0.2566	0.1044	0.7870	0.8969	0.1835	0.2394
$BCMR$	0.6081	0.7920	0.8697	0.5974	0.2040	0.9473	0.9570	0.3694	0.4656
T_2	0.6078	0.7890	0.8655	0.5957	0.2740	0.9489	0.9707	0.4435	0.5468
W	0.6058	0.7928	0.8730	0.5942	0.1579	0.9453	0.9397	0.3106	0.3938

[a] $G(0.8), G(1.5)$: the Gamma distribution with shape parameter 0.8, 1.5. [b] $F_1(x)$: the LogNormal(0, 1) distribution. [c] $F_3(x)$: $X = \sinh(Z)$, $Z \sim N(0,1)$.

1) When the alternative distribution is a non-normal distribution function, the power of the test $\widehat{ZR}_{q=0.05}$ is significantly higher than that of other tests. When the capacity is 20, the power of $\widehat{ZR}_{\mu}$ and T_2 is similar. However, when the capacity is 50, for an asymmetric alternative distribution, $BCMR$, T_2, W and $\widehat{ZR}_{\mu}$ are not much different, for a symmetric alternative distribution, T_2 is the best, coming next is $\widehat{ZR}_{\mu}$. In short, Among the opponents of the new tests, the order of superiority and inferiority of the tests is T_2, $BCMR$, Shapiro-Wilk test, A_n^2 and D_n.

2) For an alternate distribution of $N(0,1)$ the power of the test is approximately equal to the probability of the first type of error. As can be seen from Table 2, under the different sample sizes, all the tests make full use of the approximately 5% test level.

3) For the alternative distribution of the second group, the heteroscedasticity case, the power of $\widehat{ZR}_{q=0.05}$ test is significantly higher than the other tests, the

Table 2. Powers for testing the error distribution is normal, against Heteroscedasticity, outlier alternative distribution and normal alternative distributions, at significance level 5% and $n=20, n=50$.

Alternatives	$N(0,1)$	$F_1(x)$ [a]	$F_2(x)$ [a]	$F_3(x)$ [a]	$F_4(x)$ [a]	$F_5(x)$ [b]	$F_6(x)$ [b]	$F_7(x)$ [b]
				$n=20$				
$\widehat{ZR}_{q=0.05}$	0.0462	0.7786	0.2011	0.4041	0.2837	0.0461	0.4500	0.0562
$\widehat{ZR}_{q=0.50}$	0.0483	0.6500	0.1004	0.2233	0.1531	0.0493	0.5529	0.0674
$\widehat{ZR}_{\mu}$	0.0485	0.6114	0.0896	0.1909	0.1324	0.0496	0.5561	0.0678
$\widehat{ZR}_{q=0.95}$	0.0484	0.4239	0.0671	0.1037	0.0881	0.0478	0.5561	0.0684
D_n	0.0509	0.3821	0.0772	0.1203	0.0992	0.0482	0.5365	0.0647
A_n^2	0.0504	0.4066	0.0628	0.0967	0.0816	0.0477	0.5921	0.0694
$BCMR$	0.0474	0.5769	0.0809	0.1661	0.1184	0.0485	0.5528	0.0684
T_2	0.0482	0.6308	0.0968	0.2086	0.1465	0.0485	0.5560	0.0672
W	0.0479	0.5628	0.0779	0.1564	0.1122	0.0483	0.5518	0.0685
				$n=50$				
$\widehat{ZR}_{q=0.05}$	0.0534	0.9851	0.4994	0.8355	0.6521	0.1143	0.6871	0.2844
$\widehat{ZR}_{q=0.50}$	0.0497	0.9778	0.2760	0.6614	0.4302	0.1392	0.7832	0.3717
$\widehat{ZR}_{\mu}$	0.0502	0.9731	0.2351	0.6102	0.3818	0.1414	0.7865	0.3761
$\widehat{ZR}_{q=0.95}$	0.0506	0.9344	0.1195	0.3626	0.1951	0.1427	0.7933	0.3857
D_n	0.0520	0.9062	0.1757	0.3782	0.2203	0.1245	0.7096	0.3036
A_n^2	0.0505	0.9242	0.1102	0.3260	0.1632	0.1394	0.7944	0.3726
$BCMR$	0.0500	0.9672	0.1951	0.5548	0.3341	0.1413	0.7868	0.3772
T_2	0.0504	0.9781	0.2777	0.6611	0.4314	0.1410	0.7852	0.3748
W	0.0519	0.9536	0.1493	0.4584	0.2559	0.1427	0.7870	0.3838

[a]The error $\sigma\bar{\varepsilon}$, $F_1(x)$: $\bar{\varepsilon}$ the Cauchy distribution; $F_2(x)$: $\bar{\varepsilon}$ the Normal distribution; $F_3(x)$: $\bar{\varepsilon}$ the Laplace distribution; $F_4(x)$: $\bar{\varepsilon}$ the Logistic distribution $\sigma \sim U(0,50)$. [b] $F_5(x)$: $\varepsilon_1 \sim N(2,1)$, $\varepsilon_i \sim N(0,1)$, $i=2,\cdots,n$; $F_6(x)$: $\varepsilon_i \sim N(2,1)$, $i=1,\cdots,5$, $\varepsilon_i \sim N(0,1)$, $i=6,\cdots,n$; $F_7(x)$: $\varepsilon_1 \sim N(8,1)$, $\varepsilon_i \sim N(0,1)$, $i=2,\cdots,n$, $x_{11}=x_{12}=2$.

other results are similar to those in (1).

4) For the alternative distribution of the third group, where there are outliers. In the new test, $\widehat{ZR}_{q=0.95}$ performs best, but not much different from $\widehat{ZR}_{q=0.50}$, $\widehat{ZR}_{\mu}$. Compared with the comparative test, the difference is not significant, A_n^2 is slightly better, while $\widehat{ZR}_{q=0.05}$ performs worse.

4. Comments

Based on the BLUS residuals, using the difference between the residuals order statistics and the pseudo-random sample order statistics, the quantile-type and the conditional expectation-type test statistics are constructed, which are used in the error distribution Normality test of the linear regression model. The simulation results show that the power of the tests provided in this paper is better than some tests in the literature. Of course, T_2, $BCMR$ and W are also good tests.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Eubank, R.L. and Hart, J.D. (1992) Testing Goodness-of-Fit in Regression via Order Selection Criteria. *Annals of Statistics*, **20**, 1412-1425. https://doi.org/10.1214/aos/1176348775

[2] Milhem, H. (2003) Testing the Goodness-of-Fit in a Gaussian Regression. *Proceedings of the European Young Statisticians Meeting* (*EYSM*), Ovronnaz, 21-26 September 2003, 101-111.

[3] Young, S.G. and Bowman, A.W. (1995) Non-Parametric Analysis of Covariance. *Biometrics*, **51**, 920-931. https://doi.org/10.2307/2532993

[4] Durbin, J. (1973) Distribution Theory for Tests Based on the Sample Distribution Function. Society for Industrial and Applied Mathematics, Philadelphia. https://doi.org/10.1137/1.9781611970586

[5] Durbin, J., Knott, M. and Taylor, C.C. (1975) Components of Cramérvon Mises statistics II. *Journal of the Royal Statistical Society*: *Series B* (*Methodological*), **37**, 216-237. https://doi.org/10.1111/j.2517-6161.1975.tb01537.x

[6] D'Agostino, R.B. and Stephens, M.A. (1986) Goodness-of-Fit Techniques. Marcel Dekker, Inc., New York.

[7] Cox, D., Koh, E., Wahba, G. and Yandell, B.S. (1988) Testing the (Parametric) Null Model Hypothesis in (Semiparametric) Partial and Generalized Spline Models. *Annals of Statistics*, **16**, 113-119. https://doi.org/10.1214/aos/1176350693

[8] Gu, C. (1992) Diagnostics for Nonparametric Regression Models with Additive Terms. *Journal of the American Statistical Association*, **87**, 1051-1058. https://doi.org/10.1080/01621459.1992.10476260

[9] Eubank, R.L. and Hart, J.D. (1993) Commonality of Cusum, von Neumann and Smoothing-Based Goodness-of-Fit Tests. *Biometrika*, **80**, 89-98. https://doi.org/10.1093/biomet/80.1.89

[10] Eubank, R.L. and La Riccia, V.N. (1993) Testing for No Effect in Nonparametric Regression. *Journal of Statistical Planning and Inference*, **36**, 1-14. https://doi.org/10.1016/0378-3758(93)90097-P

[11] Stute, W. (1997) Nonparametric Model Checks for Regression. *Annals of Statistics*, **25**, 613-641. https://doi.org/10.1214/aos/1031833666

[12] Stute, W. and Gonzáilez Manteiga, W. (1996) NN Goodness-of-Fit Tests for Linear Models. *Journal of Statistical Planning and Inference*, **53**, 75-92. https://doi.org/10.1016/0378-3758(95)00144-1

[13] Stute, W., González Manteiga, W. and Presedo Quindimil, M. (1998) Bootstrap Approximations in Model Checks for Regression. *Journal of the American Statistical Association*, **93**, 141-149. https://doi.org/10.1080/01621459.1998.10474096

[14] Stute, W., Thies, S. and Zhu, L.X. (1998) Model Checks for Regression: An Innovation Process Approach. *Annals of Statistics*, **26**, 1916-1934. https://doi.org/10.1214/aos/1024691363

[15] Stute, W. and Zhu, L.X. (2002) Model Checks for Generalized Linear Models. *Scandinavian Journal of Statistics*, **29**, 535-545. https://doi.org/10.1111/1467-9469.00304

[16] Koul, H.L. and Ni, P. (2004) Minimum Distance Regression Model Checking. *Jour-*

nal of Statistical Planning and Inference, **119**, 109-141.
https://doi.org/10.1016/S0378-3758(02)00415-9

[17] Zhu, L., Zhu, R. and Song, S. (2008) Diagnostic Checking for Multivariate Regression Models. *Journal of Multivariate Analysis*, **99**, 1841-1859.
https://doi.org/10.1016/j.jmva.2008.01.022

[18] Theil, H. (1968) A Simplification of the BLUS Procedure for Analyzing Regression Disturbances. *Journal of the American Statistical Association*, **63**, 242-251.
https://doi.org/10.1080/01621459.1968.11009238

[19] Magnus, J.R. and Sinha, A.K. (2005) On Theil's Errors. *Econometrics Journal*, **8**, 39-54. https://doi.org/10.1111/j.1368-423X.2005.00150.x

[20] White, H. and MacDonald, G. (1980) Some Large-Sample tests for Non-Normality in the Linear Regression Model. *Journal of the American Statistical Association*, **75**, 16-28. https://doi.org/10.1080/01621459.1980.10477415

[21] Cliff, H.J. and Ben, B.W. (1974) On the Testing of Regression Disturbances for Normality. *Journal of the American Statistical Association*, **69**, 330-335.
https://doi.org/10.1080/01621459.1974.10482948

[22] Jurečková, J., Picek, J. and Sen, P.K. (2003) Goodness-of-Fit Test with Nuisance Regression and Scale. *Metrika*, **58**, 235-258. https://doi.org/10.1007/s001840300262

[23] Sen, P.K., Jurečková, J. and Picek, J. (2003) Goodness-of-Fit Test of Shapiro-Wilk Type with Nuisance Regression and Scale. *Austrian Journal of Statist*, **32**, 163-177.

[24] Shapiro, S.S. and Wilk, M.B. (1965) An Analysis of Variance Test for Normality. *Biomatrika*, **52**, 591-611. https://doi.org/10.1093/biomet/52.3-4.591

[25] Jurečková, J. and Picek, J. (2007) Shapiro-Wilk Type Test of Normality under Nuisance Regression and Scale. *Computational Statistics & Data Analysis*, **51**, 5184-5191. https://doi.org/10.1016/j.csda.2006.08.026

[26] Neumeyer, N., Dette, H. and Nagel, E.-R. (2006) Bootstrap Tests for the Error Distribution in Linear and Nonparametric Regression Models. *Australian & New Zealand Journal of Statistics*, **48**, 129-156.
https://doi.org/10.1111/j.1467-842X.2006.00431.x

[27] Hattab, M.W. and Christensen, R. (2018) Lack of Fit Tests Based on Sums of Ordered Residuals for Linear Models. *Australian & New Zealand Journal of Statistics*, **60**, 230-257. https://doi.org/10.1111/anzs.12231

[28] Christensen, R. and Sun, S.K. (2010) Alternative Goodness-of-Fit Tests for Linear Models. *Journal of the American Statistical Association*, **105**, 291-301.
https://doi.org/10.1198/jasa.2009.tm08697

[29] Christensen, R. and Lin, Y. (2015) Lack-of-fit Tests Based on Partial Sums of Residuals. *Communications in Statistics - Theory and Methods*, **44**, 2862-2880.
https://doi.org/10.1080/03610926.2013.844256

[30] Ramsey, J.B. (1969) Tests for Specification Errors in Classical Linear Least Squares Regression Analysis. *Journal of the Royal Statistical Society: Series B* (*Methodological*), **31**, 350-371. https://doi.org/10.1111/j.2517-6161.1969.tb00796.x

[31] Chinchilli, V.M. (1987) Blus Residuals in Multivariate Linear Models. In: Bozdogan, H. and Gupta, A.K., Eds., *Multivariate Statistical Modeling and Data Analysis*, Springer, Dordrecht, 61-75. https://doi.org/10.1007/978-94-009-3977-6_4

[32] Vougas, D.V. (2008) Unit Root Testing Based on BLUS Residuals. *Statistics & Probability Letters*, **78**, 1943-1947. https://doi.org/10.1016/j.spl.2008.01.063

[33] Shapiro, S.S., Wilk, M.B. and Chen, H.J. (1968) A Comparative Study of Various Tests for Normality. *Journal of the American Statistical Association*, **63**, 1343-1372.

https://doi.org/10.1080/01621459.1968.10480932

[34] De Wet, T. and Venter, J.H. (1972) Asymptotic Distributions of Certain Test Criteria of Normality. *South African Statistical Journal*, **6**, 135-149.

[35] Del Barrio, E., Couesta-Albertos, J.A., Rodríguez-Rodríguez, J.M. and del Barrio, E. (1999) Tests of Goodness of Fit Based on the L_2-Wasserstein Distance. *The Annals of Statistics*, **27**, 1230-1239. https://doi.org/10.1214/aos/1017938923

[36] Scheffe, H. and Tukey, J.W. (1945) Non-Parametric Estimation. I. Validation of Order Statistics. *The Annals of Mathematical Statistics*, **16**, 187-192. https://doi.org/10.1214/aoms/1177731119

[37] Zhao, J., Xu, X. and Ding, X. (2009) Some New Goodness-of-Fit Tests Based on Stochastic Sample Quantiles. *Communications in Statistics - Simulation and Computation*, **38**, 571-589. https://doi.org/10.1080/03610910802585832

[38] Zhao, J. and Xu, X. (2014) Goodness-of-Fit Tests for Location?Cscale Families Based on Random Distance. *Journal of Statistical Computation and Simulation*, **84**, 739-752. https://doi.org/10.1080/00949655.2012.725246

[39] Gan, F.F. and Koehler, K.J. (1990) Goodness-of-Fit Tests Based on P-P Probability Plots. *Technometrics*, **32**, 289-303. https://doi.org/10.2307/1269106

[40] Krauczi, É. (2007) A Study of the Quantile Correlation Test for Normality. *Test*, **18**, 156-165. https://doi.org/10.1007/s11749-007-0074-6

Scientific Research Publishing

Applied Mathematics, 2022, 13, 683-703
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Improving the Performance of Urban Waste Management Systems in the Context of a Closed-Loop Supply Chain

Massiva Boudane, Jean-Pierre Kenné, Lucas Hof

Mechanical Engineering Department, University of Quebec-ETS, Montreal, Canada
Email: jean-pierre.kenne@etsmtl.ca

How to cite this paper: Boudane, M., Kenné, J.-P. and Hof, L. (2022) Improving the Performance of Urban Waste Management Systems in the Context of a Closed-Loop Supply Chain. *Applied Mathematics*, **13**, 683-703.
https://doi.org/10.4236/am.2022.138043

Received: July 13, 2022
Accepted: August 28, 2022
Published: August 31, 2022

Abstract

The saving of non-renewable energies, as well as the reduction of emissions into the environment, are two crucial objectives of industrial production. The recovery of post-consumer products associated with the use of end-of-life products is part of a context of optimization of these objectives. This recovery maximizes the use of resources from end-of-life products in a circular logic while recording the environmental footprint. This study considers a recycling strategy adapted to the need and urgency to reduce greenhouse gas emissions caused by global warming. The proposed model aims to optimize the profits of the circular manufacturing strategies while minimizing operational costs (collection, sorting, recycling), transport, *GHG* emissions and recycling. In this paper, a compromise between the gains of CM and the costs associated with it was studied. The robustness of the designed model was tested using a case study based on real-world scenarios. A sensitivity analysis was carried out to study the impact of the emission cost on the overall objective, considering the two options currently offered to industries. The obtained results support companies to take the ecological aspect into account and integrate sustainable development into their strategic axes for their logistics supply chains.

Keywords

Waste Management, Closed-Loop Supply Chain, End-of-Life Products, Circular Economy, *GHG*, Sensitivity Analysis

1. Introduction

Today's current global production and consumption patterns are unsustainable. Nearly 90 billion tons of natural resources are extracted each year to meet the needs of society and it is expected to double by 2050. That same year, the world's population is expected to reach 9.7 billion and the demand for finite resources will reach 400% of the Earth's total capacity Goueland Guimbard. [1]. It is therefore, time to think about implementing more sustainable practices for tomorrow by taking advantage of the resources that already exist today. These practices are part of a concept referred to as the Circular Economy (*CE*). Its goal is to produce goods and services while strongly limiting the waste of resources. In Canada, the concept of circular economy has advantages from different points of view. Indeed, the country has prospered for centuries due to the abundance of its resources. Currently, these natural resources constitute 17% of Canada's Gross Domestic Product *GDP* and support 1.75 million jobs. In addition to providing stability and a sense of physical and mental well-being to the population, this job creation promotes the productivity and profitability of companies Bonet *et al.* [2]. In fact, it is about 236 billion dollars that fuel the Canadian economy.

In addition to the social and economic benefits mentioned above, the circular economy is intended to be restorative and regenerative. Indeed, apart from aiming to double job creation and increase gross domestic product, it is committed to reducing *GHG* emissions significantly while reducing pressure on resources and supply costs by 70% Steffen *et al.* [3]. Circular economy practices, including residual materials management (*RMM*), are carried out mainly in cities which are a hotspot for resource cycles since they consume up to 80% of the world's natural resources and produce 50% of the world's waste and 75% of greenhouse gas emissions Amrani *et al.* [4]. In this work, we are interested in the waste that consumers produce in their homes. This household waste can be considered as end-of-life (*EOL*) products containing residual value just waiting to be exploited by Alamerew and Brissaud. [5]. Hence, their efficient management would reduce environmental pollution, create new raw materials, and generate profit while creating jobs that allow people to live with dignity in society. To achieve the most optimal urbane waste management, we will first carry out a review of the existing literature dealing with the subject. Then, we will extract the problem of this research.

This article is organized as follows: In introduction, the literature review and the research problem are presented. Section 1 presents the proposed approach. The optimization model and sensibility analysis are presented in Section 2. Finally, the paper is concluded in Section 3.

The circular economy is meant to be restorative and regenerative in relation to the environmental issues of the industry. Indeed, in addition to the objective of doubling job creation and increasing gross domestic product, it is committed to reducing greenhouse gas emissions by 70% while reducing pressure on resources and supply costs Steffen *et al.* [3]. Circular economy practices, including residual materials management (*RMM*), are carried out mainly in cities which are a hotspot for resource cycles since they consume up to 80% of the world's natural resources and produce 50% of the world's waste and 75% of greenhouse gas emissions Amrani *et al.* [4]. This literature discusses existing work and studies on waste valorization in Section 1.1, ecological waste management in Section 1.2, economic challenges on waste management in Section 1.3, and current urban waste practices discussed in Section 1.4.

1.1. Valorization of Waste and Creation of Raw Materials

The high consumption of natural resources leads to a decrease in the latter. Most of these resources, like petroleum-based plastics, often must be destroyed (e.g., incinerated) when they are deemed unrecoverable. On the other hand, certain ores necessary for high-tech production are becoming increasingly rare, posing problems at other levels. As these are controlled by economically competing countries, replenishment can be a problem in the event of a shortage. Therefore, the idea of reusing material from waste as raw material in manufacturing processes could eventually solve such issues. Safaei *et al.* [6], Cariou *et al.* [7] and McDonald *et al.* [8] argue that recycling paper and cardboard is one of the best secondary materials in the world, constituting a real source of usable raw materials. Their work presents a new framework to support paperboard supply chain managers in designing an optimal closed-loop paperboard supply chain (CLSC) that considers a mix of internal and external flows for the recycling network.

Such a chain would be much more efficient thanks to the use of the Internet of Things (Zuo *et al.* [9], Misra *et al.* [10], Al-Masri *et al.* [11], and Ferrari *et al.* [12]). Similarly, Miao *et al.* [13] propose a new recycling mode based on the "$O2O$" classification (online to offline) capable of integrating upstream and downstream resources on a network platform named "$O2O$ waste recycling platform" where consumers log in, register, and complete the recycling process through online submissions by following a defined user guide. In a subsequent step, green recyclers receive instructions that allow them to contact the consumer for a collection appointment, thus contributing to the development of a circular model integrating "resources-products-waste renewable resources" where raw materials are regenerated.

1.2. Ecological Waste Management

There are many ecological problems caused by the disposal of end-of-life products. Indeed, most do not degrade alone in nature and others can release elements that until then did not exist in the environment, thus, causing irreversible damage to the health of humans and their environment. Certain quantities have a strong impact on the ecosystem but also on the living conditions of the populations located near the treatment areas. Although manufacturers are sensitive to these dangers, their points of view differ as to the solutions to be adopted. Ghahremani *et al.* [14] believe that a closed-loop system is feasible and have developed a bi-objective, multi-period, multi-product, closed-loop supply chain network taking into account environmental considerations, discounts and uncertainties while Samuel *et al.* [15] proposed a deterministic mathematical model and its robust variant in which they studied the possibility of adding pre-sorting in the *CLSC* network, which would generate costs that are generally lower than those incurred in the subsequent stages of processing. These pre-sorting centers would separate poor-quality products at the beginning of the reverse logistics cycle, thus reducing transport costs and many others. Among these, an emerging cost of emissions, called the green cost, is beginning to gain momentum. This is involved in the management of a green supply chain taking into account greenhouse gas emissions in the total costs of logistics in particular by proposing a model which not only takes into account the characteristics and performance of an entity (vehicles of manufacturers) by calculating the carbon dioxide emissions for a certain type (e.g. a truck), but also the emissions linked to transport, to the operation of the installations in terms of energy consumption, transport of employees, paper consumption and computer use (Baland Satoglu [16]). Others have tried to minimize the environmental impact (in the case of transport fleets) by building a green supply chain model integrating forward and reverse queuing logistics to optimize transport and waiting time. of the network (of these fleets) to reduce environmental impacts (Mohtashami *et al.* [17]).

1.3. Economic Challenge of Waste Management

The high consumption of natural resources also leads to a decrease in its supply and therefore an increase in prices in parallel with the high rate of demand and although manufacturers are aware of the need to adopt a production that is more respectful of the environment, they nonetheless remain dependent on the economic profitability of their businesses Jiang *et al.* [18] presented a general closed-loop supply chain network including various recovery options and formulated a mixed-integer multi-objective linear programming model considering the firm's profit as well as its service level. In addition, waste generates costs, regardless of its origin (transformation process or the obligation to treat the production at the end of its life). It is therefore essential to consider adopting good waste management to optimize the costs associated with it. Aware of this, Ghahremani-Nahr *et al.* [14] designed a mathematical model of a multi-period multi-echelon closed-loop supply chain under uncertainty whose objective is to determine the quantities of products and raw materials transported between entities in the chain during each period by minimizing the total logistics cost which includes fixed, opening and closing costs, transportation costs between facilities, inventory holding costs, operation in all facilities (production costs, distribution costs, collection costs, repair costs, recovery/decomposition costs, disposal costs) and shortage costs.

Finally, the need to justify the use of such management encourages researchers to compare sales revenues with the various costs of the logistics adopted. This example was followed by Scheller *et al.* [19] who, in the case of lithium-ion batteries, proposed models for optimizing production and recycling planning that generate a maximum revenue-cost margin.

1.4. Urban Society and Waste

The design of a model stipulates that one must first estimate the value of the chosen criteria before being able to assess whether the constraints are respected. Among these criteria is the international standard for guidance on social responsibility (*ISO* 26000) for the identification of social criteria around seven major themes and five categories of social factors. The major themes include organizational governance, human rights, work practices, environment, fair operating practices, consumer issues, community involvement and development considered by Bal and Satoglu. [16]. The categories include demand satisfaction, resource equity, development opportunities employment, regional development, level of safety on site, level of access to medical facilities. Such categories are described as the local development objective (Anvari and Turkey [20]). And then accompanied by maintaining the number of workers at a certain level for an operating facility as a social goal. In this work, we are interested in the waste that the consumer produces in his home. These household wastes are end-of-life (*EOL*) products containing residual value just waiting to be exploited (Alamerew and Brissaud [5]). Thus, their good management would reduce environmental pollution, create new raw materials, and generate profit while creating jobs that allow people to live with dignity in society. To do this, we will first carry out a synthesis of the literature reviewed dealing with the subject. Then, we will extract the problem of this research.

A significant amount of sustainable supply chain research has been conducted considering various sustainability indicators in decision-making and operations management. Compared to extensive research on environmental aspects and especially economic issues, social and resource supply aspects are often overlooked in the urban waste management (*UWM)* literature. In the following, we summarize recent work by looking at different aspects of the problem (raw materials, ecological aspect, economic and social aspect). **Table 1** summarizes the recent literature review in the field of waste management; it seems clear that the ecological, social, economic and resource aspects, although they are strongly linked and sometimes mutually impacted, do not appear together in the studies carried out to date. The objective of this work is to study, model and optimize the impact on *UWM* of these four axes.

1.5. Issues and Objectives

As shown in the previous table, the different aspects (ecological, economic, social and resources) have been treated separately in the literature. No study has, to date, investigated the possibility of monitoring the impact of urban waste recovery on society, the environment, the availability of resources and the economy. However, the researchers did not rule out the idea that there is a link between them and strongly encouraged their joint integration in future decision models to be proposed. Thus, the research question to answer is: how to manage domestic waste to create new resources, protect the environment, and reduce the

Table 1. Summary of the literature review.

Paper	Aspect			
	Ecologic	Economic	Social	Resources
Gouel and Guimbard [1]	✓		✓	
Bonet *et al.* [2]			✓	✓
Steffen *et al.* [3]	✓	✓		
Amrani *et al.* [4]	✓		✓	
Alamerew and Brissaud [5]	✓			✓
Safaei *et al.* [6]		✓		✓
Cariou *et al.* [7]		✓		✓
McDonald *et al.* [8]	✓			✓
Zuo *et al.* [9]	✓	✓	✓	
Misra *et al.* [10]	✓		✓	
Al-Masri *et al.* [11]		✓		✓
Ferrari *et al.* [12]	✓	✓		
Miao *et al.* [13]	✓			✓
Ghahremani-Nahr *et al.* [14]	✓	✓		
Samuel *et al.* [15]	✓	✓		
Bal and Satoglu [16]	✓	✓	✓	
Mohtashami *et al.* [17]	✓			✓
Jiang *et al.* [18]		✓	✓	
Scheller *et al.* [19]	✓	✓		
Anvari and Turkay [20]	✓	✓	✓	
Proposed research	✓	✓	✓	✓

unemployment rate and all while making a profit?

This study will deal more in-depth with the issue of household waste, the material, social, ecological, and economic objectives to be achieved by developing the model and the methodology used to achieve it. It is therefore, necessary to identify the environmental and economic indicators, to understand the current problem of urban waste and the ideal scenario to be achieved. Finally, it would be relevant to look at a particular case to verify how such management is usually exercised to test performance and validate the logic and feasibility of the model developed. A sensitivity analysis should support the decisions proposed by the model. This one, although elaborated for a local territory, appears appropriate in the current context to solve this kind of problem on other scales.

2. Proposed Approach

This study aims at designing and presenting a model based on the ecological, economic, social and resource axes to respond to the problem mentioned above and propose an alternative to conventional *UWM* which would help decision makers to better choose between the possible alternatives (impose a pre-sorting or not). The proposed approach consists of seven steps S1 to S7) related to the following stages shown in Figure 1.

Stage 1. Modeling and optimizing the *UWM* (case with vs. without pre-sorting). The objective is to prove the economic, social, and ecological efficiency of the model by comparing the total costs generated by the current *UWM* with those of the proposed method in which various costs related to management, the environment, and the conservation of resources are considered (S1 to S3).

Stage 2. Anticipate possible decisions by governments decreeing an increase in dissipation penalties by implementing the model and making a sensitivity analysis through increasing scenarios which aims to determine the impact that such decisions can have on the cost of issuance and therefore, on the overall objective (S5 to S7).

The 7 steps considered in this research are:

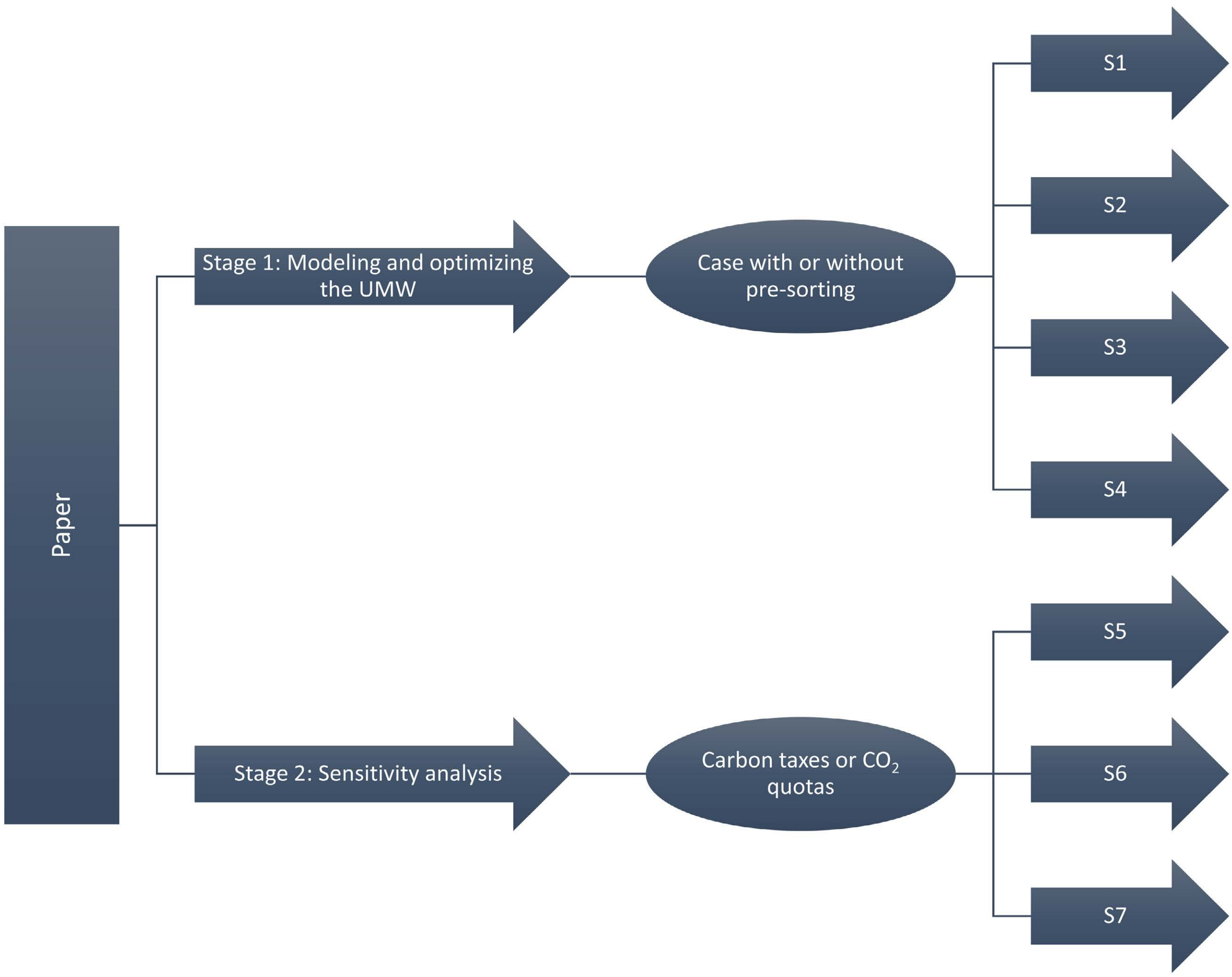


Figure 1. Proposed approach.

- S1: Presenting the reference case.
- S2: Transformation of the model by including a first selective sorting at the consumer's home (the proposed case).
- S3: Formulation of the optimization problem.
- S4: Implementation of the two models in the basic case and construction of the corresponding code.
- S5: Carry out a sensitivity analysis involving new *GHG* emission policies to choose among the recent strategies proposed by *COP*26.
- S6: Extend the model by including the new *COP*26 constraints and objectives.
- S7: Numeral application and discussion of the results.

The developments associated with these steps constitute the content of the following section which presents the proposed mathematical model as well as the obtained results.

3. Modeling and Optimizing the UWM (Case with vs. without Pre-Sorting)

The objective of this part is to design and develop a compact model capable of managing municipal waste (cardboard, plastic, metal, glass, etc.) while considering their impact on the preservation of resources, ecological, economic, environmental, and social. The literature reviewed allowed us to separately identify the different costs and parameters related to the axes studied. We propose the study of a reference case taken from the literature and from which we will build a first model corresponding to the case of municipal waste management without the possibility of pre-sorting at home Bal and Satoglu. [16]. To validate this, we will incorporate costs omitted from the base case, such as inventory costs, and then forecast the demand for recycled materials for the next 12 months. Then, we will improve the performance of the model designed from the reference case with, this time, the possibility of carrying out a first selective sorting directly in the consumer's household.

Step 1. The model designed in the studied article encompasses four objectives determined according to the minimization of costs, the reduction of environmental effects, the balance of manpower and equal objectives. Thus, **Figure 2** corresponds to the block diagram illustrating the reference case from which we drew our inspiration.

Storage costs are ignored, and the income generated by the satisfaction of demand (often stochastic), although cited, is not considered in the proposed model. Adding inventory and shortage costs to the model in **Figure 1** in a municipal waste management context will allow us to obtain the model corresponding to the case without selective sorting at home illustrated in **Figure 3**. The logistics network based on the reference case and considering a *UWM* without pre-sorting is illustrated in **Figure 3**.

Figure 3 is inspired by the reference case presented in Bal and Satoglu [16] to

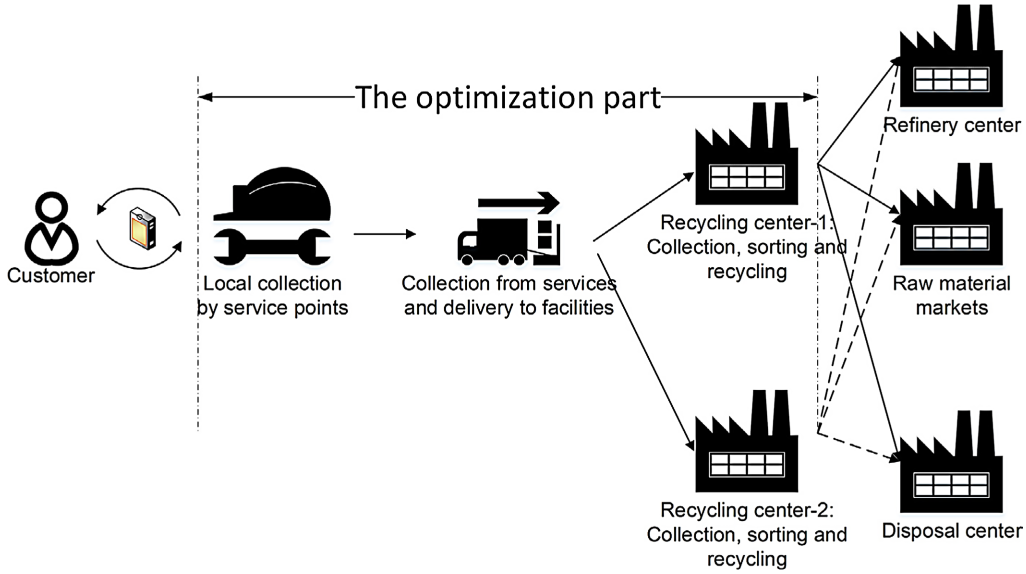


Figure 2. Reverse logistics network (from [16]).

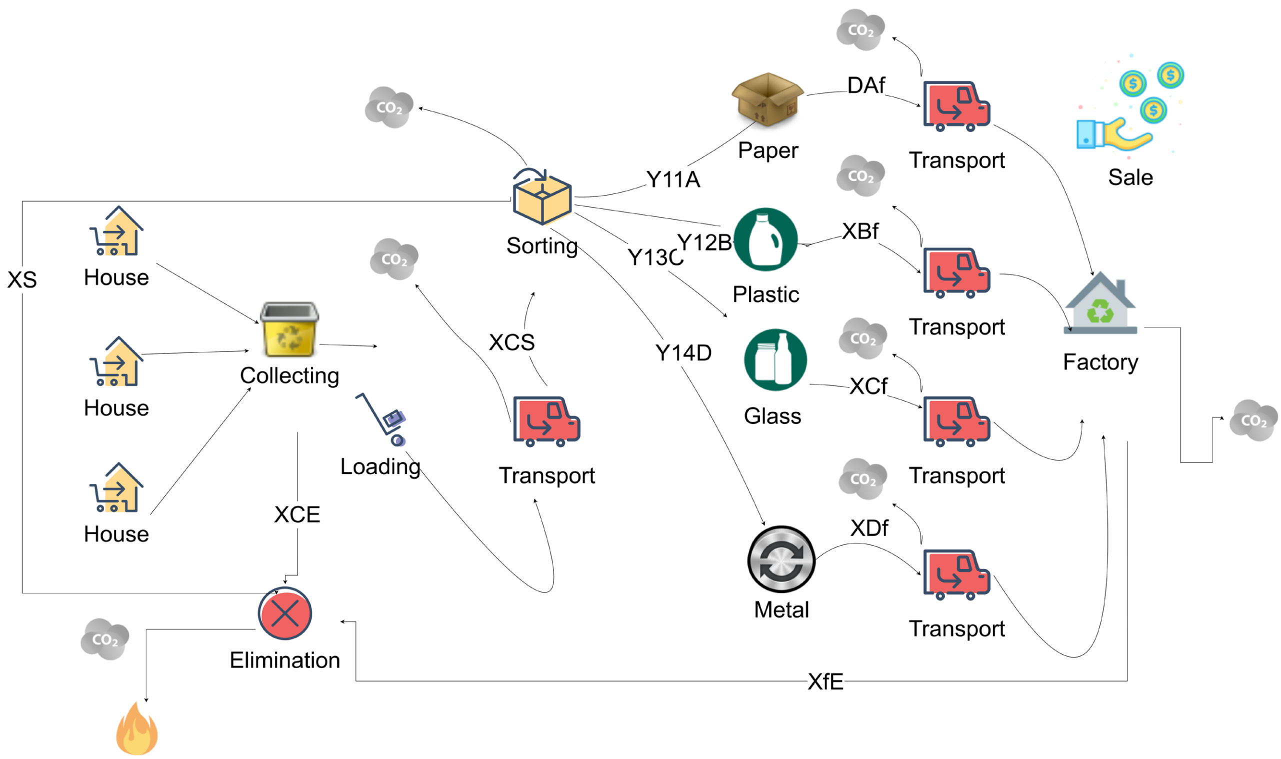


Figure 3. Model inspired by the reference case (without pre-sorting).

model the logistics of municipal waste management at the scale of a city like Montreal (which can extend over a province like Quebec) with the goal of maximizing profits corresponding to the difference between sales revenue and associated costs. This model (case without selective sorting) will consider the stochastic aspect of the demand which makes it possible to guarantee the supply. It will also consider more than one type of product and will set up an overstock management policy.

Step 2. In this part, we carry out modeling of the logistic network for the activity of household waste management in the case where a first sorting is carried out at the consumer's house. According to Figure 3, waste from each household (h) is collected in the different sectors of Montreal, at the curb, then transferred to sorting centers where manual and/or automatic sorting is carried out to design mono-materials (cardboard, plastic, metal, glass, etc.). Depending on the contamination level of the material, the next decision is to send the waste to recycling or disposal centers. Then recycling centers also have the option of recycling the quantities received or sending them to disposal centers. It should also be noted that the rate of materials to be recycled is a fixed parameter of the proposed model. This percentage is the objective of the annual *UWM* targeted by the various government awareness campaigns and implied in the proposed model. The diagram in Figure 4 represents the logistics adopted for such a *UWM*.

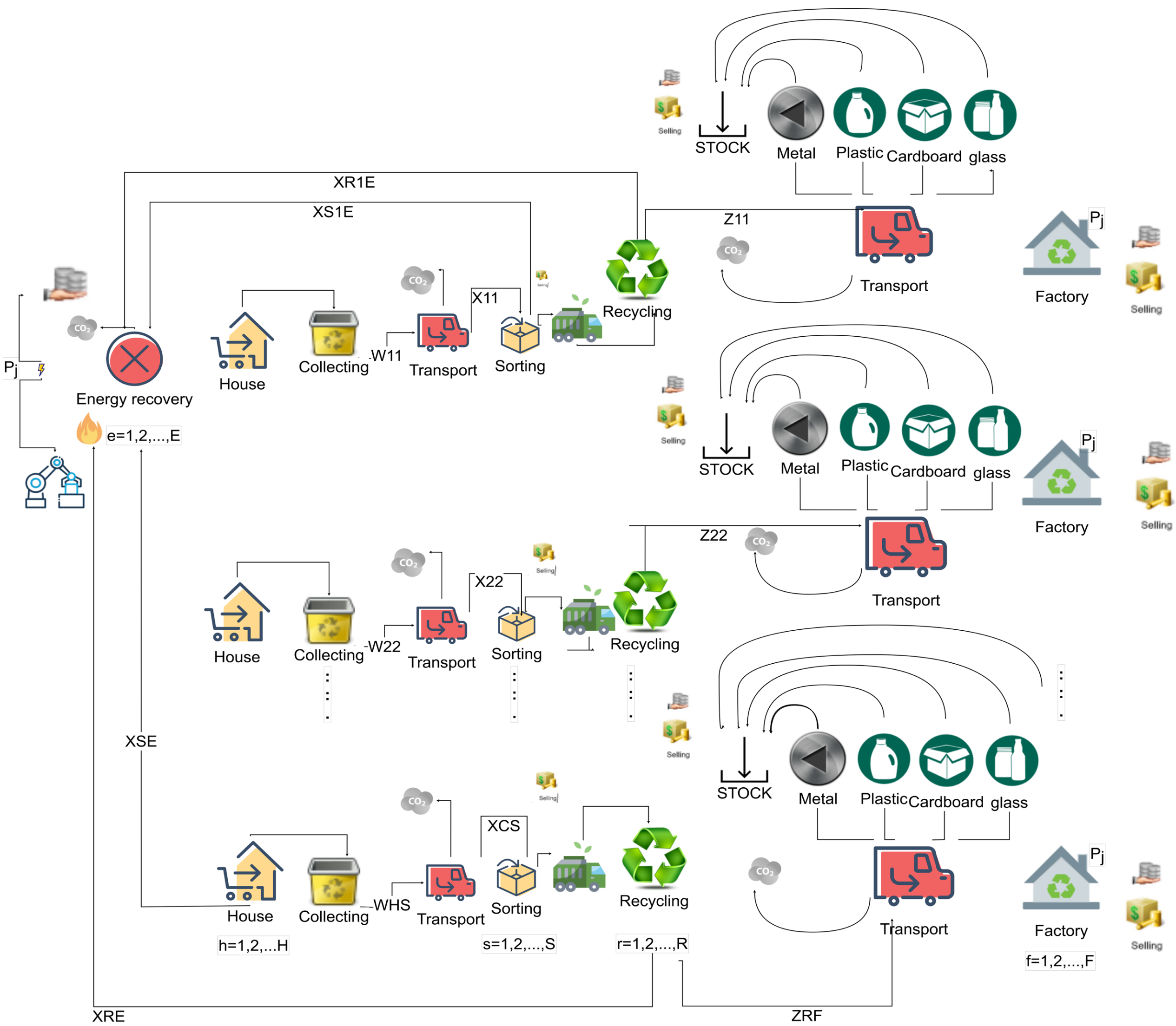


Figure 4. Diagram of the proposed urban waste management model (with selective sorting).

Nodes from left to right, represent consumer households $h \in \{1, 2, \cdots, H\}$ as well as waste bins $c \in \{1, 2, \cdots, C\}$ linked by arcs illustrating material flow Xhs collected at this level. Once loaded into trucks, these quantities Xhs will be transported to sorting centers $s \in \{1, 2, \cdots, S\}$, then to recycling $r \in \{1, 2, \cdots, R\}$ from which will result in a characterization and classification of materials: paper, plastic (*PET*, *HDPE*, *PVC*, *LDPE*, *PP*, *PS*), glass and metal. This classification will identify the right customer (the highest bidder) for the right product. Such a decision is represented by the binary variable Yf where $f \in \{1, 2, \cdots, F\}$ represents a potential customer. This will be activated (will take the value 1) if the quantities of materials transported in the truck will be sold to the customer $f \in F$ (and 0, otherwise). The unsold sorted units will be stored to prevent a possible shortage of materials in the future or will be sold in bundles to the various recycling centers. At this level, other profits will be considered. As for materials that have been contaminated or whose life cycle turns out to be shorter than expected, the energy recovery process will allow them to act as an equally essential element in the recycling chain because this will release enough heat and energy to power the machinery.

Step 3. This step includes a mathematical formulation of the problem to be optimized by considering the two cases (with or without selective sorting). It is a compact mixed linear program including integer variables and binary variables (*MILP*) that can be implemented and run with *LINGO* software. The objective will be to optimize the flow of materials to be transported to customers while minimizing the main costs linked to their logistics process: the operational cost of treatment, transport (which is proportional to the distance traveled and represented the purchase and fuel consumption), as well as the green cost (vehicle greenhouse gas emissions). The optimum quantities, as well as the designation of the centers to which the waste is sent will be determined by the model presented below.

The sets, the parameters, and the decision variables on which the model is based are the following.

Sets.

H: All homes.

S: All sorting centers.

A: All recycling centers.

E: All elimination centers.

F: All customer factories.

Parameters.

C_s: Sorting center capacity s.

C_e: Disposal center capacity e.

P_i: Unit sale price of recycled product i (\$/t).

a: Unit cost of treatment-sorting center (\$\t).

η: Unit disposal cost (\$\t).

u: Unit collection cost (\$\t).

θ: Unit cost of GHG emissions ($).

m: Recycling unit cost ($/t).

c: Unit transport cost ($/t-km).

f: GHG emission factor (t Co2 eq/km).

d_s: Distance between h and center s (km).

d_{sr}: Distance between center s and center r (km).

d_{rf}: Distance between center r and factory f (km).

ρ: Storage unit cost ($\t).

Ω: Percentage of waste to be recovered (%).

D_{fi}: Request from customer f for product i(t).

k_r: Amount of waste available in recycling center r(t).

Q: Quantity of waste (t) requiring a threshold of 70% of employees (workforce).

Decision variables.

X_s: Amount of waste sent from h to center s(t).

X_{sr}: Amount of waste sent from s to r center r(t).

X_{se}: Amount of waste sent from s to center e(t).

X_{re}: Amount of waste sent from r to center e(t).

X_{Arf}: Amount of paper waste i sent from r to plant f(t).

X_{Brf}: Quantity of plastic waste i sent from r to plant f(t).

X_{Crf}: Amount of glass waste i sent from r to plant f(t).

X_{Drf}: Amount of metal waste i sent from r to plant f(t).

X_{rti}: Amount of i waste stored in the g center r(t).

Y_{fi}: Factory selection (1 if factory f is a customer of product i, 0 otherwise}.

Objective function

$$
\begin{aligned}
MAX = & \sum_{f\in F}\sum_{r\in R}\left(P_A x_{Arf} y_f + P_B x_{Brf} y_f + P_C x_{Crf} y_f + P_D x_{Drf} y_f\right) \\
& - c\left(\sum_{s\in S}\sum_{h\in H} d_{hs} + \sum_{e\in E}\sum_{s\in S} d_{se} + \sum_{r\in R}\sum_{s\in S} d_{sr} + \sum_{e\in E}\sum_{r\in R} d_{re} + \sum_{f\in F}\sum_{r\in R} d_{rf}\right) \\
& - u\left(\sum_{s\in S}\sum_{h\in H} x_{hs}\right) - a\left(\sum_{r\in R}\sum_{s\in S} x_{sr}\right) - m\left(\sum_{f\in F}\sum_{r\in R} x_{rf}\right) - \eta\left(\sum_{e\in E}\sum_{s\in S} x_{se}\right. \\
& \left. + \sum_{e\in E}\sum_{r\in R} x_{re}\right) - \theta f\left(\sum_{s\in S}\sum_{h\in H} d_{hs} + \sum_{r\in R}\sum_{s\in S} d_{sr} + \sum_{f\in F}\sum_{r\in R} d_{rf} + \sum_{e\in E}\sum_{s\in S} d_{se}\right. \\
& \left. + \sum_{e\in E}\sum_{r\in R} d_{re}\right) - \rho\sum_{f\in F}\sum_{r\in R}\Big(\left(x_{Arf} - D_{Af} + k_{Ar}\right) + \left(x_{Brf} - D_{Bf} + k_{Br}\right) \\
& + \left(x_{Crf} - D_{Cf} + k_{Cr}\right) + \left(x_{Drf} - D_{Df} + k_{Dr}\right)\Big)
\end{aligned}
\tag{1}
$$

Constraints.

Sorting center capacity constraint:

$$\sum_{s\in S}\sum_{h\in H} x_{hs} \le c_s \quad \forall s\in S \tag{2}$$

Capacity constraint of recycling centers:

$$\sum_{r\in R}\sum_{s\in S} x_{sr} \le c_r \quad \forall r\in R \tag{3}$$

Capacity constraint of disposal centers:

$$\sum_{e\in E}\sum_{s\in S} x_{se} + \sum_{e\in E}\sum_{r\in R} x_{re} \le c_e \quad \forall e \in E \tag{4}$$

Flow conservation constraints:

$$\sum_{s\in S}\sum_{h\in H} x_{hs} = \sum_{r\in R}\sum_{s\in S} x_{sr} + \sum_{e\in E}\sum_{s\in S} x_{se} \quad \forall s \in S \tag{5}$$

$$\sum_{r\in R}\sum_{s\in S} x_{sr} + \sum_{r\in R} x_r = \sum_{f\in F}\sum_{r\in R} x_{rf} + \sum_{e\in E}\sum_{r\in R} x_{re} + \sum_{r\in R} k_r \quad \forall r \in R$$

Demand Fulfillment Constraint:

$$\sum_{r\in R}\left(x_{Mr} + k_{Mr}\right) + \sum_{f\in F}\sum_{r\in R} x_{rMf} = D_{iMf} \tag{6}$$

$$\forall f \in F, \forall M \in \{A, B, C, D\}, \forall i \in \{jan, fev, \cdots, dec\}$$

Employment constraint:

$$\sum_{s\in S}\sum_{h\in H} x_{hs} \ge \Omega Q \tag{7}$$

Constraint of non-negativity

$$x_{ijk} \ge 0 \quad \forall (i, j, k) \tag{8}$$

Step 4.

In this part, we will test the robustness of the model in a simple case where we consider a basic structure inspired by **Figure 3** and containing an element of each set (recycled material: paper, a building, a sorting center, a recycling center, and a customer factory). To control the stochastic aspect of demand, we will make forecasts for the next 12 months using the *ARIMA* method based on *R language*. **Table 2** represents the recent demands for paper constituting the time series on which the forecasts are based.

The *R language* instructions Demand <- ts (data = Y, start = 1, end = c(9), frequency = 3) and Arima (Demand) allow to observe the crooks for the year 2021 which are given by **Table 3**.

The resolution was made using *LINGO* and the results are presented in **Table 4**.

The results obtained in **Table 4** are represented in **Figure 5** which shows the evolution of the stock, the environmental cost, and the monthly demand according to the decisions to be made.

The target indicated for each case in **Figure 5** is the result of the difference between the sales revenue and the various associated costs whose value is presented in **Table 5** and **Table 6** where:

Table 2. Quantities (tons) of paper sold in 2020.

Month\Demand	Jan	Feb	Mar	Apr	May	Juin
	10,600	13,400	18,300	17,200	11,600	10,600
	Jul	Aug	Sept	Oct	Nov	Dec
	14,200	11,100	14,900	16,600	12,100	16,300

Table 3. Demand forecast for 2021.

Month	Dem (tons)	Unit Price
Jan	12,756	
Feb	12,203	
Mar	13,626	
Apr	13,552	
May	12,473	
Jun	13,105	67$
Jul	13,207	
Aug	13,126	
Sept	12,855	
Oct	13,019	
Nov	13,471	
Dec	12,580	

Table 4. Case study results. A. Case without pre-sorting. B. Case with pre-sorting.

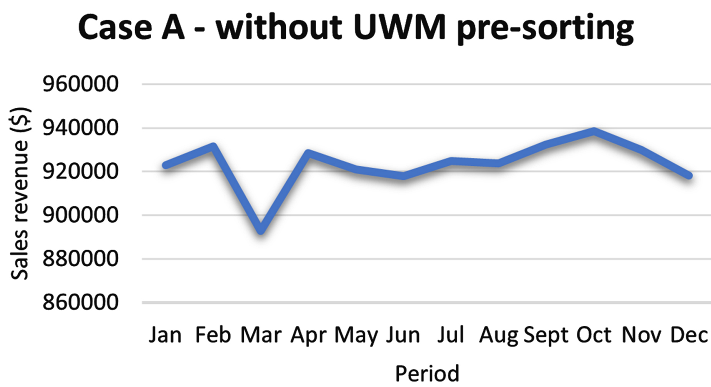
Case A			Case B		
Month	Q_Month/t	Rev_vente/$	Month	Q_Month/t	Rev_vente/$
Jan	13,776	922,992	Jan	45,604	3,055,468
Feb	13,903	931,501	Feb	51,063	3,421,221
Mar	13,324	922,708	Mar	42,915	2,875,305
Apr	13,856	928,352	Apr	40,370	2,704,790
May	13,746	920,982	May	41,209	2,761,003
Juin	13,700	917,900	Juin	48,661	3,260,287
Jul	13,804	924,868	Jul	40,205	2,693,735
Aug	13,788	923,796	Aug	51,134	3,425,978
Sept	13,915	932,305	Sept	39,723	2,661,441
Oct	14,008	938,536	Oct	40,277	2,698,559
Nov	13,880	929,960	Nov	53,947	3,614,449
Dec	13,704	918,168	Dec	46,053	3,085,551

Table 5. Result of case A (without pre-sorting).

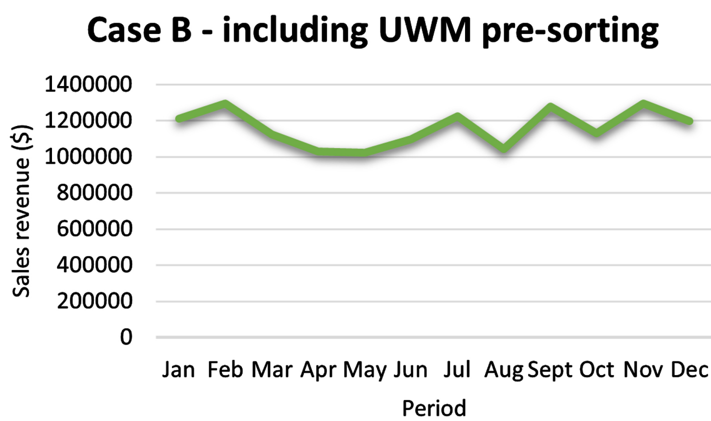
CASE A: Objective = −343735.00$							
C_Trs/$	C_Em/$	C_Op/$	C_Elim	C_St	Q_Recup/t	Q-Elim	Rev_sell/$
289,118	116,835	8,902,739	2,064,526	52583.4	424,440	206,452	11,082,068

Table 6. Result of case B (with pre-sorting).

CAS B: Objectif = 490202.00$							
C_Trs/$	C_Em/$	C_Op/$	C_Elim	C_St	Q_Recup/t	Q-Elim	Rev_sell/$
327240.84	134,730	12,570,700	0	420687.16	628,800	0	13,943,560



(a)



(b)

Figure 5. Result of case study (a). Case without pre-sorting. (b). Case with pre-sorting.

C_Trs: is the transport cost ($).

C_Em: is the CO_2 emission cost ($).

C_Op: is the operational cost ($).

C_Elim: is the elimination cos ($).

Q_Recup: is the amount of waste collected (ton).

Q_Elim: is the amount of waste eliminated (ton).

Rev_sell: is the sales revenue ($).

According to Figure 5 as well as Table 5 and Table 6, in case B (with selective sorting), the income from the sale is worth $13,943,560, while in the absence of pre-sorting (case A), it is worth $11,082,068. This is since municipal waste bins are less contaminated when they are pre-sorted by consumers (case B) and therefore, the quantity to be recovered for recycling purposes is greater and generates a higher income than in the case where no pre-sorting is performed (case A). The operational cost of collection, sorting, recycling and disposal amounts to $12,570,700 in the presence of pre-sorting (case B) and to $8902739.14 when there is no pre-sorting (case A). This is since the quantities of waste recovered and requiring treatment on the sites are higher in the case with selective sorting (case B) while in case A, the contaminated waste no longer having any residual value does not require any treatment but will increase disposal cost. The cost of *GHG* emissions is only worth $134,730 in the presence of selective sorting and $116835.77 in the case without pre-sorting, to which we add an elimination cost of up to $2,064,526. This emission cost (better known as the *Green Cost*) seems negligible compared to the operational cost. This is because the green strategy imposed by governments is emerging and gradually taking hold to let decision-makers embrace green costs in their direct and return supply chains. Given this progression to become, in what follows, we will study the impact of the progressive variation of this green cost on the overall objective through a sensitivity analysis.

Step 5. The interpretation of the cost and greenhouse gas emission limits plays a role in this sensitivity analysis. The goal here is to anticipate possible decisions by governments decreeing an increase in dissipation penalties. Through increasing scenarios, this analysis aims to determine the impact that such decisions can have on the overall objective. Canadian ministers set the goal in March 2016: a 30 percent reduction below 2005 greenhouse gas (*GHG*) emission levels by 2030. They agreed to the pricing of carbon as a policy strategy for reducing *GHG* emissions.

Two carbon pricing options have gained traction:

- A carbon tax, levied on the volume of emissions (per ton),
- A cap and trade system that sets a limit on emissions and requires emitters to buy allowances under a fixed offer.

The federal government has proposed that the price per ton start at $10 in 2018 and rise to $50 per ton in 2022. In this part, we study the impact of the two strategies on the total cost by performing an analysis of sensitivity. To do this, we first vary the ceiling limit provided in the text of the Canada's legislation until the objective coveted by the government for 2030 is reached. Then, we vary the allowances purchase price defined above as one of the two options and which grows annually (from $10-$50 between 2018 and 2022). The Copenhagen accord stipulates that the value of CO_2 emissions from heavy-duty engines cannot exceed the CO_2 standard of 627 g/BHP-Hour during their useful life. This standard is defined according to the maximum quantity of emissions according to distance in the case of light vehicles (less than 3855 kg) or according to brake horsepower per hour (bhp/h) in the case of heavy vehicles. The following step should allow us to identify the new parameters to include in the model.

Step 6.

In this step, we extend the flexibility of the model presented in stage 1, by adding the formulas taken from the new *GHG* emission policies cited in step 5 of stage 2. Thus, we assume that:

Nr: Heavy-Duty Engine CO_2 Emission Standard (627 g/BHP-hr).

V: Speed of class 8 heavy vehicle (105 km/h).

E: The emission of CO_2 per class 8 heavy vehicle is 99.7 g CO_2 e/km.

Tx: Carbon tax ($/ton).

BHP: Heavy vehicle brake power.

Qta: Quota purchase price (from \$10-\$50 between 2018 and 2022).

T: Maximum number of vehicles.

In this study, we design heavy-duty vehicles specialized in the collection and transport of urban waste whose *X*15 *Volvo-series* 505 - 605 *hp*, 1850 - 2050 *lb-ft* engines are equipped with a brake power that can reach 600 hp. In the case of the Quebec-California market, each unit (equivalent to one metric ton of *CO*2) has a floor price in CAD that increases by 5% per year.

According to the data extracted from the context studied, the vehicles used in this study must satisfy Equation (9).

$$\frac{E \times V}{BHP} \le Nr \tag{9}$$

Based on preliminary recommendations in the literature, it is reasonable to estimate emission cost values at $Tx =$ \$28.44/ton CO_2 increasing each year based on expected damage growth.

Exceeding this standard will result in costs ranging from \$28.44 to almost \$50 per ton of CO_2 emitted in 2050 and the cost to be added to the objective function (if the constraint 9 is not satisfied) is given by Equation (10).

$$\begin{cases} Tx\sum_{t=1}^{T}\left(\frac{E \times V}{BHP} - Nr\right)z_1 + Qta\sum_{t=1}^{T}\left(\frac{E \times V}{BHB} - Nr\right)z_2, \text{ si } \frac{E \times V}{BHP} > Nr \\ 0 \text{ otherwise} \end{cases} \tag{10}$$

with $z_i = \begin{cases} 1 \text{ if the strategy } i \text{ is chosen} \\ 0 \text{ otherwise} \end{cases} \quad \forall i \in \{1,2\}$

$$z_1 + z_2 = 1$$

And $i \in \{1:\text{Carbon taxes}, 2:\text{Redemption of allowances}\}$

Step 7.

At first glance, the strategy of buying allowances seems more attractive given the difference between their price and that of carbon taxes. However, this price increased by 5% per year, it would possibly be judicious and less appropriate (even more commendable) for certain companies, to consider adopting the first plan to reduce their costs in the long term. To prove it, we carry out a numeral application which may help the decision makers to choose among these two strategies, the one which will guarantee an optimal cost. Through 6 scenarios, we will gradually reduce the cap limits from 627 g/bhp-h to 500 g/bhp-h (**Table 7**). Then, in 6 other scenarios (**Table 8**), we will increase the price by 5% annually and purchase one unit (in tons) of allowances. The results obtained through the increase in the purchase price of allowances are listed in **Table 7**, while those of the variation of the Nr standard appear in **Table 8**. The 5% annual increase in the allowance purchase price gave the results shown in **Table 7**. Gradually decreasing the cap limits from 627 g/bhp-h to 500 g/bhp-h gave the results shown in **Table 8**.

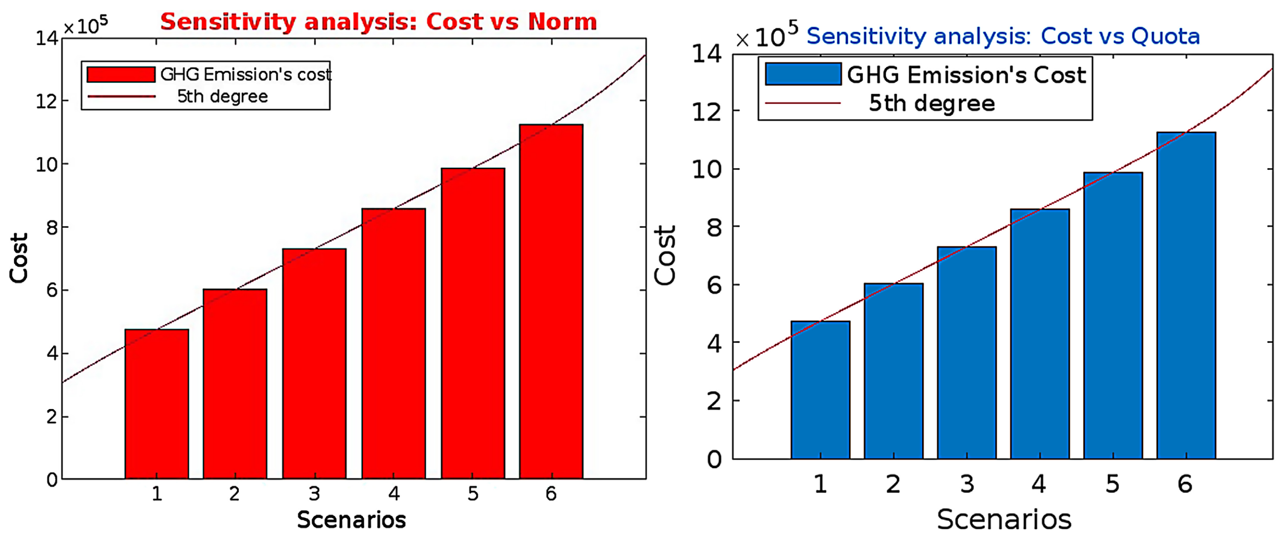
Through **Figure 6**, we observe the evolution of the cost of greenhouse gas

Table 7. Results of the variation of GHG emission standard.

Scenario	Varied parameter	Unit	Value	Emissions	Emission costs
1			28.44		474663.6$
2			32		54080.0$
3	Qta	$/t co2e	36	77.73t co2e	600840.0$
4			40		667600.0$
5			44		734360.0$
6			50		834500.0$

Table 8. Results of the variation in the purchase price of allowances.

Scenario	Varied parameter	Unit	Value	Emissions	Emission costs
1			627		474663.6$
2			602		602643.6$
3	Nr	g/bhp-h	577	77.73t co2e	730623.6$
4			552		858603.6$
5			527		986583.6$
6			500		1,124,802$



(a) (b)

Figure 6. Sensitivity analysis.

emissions according to the variation of the threshold described by the imposed standard (strategy 1). It clearly appears in this graph that as the scenarios represent a gradual drop of almost 4% of the tolerated threshold per g/bhp-h, the green cost generated increases rapidly until it reaches a value between 1 million and $1.2 million. This amount corresponds to the cost of $1,124,802 observed in scenario 6 of **Table 8**. **Figure 6** shows the evolution of the cost of greenhouse gas emissions according to the variation in the price of allowances (strategy 2). It also appears clearly in this graph that as the scenarios represent a gradual increase of nearly $4 per ton of CO_2 emitted, the green cost generated increases gradually until it reaches a value between 800,000 and 1 million bucks. This amount corresponds to the cost of $834,500 observed in scenario 6 of **Table 7**.

In view of these results, we see that it would be less costly in the long term for companies whose activities emit large quantities of *GHGs* to opt for the second strategy and consider buying allowances rather than having to pay the carbon tax, the generated costs of which shown in **Figure 6** increase more rapidly compared to the price of buying allowances.

4. Conclusions

The purpose of this work was to develop a compact urban waste management plan based on the principle of circular manufacturing and which can be adopted by different industries to meet their constraints optimize their profit while preserving planetary resources. To do this, we have gone through the recent literature studying the subject and have deduced a problem and issued objectives through its synthesis. The work done during this research identified the problem through a review of the synthesized literature and highlighted some recommendations. Indeed, the results obtained emphasized the need to carry out pre-sorting at the consumer's home. In practice, such a voluntary contribution should, in one way or another, be rewarded. Some manufacturers, such as *COCORICO* in France, have drawn up a contract binding them to the customer. It is stipulated that on the purchase of an item, a reduction in the price is made if the customer agrees to return the product when it becomes obsolete. This allows the manufacturer to recycle it to make new finished products.

Finally, the case study also demonstrated the importance of barcodes on packaging to identify the type of material to be recovered. Indeed, the proposed model had taken into consideration the different categories of plastics (*PET*, *HDPE*, *PVC*, *LDPE*, *PP* and *PS*) to make mono-material products. On the other hand, this cannot be envisaged for other types of materials, such as metals, for example, which nowadays do not have codes allowing their identification for recycling purposes. In this perspective, the proposed and implemented model can be further improved depending on the context. For example, a game theory could intervene in the optimization of its performance since consumers are Rational Entities that can act with the aim of achieving a gain.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Gouel, C. and Guimbard, H. (2017) La demande alimentaire mondiale en 2050. La Lettre du CEPII, Centre d'Etudes Prospectives et d'Informations Internationales, Paris, 5.

[2] Bonet Fernandez, D., Petit, I. and Lancini, A. (2014) L'économie circulaire: quelles mesures de la performance économique, environnementale et sociale? *Revue Française De Gestion Industrielle*, **33**, 23-43. https://doi.org/10.53102/2014.33.04.791

[3] Steffen, W., Richardson, K., Rockström, J., Cornell, S.E., Fetzer, I., Bennett, E.M., Biggs, R., *et al*. (2015) Planetary Boundaries: Guiding Human Development on a Changing Planet. *Science*, **347**, Article ID: 1259855. https://doi.org/10.1126/science.1259855

[4] Amrani, F., Prud'homme, J., Maesano, C. and Annesi-Maesano, I. (2019) Évolution de la pollution atmosphérique urbaine dans 13 grandes villes françaises entre 2008 et 2015. *Revue des Maladies Respiratoires*, **36**, 1096-1106. https://doi.org/10.1016/j.rmr.2019.09.003

[5] Alamerew, Y.A. and Brissaud, D. (2019) Circular Economy Assessment Tool for End of Life Product Recovery Strategies. *Journal of Remanufacturing*, **9**, 169-185. https://doi.org/10.1007/s13243-018-0064-8

[6] Safaei, A.S., Roozbeh, A. and Paydar, M.M. (2017) A Robust Optimization Model for the Design of a Cardboard Closed-Loop Supply Chain. *Journal of Cleaner Production*, **166**, 1154-1168. https://doi.org/10.1016/j.jclepro.2017.08.085

[7] Cariou, N., Vergnaud, S., Le Galliot, S., Berger, V., Rigous-Lemercier, C. and Ouattara, A. (2019) Recyclage du papier au bloc opératoire, un enjeu pour l'avenir. *Interbloc*, **38**, 168-171. https://doi.org/10.1016/j.bloc.2019.07.006

[8] McDonald, M., Normandin, D. and Sauvé, S. (2016) L'économie circulaire: Une transition incontournable. Presses de l'Université de Montréal, Montreal. https://doi.org/10.4000/books.pum.4151

[9] Zuo, L., Wang, C. and Sun, Q. (2020) Sustaining WEEE Collection Business in China: The Case of Online to Offline (O2O) Development Strategies. *Waste Management*, **101**, 222-230. https://doi.org/10.1016/j.wasman.2019.10.008

[10] Misra, D., Das, G., Chakrabortty, T. and Das, D. (2018) An IoT-Based Waste Management System Monitored by Cloud. *Journal of Material Cycles and Waste Management*, **20**, 1574-1582. https://doi.org/10.1007/s10163-018-0720-y

[11] Al-Masri, E., Diabate, I., Jain, R., Lam, M.H. and Reddy Nathala, S. (2018) An IoT-Enabled Framework for Urban Waste Management 2018 *IEEE International Conference on Big Data* (*Big Data*), Seattle, 10-13 December 2018, 5285-5287. https://doi.org/10.1109/BigData.2018.8622117

[12] Ferrari, F., Striani, R., Minosi, S., De Fazio, R., Visconti, P., Patrono, L., *et al*. (2020) An Innovative IoT-Oriented Prototype Platform for the Management and Valorisation of the Organic Fraction of Municipal Solid Waste. *Journal of Cleaner Production*, **247**, Article ID: 119618. https://doi.org/10.1016/j.jclepro.2019.119618

[13] Miao, S., Liu, D. and Ma, J. (2020) O2O Recycling Closed-Loop Supply Chain Modeling Based on Classification Process considering Environmental Index. *Discrete Dynamics in Nature and Society*, **2020**, Article ID: 5737514. https://doi.org/10.1155/2020/5737514

[14] Ghahremani-Nahr, J., Nozari, H. and Najafi, S.E. (2020) Design a Green Closed Loop Supply Chain Network by Considering Discount under Uncertainty. *Journal*

of Applied Research on Industrial Engineering, **7**, 238-266.

[15] Samuel, C.N., Venkatadri, U., Diallo, C. and Khatab, A. (2020) Robust Closed-Loop Supply Chain Design with Presorting, Return Quality and Carbon Emission Considerations. *Journal of Cleaner Production*, **247**, Article ID: 119086. https://doi.org/10.1016/j.jclepro.2019.119086

[16] Bal, A. and Satoglu, S.I. (2018) A Goal Programming Model for Sustainable Reverse Logistics Operations Planning and an Application. *Journal of Cleaner Production*, **201**, 1081-1091. https://doi.org/10.1016/j.jclepro.2018.08.104

[17] Mohtashami, Z., Aghsami, A. and Jolai, F. (2020) A Green Closed Loop Supply Chain Design Using Queuing System for Reducing Environmental Impact and Energy Consumption. *Journal of Cleaner Production*, **242**, Article ID: 118452. https://doi.org/10.1016/j.jclepro.2019.118452

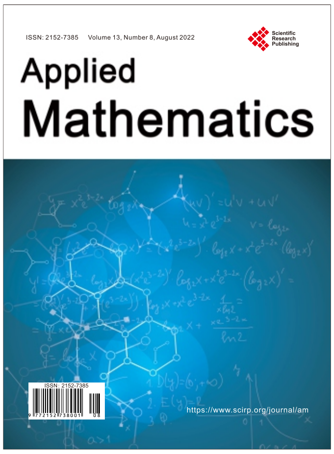
[18] Jiang, G., Wang, Q., Wang, K., Zhang, Q. and Zhou, J. (2020) A Novel Closed-Loop Supply Chain Network Design Considering Enterprise Profit and Service Level. *Sustainability*, **12**, Article No. 544. https://doi.org/10.3390/su12020544

[19] Scheller, C., Schmidt, K. and Spengler, T.S. (2021) Effects of CO_2-Penalty Costs on the Production and Recycling Planning of Lithium-Ion Batteries. *Procedia CIRP*, **98**, 643-647. https://doi.org/10.1016/j.procir.2021.01.168

[20] Anvari, S. and Turkay, M. (2017) The Facility Location Problem from the Perspective of Triple Bottom Line Accounting of Sustainability. *International Journal of Production Research*, **55**, 6266-6287. https://doi.org/10.1080/00207543.2017.1341064

Call for Papers

Applied Mathematics (AM)



ISSN Print: 2152-7385 ISSN Online: 2152-7393
https://www.scirp.org/journal/am

Applied Mathematics **(AM)** is an international journal dedicated to the latest advancements in applied mathematics. The goal of this journal is to provide a platform for scientists and academicians all over the world to promote, share, and discuss various new issues and developments in different areas of applied mathematics.

Subject Coverage

All manuscripts must be prepared in English, and are subject to a rigorous and fair peer-review process. Accepted papers will immediately appear online followed by printed hard copy. The journal publishes original papers including but not limited to the following fields:

- Applied Probability
- Applied Statistics
- Approximation Theory
- Chaos Theory
- Combinatorics
- Complexity Theory
- Computability Theory
- Computational Methods in Mechanics and Physics
- Continuum Mechanics
- Control Theory
- Cryptography
- Discrete Geometry
- Dynamical Systems
- Elastodynamics
- Evolutionary Computation
- Financial Mathematics
- Fuzzy Logic
- Game Theory
- Graph Theory
- Information Theory
- Inverse Problems
- Linear Programming
- Mathematical Biology
- Mathematical Chemistry
- Mathematical Economics
- Mathematical Physics
- Mathematical Psychology
- Mathematical Sociology
- Matrix Computations
- Neural Networks
- Nonlinear Processes in Physics
- Numerical Analysis
- Operations Research
- Optimal Control
- Optimization
- Ordinary Differential Equations
- Partial Differential Equations
- Probability Theory
- Statistical Finance
- Stochastic Processes
- Theoretical Statistics

We are also interested in: 1) Short Reports—2-5 page papers where an author can either present an idea with theoretical background but has not yet completed the research needed for a complete paper or preliminary data; 2) Book Reviews—Comments and critiques.

Notes for Intending Authors

Submitted papers should not have been previously published nor be currently under consideration for publication elsewhere. Paper submission will be handled electronically through the website. All papers are refereed through a peer review process. For more details about the submissions, please access the website.

Website and E-mail

https://www.scirp.org/journal/am **E-mail: am@scirp.org**