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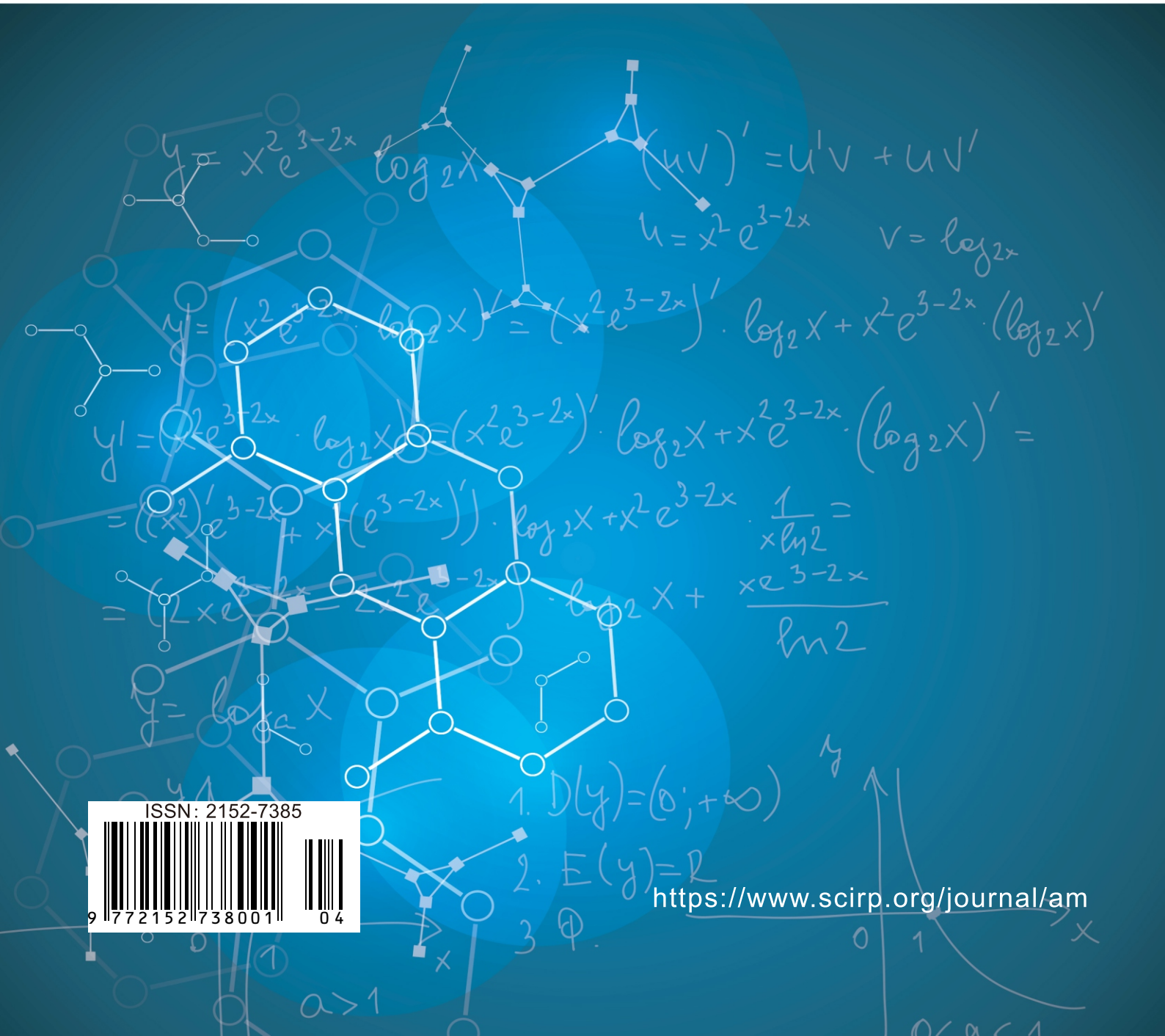

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Table of Contents

Volume 13 Number 4 April 2022

Production of the Reduction Formula of Seventh Order Runge-Kutta Method with Step Size Control of an Ordinary Differential Equation

G. D. Trikkaliotis, M. Ch. Gousidou-Koutita…325

Derivation of the Reduction Formula of Sixth Order and Seven Stages Runge-Kutta Method for the Solution of an Ordinary Differential Equation

G. D. Trikkaliotis, M. Ch. Gousidou-Koutita…338

A Kind of Time-Delayed COVID-19 Dynamical Model with Vaccination

C. A. Li, J. L. Lu…356

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Journal Information

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Production of the Reduction Formula of Seventh Order Runge-Kutta Method with Step Size Control of an Ordinary Differential Equation

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Abstract

The purpose of the present work is to construct a nonlinear equation system (85 × 53) using Butcher's Table and then by solving this system to find the values of all set parameters and finally the reduction formula of the Runge-Kutta (7,9) method (7^{th} order and 9 stages) for the solution of an Ordinary Differential Equation (ODE). Since the system of high order conditions required to be solved is too complicated, we introduce a subsystem from the original system where all coefficients are found with respect to 9 free parameters. These free parameters, as well as some others in addition, are adjusted in such a way to furnish more efficient R-K methods. We use the MATLAB software to solve several of the created subsystems for the comparison of our results which have been solved analytically.

Keywords

Initial Value Problem, Runge-Kutta Methods, Ordinary Differential Equations

1. Fundamental Principles

From the research carried out in publications of works related to the creation of R-K methods for solving Ordinary Differential Equations it was observed that the description of the creation of these methods was done in a general way and for all classes of normal differential equations. The approach to create this method was not simple but was partly complex. So it was decided to create and propose a process for creating a R-K method that will be simple, understandable

and applicable.

A system of ordinary differential equations of the form

$$y' = f(x, y),\ y(x_0) = y_0 \tag{1}$$

with $x_0 \in \mathbb{R}$, $y, y' \in \mathbb{R}^m$ and $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, is called Initial Value Problem (IVP).

Runge-Kutta methods are commonly used numerical methods for addressing (1). They usually presented in a so-called Butcher table (**Table 1**) [2] [3]:

The table contains on the 1st column the coefficients c_i, the matrix A with the coefficients of a_{ij}, which appear in the Formulae of K_i, and w_i the coefficients in Formula of y_{i+1}.

In this type of table, we have $\boldsymbol{w}^{\mathrm{T}}, \boldsymbol{c} \in \mathbb{R}^m$ while $\boldsymbol{A} \in \mathbb{R}^{m \times m}$. Then, the method shares m stages and in case that $c_1 = 0$ and $\boldsymbol{A}$ is strictly lower triangular, it is evaluated explicitly.

The solution of a differential equation is a continuous curve $y(x)$ that passes through the point (x_0, y_0) and satisfies $y'(x) = f(x, y)$. Numerical solution of a differential equation is a distinct set of values of $y(x)$ which is an approach to the continuous solution of the $y(x)$ curve.

Carl David Tolmé Runge [4] and Martin Wilhelm Kutta [5] introduced the methods bearing their names almost in the turning of the 19th century. Runge and Kutta observed that the derivation of high-order derivatives that appear in the Taylor method can be avoided. In this method we place the problem with indeterminate parameters and make the result at the highest order using calculations of $f(x,y)$ inside (x_n, y_n) and (x_{n+1}, y_{n+1}) intervals. The derivatives in the Taylor form are replaced by calculating $f(x,y)$ at a number of points inside (x_n, y_n) and (x_{n+1}, y_{n+1}) intervals.

Runge was the first to present a 2nd order R-K method by combining a sequence of Euler formulas [4]. Some years later, Kutta managed to construct a 4 stages 4th order method [5]. Nyström showed a method (5,6) of 5th order and 6 stages [6]. Fehlberg [7], Shanks [8] and Lawson [9] showed 5th order methods of 6 stages too. 6th order methods have been presented by Butcher [2], Fehlberg [7], Shanks [8] and Lawson [9]. Huta's 6th order method of 8 stages is the most popular [10]. Higher order R-K methods have been presented by Shanks [8], Felhberg [7], Feagin [11] [12] Hairer [13], Butcher [14] [15], Curtis [16], Famelis [17], Papakostas [17] [18], Tsitouras [17] [18] [19] and others.

Some problems that could be solved in this paper:

- We want with analytic way to derive the RK(7,9) method and we introduce our method for that.
- We give first arbitrary variables with values of the existing table of RK(7,9) method in order to compare our method of solving the non-linear system.

Table 1. The so-called butcher table.

c	A
	w^{T}

- We suggest and some others arbitrary variables which lead to desired Tables of the RK(7,9) method, because not all the arbitrary values lead to desired results of the method.

Firstly, we present the Introduction of the paper with historical references, and then in section 2 we give an analytic approach of the Runge-Kutta method (7,9) 7th order with 9 stages method. Finally, we give the conclusions of our work of a certain set of values of the parameters of the method.

2. Presentation of the Runge-Kutta 7th Order 9 Stages Method

The reduction formula of R-K methods for an ordinary differential equation is given by the relation $y_{n+1} = y_n + \sum_{i=1}^{\nu} w_i K_i$ (2), with w_i acting as coefficients of weight, ν the number of steps and $K_i = hf\left(x_n + c_i h, y_n + \sum_{j=1}^{i-1} \alpha_{ij} K_j\right), c_1 = 0, i = 2,3,\cdots,\nu$ (3) with h the step of the method. The parameters w_i, c_i and α_{ij} must be specified. In every R-K method the relations $\sum_{i=1}^{\nu} w_i = 1$ (4) και $c_i = \sum_{j=1}^{i-1} \alpha_{ij}, i = 2,3,\cdots,\nu$ (5) must be valid.

Runge-Kutta (7,9) method is a method of 7th order and 9 stages and we use the coefficients obtained for $r = 1,2,\cdots,7$ (6), where r is the order of ODE, from Butcher's Table [1] from whom the equations of the nonlinear 85 × 53 system result.

The values of w_i, c_i and α_{ij} will be found by the solving this system as well as the K_i and the reduction formula for the solution of the differential equation.

The equations of the system are numbered from (8), (9), ···, (92) and introducing the abbreviation:

$$P_{\kappa\lambda} = \alpha_{\kappa 2} c_2^{\lambda} + \alpha_{\kappa 3} c_3^{\lambda} + \cdots + \alpha_{\kappa\kappa-1} c_{\kappa-1}^{\lambda} \text{ with } \kappa = 3,4,\cdots,9 \text{ and } \lambda = 1,2,3,4,5 \quad (7)$$

[7] the following system is obtained:

$$\sum_{\kappa=1}^{9} w_\kappa = 1 \quad (8)$$

$$\sum_{\kappa=2}^{9} w_\kappa c_\kappa = \frac{1}{2} \quad (9)$$

$$\sum_{\kappa=3}^{9} w_\kappa P_{\kappa 1} = \frac{1}{6} \quad (10)$$

$$\sum_{\kappa=2}^{9} w_\kappa c_\kappa^2 = \frac{1}{3} \quad (11)$$

$$\sum_{\kappa=4}^{9} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{24} \quad (12)$$

$$\sum_{\kappa=3}^{9} w_\kappa P_{\kappa 2} = \frac{1}{12} \quad (13)$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa P_{\kappa 1} = \frac{1}{8} \quad (14)$$

$$\sum_{\kappa=2}^{9} w_\kappa c_\kappa^3 = \frac{1}{4} \quad (15)$$

$$\sum_{\kappa=5}^{9} w_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 1}\right)\right]=\frac{1}{120} \tag{16}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 2}\right)=\frac{1}{60} \tag{17}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}P_{\lambda 1}\right)=\frac{1}{40} \tag{18}$$

$$\sum_{\kappa=3}^{9} w_{\kappa}P_{\kappa 3}=\frac{1}{20} \tag{19}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}c_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 1}\right)=\frac{1}{30} \tag{20}$$

$$\sum_{\kappa=3}^{9} w_{\kappa}c_{\kappa}P_{\kappa 2}=\frac{1}{15} \tag{21}$$

$$\sum_{\kappa=3}^{9} w_{\kappa}P_{\kappa 1}^{2}=\frac{1}{20} \tag{22}$$

$$\sum_{\kappa=3}^{9} w_{\kappa}c_{\kappa}^{2}P_{\kappa 1}=\frac{1}{10} \tag{23}$$

$$\sum_{\kappa=2}^{9} w_{\kappa}c_{\kappa}^{4}=\frac{1}{5} \tag{24}$$

$$\sum_{\kappa=6}^{9} w_{\kappa}\left\{\sum_{\lambda=5}^{\kappa-1}\alpha_{\kappa\lambda}\left[\sum_{\mu=4}^{\lambda-1}\alpha_{\lambda\mu}\left(\sum_{\nu=3}^{\mu-1}\alpha_{\mu\nu}P_{\nu 1}\right)\right]\right\}=\frac{1}{720} \tag{25}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 2}\right)\right]=\frac{1}{360} \tag{26}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}c_{\mu}P_{\mu 1}\right)\right]=\frac{1}{240} \tag{27}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 1}\right)=\frac{1}{120} \tag{28}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 1}\right)\right]=\frac{1}{180} \tag{29}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}P_{\lambda 2}\right)=\frac{1}{90} \tag{30}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 1}^{2}\right)=\frac{1}{120} \tag{31}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}^{2}P_{\lambda 1}\right)=\frac{1}{60} \tag{32}$$

$$\sum_{\kappa=3}^{9} w_{\kappa}P_{\kappa 4}=\frac{1}{30} \tag{33}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}c_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 1}\right)\right]=\frac{1}{144} \tag{34}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}c_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 2}\right)=\frac{1}{72} \tag{35}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}c_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}P_{\lambda 1}\right)=\frac{1}{48} \tag{36}$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa P_{\kappa 3} = \frac{1}{24} \tag{37}$$

$$\sum_{\kappa=4}^{9} w_\kappa P_{\kappa 1} \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1} \right) = \frac{1}{72} \tag{38}$$

$$\sum_{\kappa=3}^{9} w_\kappa P_{\kappa 1} P_{\kappa 2} = \frac{1}{36} \tag{39}$$

$$\sum_{\kappa=4}^{9} w_\kappa c_\kappa^2 \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1} \right) = \frac{1}{36} \tag{40}$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa^2 P_{\kappa 2} = \frac{1}{18} \tag{41}$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa P_{\kappa 1}^2 = \frac{1}{24} \tag{42}$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa^3 P_{\kappa 1} = \frac{1}{12} \tag{43}$$

$$\sum_{\kappa=2}^{9} w_\kappa c_\kappa^5 = \frac{1}{6} \tag{44}$$

$$\sum_{\kappa=7}^{9} w_\kappa \left\langle \sum_{\lambda=6}^{\kappa-1} \alpha_{\kappa\lambda} \left\{ \sum_{\mu=5}^{\lambda-1} \alpha_{\lambda\mu} \left[\sum_{\nu=4}^{\mu-1} \alpha_{\mu\nu} \left(\sum_{\rho=3}^{\nu-1} \alpha_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle = \frac{1}{5040} \tag{45}$$

$$\sum_{\kappa=6}^{9} w_\kappa \left\{ \sum_{\lambda=5}^{\kappa-1} \alpha_{\kappa\lambda} \left[\sum_{\mu=4}^{\lambda-1} \alpha_{\lambda\mu} \left(\sum_{\nu=3}^{\mu-1} \alpha_{\mu\nu} P_{\nu 2} \right) \right] \right\} = \frac{1}{2520} \tag{46}$$

$$\sum_{\kappa=6}^{9} w_\kappa \left\{ \sum_{\lambda=5}^{\kappa-1} \alpha_{\kappa\lambda} \left[\sum_{\mu=4}^{\lambda-1} \alpha_{\lambda\mu} \left(\sum_{\nu=3}^{\mu-1} \alpha_{\mu\nu} c_\nu P_{\nu 1} \right) \right] \right\} = \frac{1}{1680} \tag{47}$$

$$\sum_{\kappa=5}^{9} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 3} \right) \right] = \frac{1}{840} \tag{48}$$

$$\sum_{\kappa=6}^{9} w_\kappa \left\{ \sum_{\lambda=5}^{\kappa-1} \alpha_{\kappa\lambda} \left[\sum_{\mu=4}^{\lambda-1} \alpha_{\lambda\mu} c_\mu \left(\sum_{\nu=3}^{\mu-1} \alpha_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{1260} \tag{49}$$

$$\sum_{\kappa=5}^{9} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} c_\mu P_{\mu 2} \right) \right] = \frac{1}{630} \tag{50}$$

$$\sum_{\kappa=5}^{9} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 1}^2 \right) \right] = \frac{1}{840} \tag{51}$$

$$\sum_{\kappa=5}^{9} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} c_\mu^2 P_{\mu 1} \right) \right] = \frac{1}{420} \tag{52}$$

$$\sum_{\kappa=4}^{9} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 4} \right) = \frac{1}{210} \tag{53}$$

$$\sum_{\kappa=6}^{9} w_\kappa \left\{ \sum_{\lambda=5}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda \left[\sum_{\mu=4}^{\lambda-1} \alpha_{\lambda\mu} \left(\sum_{\nu=3}^{\mu-1} \alpha_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{1008} \tag{54}$$

$$\sum_{\kappa=5}^{9} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 2} \right) \right] = \frac{1}{504} \tag{55}$$

$$\sum_{\kappa=5}^{9} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} c_\mu P_{\mu 1} \right) \right] = \frac{1}{336} \tag{56}$$

$$\sum_{\kappa=4}^{9} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda P_{\lambda 3} \right) = \frac{1}{168} \tag{57}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 1}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 1}\right)\right]=\frac{1}{504} \tag{58}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 1}P_{\lambda 3}\right)=\frac{1}{252} \tag{59}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}^{2}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 1}\right)\right]=\frac{1}{252} \tag{60}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}^{2}P_{\lambda 2}\right)=\frac{1}{126} \tag{61}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}P_{\lambda 1}^{2}\right)=\frac{1}{168} \tag{62}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}^{3}P_{\lambda 1}\right)=\frac{1}{84} \tag{63}$$

$$\sum_{\kappa=3}^{9} w_{\kappa}P_{\kappa 5}=\frac{1}{42} \tag{64}$$

$$\sum_{\kappa=6}^{9} w_{\kappa}c_{\kappa}\left\{\sum_{\lambda=5}^{\kappa-1}\alpha_{\kappa\lambda}\left[\sum_{\mu=4}^{\lambda-1}\alpha_{\lambda\mu}\left(\sum_{\nu=3}^{\mu-1}\alpha_{\mu\nu}P_{\nu 1}\right)\right]\right\}=\frac{1}{840} \tag{65}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}c_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 2}\right)\right]=\frac{1}{420} \tag{66}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}c_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}c_{\mu}P_{\mu 1}\right)\right]=\frac{1}{280} \tag{67}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}c_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 3}\right)=\frac{1}{140} \tag{68}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}c_{\kappa}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 1}\right)\right]=\frac{1}{210} \tag{69}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}c_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}P_{\lambda 2}\right)=\frac{1}{105} \tag{70}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}c_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 1}^{2}\right)=\frac{1}{140} \tag{71}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}c_{\kappa}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}^{2}P_{\lambda 1}\right)=\frac{1}{70} \tag{72}$$

$$\sum_{\kappa=3}^{9} w_{\kappa}c_{\kappa}P_{\kappa 4}=\frac{1}{35} \tag{73}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}P_{\kappa 1}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 1}\right)\right]=\frac{1}{336} \tag{74}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}P_{\kappa 1}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 2}\right)=\frac{1}{168} \tag{75}$$

$$\sum_{\kappa=4}^{9} w_{\kappa}P_{\kappa 1}\left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}c_{\lambda}P_{\lambda 1}\right)=\frac{1}{112} \tag{76}$$

$$\sum_{\kappa=3}^{9} w_{\kappa}P_{\kappa 1}P_{\kappa 3}=\frac{1}{56} \tag{77}$$

$$\sum_{\kappa=5}^{9} w_{\kappa}c_{\kappa}^{2}\left[\sum_{\lambda=4}^{\kappa-1}\alpha_{\kappa\lambda}\left(\sum_{\mu=3}^{\lambda-1}\alpha_{\lambda\mu}P_{\mu 1}\right)\right]=\frac{1}{168} \tag{78}$$

$$\sum_{\kappa=4}^{9} w_\kappa c_k^2 \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 2} \right) = \frac{1}{84} \tag{79}$$

$$\sum_{\kappa=4}^{9} w_\kappa c_k^2 \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda P_{\lambda 1} \right) = \frac{1}{56} \tag{80}$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa^2 P_{\kappa 3} = \frac{1}{28} \tag{81}$$

$$\sum_{\kappa=4}^{9} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1} \right)^2 = \frac{1}{252} \tag{82}$$

$$\sum_{\kappa=4}^{9} w_\kappa P_{\kappa 2} \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1} \right) = \frac{1}{126} \tag{83}$$

$$\sum_{\kappa=3}^{9} w_\kappa P_{\kappa 2}^2 = \frac{1}{63} \tag{84}$$

$$\sum_{\kappa=4}^{9} w_\kappa c_\kappa P_{\kappa 1} \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1} \right) = \frac{1}{84} \tag{85}$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa P_{\kappa 1} P_{\kappa 2} = \frac{1}{42} \tag{86}$$

$$\sum_{\kappa=4}^{9} w_\kappa c_\kappa^3 \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1} \right) = \frac{1}{42} \tag{87}$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa^3 P_{\kappa 2} = \frac{1}{21} \tag{88}$$

$$\sum_{\kappa=3}^{9} w_\kappa P_{\kappa 1}^3 = \frac{1}{56} \tag{89}$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa^2 P_{\kappa 1}^2 = \frac{1}{28} \tag{90}$$

$$\sum_{\kappa=3}^{9} w_\kappa c_\kappa^4 P_{\kappa 1} = \frac{1}{14} \tag{91}$$

$$\sum_{\kappa=2}^{9} w_\kappa c_\kappa^6 = \frac{1}{7} \tag{92}$$

In the system of (8), (9), (11), (15), (24), (44) and (92) equations *we set as* $c_2 = c_3 = 1/12$, $c_4 = 1/6$, $c_5 = 2/6$, $c_6 = 3/6$, $c_7 = 4/6$, $c_8 = 5/6$, $c_9 = 6/6$. The values of c_2, c_3, $\cdots$, c_9 are chosen to be in ascending order and as small and different from each other as possible. We set in addition $w_2 = w_3 = 0$ (93) and the resulting solution is:

$$w_1 = w_9 = \frac{41}{840} \tag{94}$$

$$w_4 = w_8 = \frac{216}{840} \tag{95}$$

$$w_5 = w_7 = \frac{27}{840} \tag{96}$$

and

$$w_6 = \frac{272}{840} \tag{97}$$

Since the above equations become somewhat lengthy, we introduce the following abbreviations: [7]

$$P_{41} = \alpha_{42}c_2 + \alpha_{43}c_3 = \frac{\alpha_{42} + \alpha_{43}}{12} = \frac{S_4}{12} \tag{98}$$

$$P_{51} = \alpha_{52}c_2 + \alpha_{53}c_3 + \alpha_{54}c_4 = \frac{\alpha_{52} + \alpha_{53} + 2\alpha_{54}}{12} = \frac{S_5}{12} \tag{99}$$

$$P_{61} = \frac{\alpha_{62} + \alpha_{63} + 2\alpha_{44} + 4\alpha_{65}}{12} = \frac{S_6}{12} \tag{100}$$

$$P_{71} = \frac{\alpha_{72} + \alpha_{73} + 2\alpha_{74} + 4\alpha_{75} + 6\alpha_{76}}{12} = \frac{S_7}{12} \tag{101}$$

$$P_{81} = \frac{\alpha_{82} + \alpha_{83} + 2\alpha_{84} + 4\alpha_{85} + 6\alpha_{86} + 8\alpha_{87}}{12} = \frac{S_8}{12} \tag{102}$$

$$P_{91} = \frac{\alpha_{92} + \alpha_{93} + 2\alpha_{94} + 4\alpha_{95} + 6\alpha_{96} + 8\alpha_{97} + 10\alpha_{98}}{12} = \frac{S_9}{12} \tag{103}$$

$$P_{42} = \alpha_{42}c_2^2 + \alpha_{43}c_3^2 = \frac{\alpha_{42} + \alpha_{43}}{144} = \frac{S_4}{144} \tag{104}$$

$$P_{52} = \alpha_{52}c_2^2 + \alpha_{53}c_3^2 + \alpha_{54}c_4^2 = \frac{\alpha_{52} + \alpha_{53} + 4\alpha_{54}}{144} = \frac{S_5 + 2\alpha_{54}}{144} \tag{105}$$

$$P_{62} = \alpha_{62}c_2^2 + \alpha_{63}c_3^2 + \alpha_{64}c_4^2 + \alpha_{65}c_5^2 = \frac{S_6 + 2\alpha_{64} + 12\alpha_{65}}{144} \tag{106}$$

$$P_{72} = \alpha_{72}c_2^2 + \alpha_{73}c_3^2 + \alpha_{74}c_4^2 + \alpha_{75}c_5^2 + \alpha_{76}c_6^2 = \frac{S_7 + 2\alpha_{74} + 12\alpha_{75} + 30a_{76}}{144} \tag{107}$$

$$P_{82} = \frac{S_8 + 2\alpha_{84} + 12\alpha_{85} + 30a_{86} + 56a_{87}}{144} \tag{108}$$

$$P_{92} = \frac{S_9 + 2\alpha_{94} + 12\alpha_{95} + 30a_{96} + 56a_{97} + 90a_{98}}{144} \tag{109}$$

$$P_{43} = \frac{a_{42} + a_{43}}{1728} = \frac{S_4}{1728} \tag{110}$$

$$P_{53} = \frac{a_{52} + a_{53} + 8a_{54}}{1728} = \frac{S_5 + 6\alpha_{54}}{1728} \tag{111}$$

$$P_{63} = \frac{S_6 + 6\alpha_{64} + 60\alpha_{65}}{1728} \tag{112}$$

$$P_{73} = \frac{S_7 + 6\alpha_{74} + 60\alpha_{75} + 210a_{76}}{1728} \tag{113}$$

$$P_{83} = \frac{S_8 + 6\alpha_{84} + 60\alpha_{85} + 210a_{86} + 504a_{87}}{1728} \tag{114}$$

$$P_{93} = \frac{S_9 + 6\alpha_{94} + 60\alpha_{95} + 210a_{96} + 504a_{97} + 990a_{98}}{1728} \tag{115}$$

$$P_{44} = \frac{S_4}{20736} \tag{116}$$

$$P_{54} = \frac{S_5 + 14\alpha_{54}}{20736} \tag{117}$$

$$P_{64} = \frac{S_6 + 14a_{64} + 252\alpha_{65}}{20736} \tag{118}$$

$$P_{74} = \frac{S_7 + 14a_{74} + 252\alpha_{75} + 1290a_{76}}{20736} \tag{119}$$

$$P_{84} = \frac{S_8 + 14a_{84} + 252\alpha_{85} + 1290a_{86} + 4088\alpha_{87}}{20736} \tag{120}$$

$$P_{94} = \frac{S_9 + 14a_{94} + 252\alpha_{95} + 1290a_{96} + 4088\alpha_{97} + 9990\alpha_{98}}{20736} \tag{121}$$

$$P_{45} = \frac{S_4}{248832} \tag{122}$$

$$P_{55} = \frac{S_5 + 30\alpha_{54}}{248832} \tag{123}$$

$$P_{65} = \frac{S_6 + 30\alpha_{64} + 1020\alpha_{65}}{248832} \tag{124}$$

$$P_{75} = \frac{S_7 + 30\alpha_{74} + 1020\alpha_{75} + 7770\alpha_{76}}{248832} \tag{125}$$

$$P_{85} = \frac{S_8 + 30\alpha_{84} + 1020\alpha_{85} + 7770\alpha_{86} + 32760\alpha_{87}}{248832} \tag{126}$$

$$P_{95} = \frac{S_9 + 30\alpha_{94} + 1020\alpha_{95} + 7770\alpha_{96} + 32760\alpha_{97} + 99990\alpha_{98}}{248832} \tag{127}$$

Then we substitute the defined abbreviations in the original system, as well as the found values of c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, and as a result the system is simplified.

In the system of (10), (14), (23), (43) and (91) we express S_5, S_6, S_7, S_8 and S_9 as a function of S_4 and by substituting them in (22) we find that: $S_4 = \frac{1}{6}$ (128) and $S_5 = \frac{4}{6}$ (129), $S_6 = \frac{9}{6}$ (130), $S_7 = \frac{16}{6}$ (131), $S_8 = \frac{25}{6}$ (132), $S_9 = \frac{36}{6} = 6$ (133).

To continue we set $\alpha_{42} = 0$ (134). From the abbreviation $S_4 = \alpha_{42} + \alpha_{43}$ and from the relation $\alpha_{41} + \alpha_{42} + \alpha_{43} = c_4$ we obtain that $\alpha_{43} = \frac{2}{12}$ (135) and $\alpha_{41} = 0$ (136).

In the system of equations (13), (18), (21), (32), (33), (36), (37), (39), (41), (62), (63), (64), (72), (73), (76), (77), (80), (81), (86) and (88) we substitute the values found above, omitting the equations which are a linear combination of equations of the system and also considering α_{94}, α_{95}, α_{96}, α_{97}, α_{98} as parameters, the 10 × 15 linear system (A) is obtained:

$$27\alpha_{54} = -822 + 41\left(\alpha_{94} + 6\alpha_{95} + 15\alpha_{96} + 28\alpha_{97} + 45\alpha_{98}\right) \tag{137}$$

$$272\alpha_{64} = 102 - 41\left(4\alpha_{94} + 18\alpha_{95} + 30\alpha_{96} + 28\alpha_{97}\right) \tag{138}$$

$$272\alpha_{65} = 653 - 41\left(\alpha_{95} + 5\alpha_{96} + 14\alpha_{97} + 30\alpha_{98}\right) \tag{139}$$

$$27\alpha_{74} = 102 + 41\left(6\alpha_{94} + 18\alpha_{95} + 15\alpha_{96}\right) \tag{140}$$

$$27\alpha_{75} = -208 + 41\left(3\alpha_{95} + 10\alpha_{96} + 14\alpha_{97}\right) \quad \text{SYSTEM (A)} \tag{141}$$

$$270\alpha_{76} = -2560 + 41\left(10\alpha_{96} + 56\alpha_{97} + 180\alpha_{98}\right) \tag{142}$$

$$216\alpha_{84} = -822 - 41\left(4\alpha_{94} + 6\alpha_{95}\right) \tag{143}$$

$$216\alpha_{85} = 653 - 41\left(3\alpha_{95} + 5\alpha_{96}\right) \tag{144}$$

$$216\left(5\alpha_{86}\right) = -1280 - 41\left(10\alpha_{96} + 28\alpha_{97}\right) \tag{145}$$

$$216\left(7\alpha_{87}\right) = 1683 - 41\left(7\alpha_{97} + 45\alpha_{98}\right) \tag{146}$$

From (20) and (85) equations we obtain:

$$\begin{aligned} &27\left(\alpha_{54}\right) + 272\left[4\left(\alpha_{64} + 4\alpha_{65}\right)\right] + 27\left[10\left(\alpha_{74} + 4\alpha_{75} + 9\alpha_{76}\right)\right] \\ &+ 216\left[20\left(\alpha_{84} + 4\alpha_{85} + 9\alpha_{86} + 16\alpha_{87}\right)\right] \\ &+ 41\left[35\left(\alpha_{94} + 4\alpha_{95} + 9\alpha_{96} + 16\alpha_{97} + 25\alpha_{98}\right)\right] = 49824 \end{aligned} \tag{147}$$

which along with the equations of system (A) and after setting:
$\alpha_{65} = \alpha_{75} = \alpha_{84} = \alpha_{86} = 0$ (148) results that:

$$\alpha_{54} = \frac{160}{9} \tag{149}$$

$$\alpha_{64} = -\frac{297}{68} \tag{150}$$

$$\alpha_{65} = 0 \tag{151}$$

$$\alpha_{74} = \frac{157}{3} \tag{152}$$

$$\alpha_{75} = 0 \tag{153}$$

$$\alpha_{76} = \frac{157}{45} \tag{154}$$

$$\alpha_{84} = 0 \tag{155}$$

$$\alpha_{85} = -\frac{65}{72} \tag{156}$$

$$\alpha_{86} = 0 \tag{157}$$

$$\alpha_{87} = \frac{29}{90} \tag{158}$$

$$\alpha_{94} = -\frac{1211}{326} \tag{159}$$

$$\alpha_{95} = -\frac{205}{237} \tag{160}$$

$$\alpha_{96} = \frac{419}{90} \tag{161}$$

$$\alpha_{96} = -\frac{25}{9} \tag{162}$$

$$\alpha_{98} = \frac{27}{25} \tag{163}$$

We found above $a_{43} = 1/6$ and setting $\alpha_{32} = \frac{11}{12}$ (164) ($\alpha_{31} = -\frac{10}{12}$ (165)), from the system of (30), (34), (58), (60) equations, implies that: $\alpha_{53} = \frac{4}{9}$ (166) $\alpha_{63} = \frac{18}{5}$ (167) $\alpha_{73} = -\frac{23}{2}$ (168) $\alpha_{83} = -\frac{3}{14}$ (169) and from Equation (12) result $\alpha_{93} = -\frac{119}{6}$ (170).

From the abbreviations: $S_5 = 4/6$, $S_6 = 9/6$, $S_7 = 16/6$, $S_8 = 25/6$ and $S_9 = 36/6 = 6$ results that: $\alpha_{52} = -\frac{106}{3}$ (171) $\alpha_{62} = \frac{199}{30}$ (172) $\alpha_{72} = -\frac{683}{6}$ (173) $\alpha_{82} = \frac{379}{70}$ (174) and $\alpha_{92} = \frac{283}{14}$ (175).

From relations $\sum_{j=1}^{i-1} \alpha_{ij} = c_i$, $i = 2,3,4,5,6,7,8,9$ we obtain: $\alpha_{51} = \frac{157}{9}$ (176) $\alpha_{61} = -\frac{161}{30}$ (177) $\alpha_{71} = \frac{3158}{45}$ (178) $\alpha_{81} = -\frac{53}{14}$ (179) and $\alpha_{91} = \frac{56}{25}$ (180).

According to the so-called Butcher's Table (Table 2) the (7,9) R-K method is given as below:

therefore

$$K_1 = hf\left(x_n, y_n\right) \tag{181}$$

$$K_2 = hf\left(x_n + \frac{h}{12}, y_n + \frac{K_1}{12}\right) \tag{182}$$

$$K_3 = hf\left(x_n + \frac{h}{12}, y_n + \frac{-10K_1 + 11K_2}{12}\right) \tag{183}$$

$$K_4 = hf\left(x_n + \frac{2h}{12}, y_n + \frac{2K_3}{12}\right) \tag{184}$$

Table 2. For choices values of arbitrary constants. $c_2 = c_3 = 1/12$, $c_4 = 1/6$, $c_5 = 2/6$, $c_6 = 3/6$, $c_7 = 4/6$, $c_8 = 5/6$, $c_9 = 6/6$.

0									
1/12	1/12								
1/12	−10/12	11/12							
2/12	0	0	2/12						
4/12	157/9	−318/9	4/9	160/9					
6/12	−161/30	199/30	108/30	−131/30	0				
8/12	3158/45	−683/6	−69/6	314/6	0	157/45			
10/12	−265/70	379/70	−15/70	0	−65/72	0	29/90		
12/12	56/25	849/42	−833/42	−156/42	−39/45	149/32	−125/45	27/25	
	41/840	0	0	216/840	27/840	272/840	27/840	216/840	41/840

$$K_5 = hf\left(x_n + \frac{4h}{12}, y_n + \frac{157K_1 - 318K_2 + 4K_3 + 160K_4}{9}\right) \quad (185)$$

$$K_6 = hf\left(x_n + \frac{6h}{12}, y_n + \frac{-322K_1 + 199K_2 + 108K_3 - 131K_5}{30}\right) \quad (186)$$

$$K_7 = hf\left(x_n + \frac{8h}{12}, y_n + \frac{3158K_1}{45} - \frac{638K_2}{6} - \frac{23K_3}{2} + \frac{157K_4}{3} + \frac{157K_6}{45}\right) \quad (187)$$

$$K_8 = hf\left(x_n + \frac{10h}{12}, y_n - \frac{53K_1}{14} + \frac{38K_2}{7} - \frac{3K_3}{14} - \frac{65K_5}{72} + \frac{29K_7}{90}\right) \quad (188)$$

$$K_9 = hf\left(x_n + h, y_n + \frac{56K_1}{25} + \frac{283k_2}{14} - \frac{119K_3}{6} - \frac{26K_4}{7} - \frac{13K_5}{15} + \frac{149K_6}{32} - \frac{25K_7}{9} + \frac{27K_8}{25}\right) \quad (189)$$

and the reduction formula for the solution of the Differential Equation is:

$$y_{n+1} = y_n + \frac{41K_1 + 216K_4 + 27K_5 + 272K_6 + 27K_7 + 216K_8 + 41K_9}{840} \quad (190)$$

3. Conclusion

This paper is concerned with training the coefficients of a 7th order and 9 stages Runge-Kutta method for addressing initial value problems. As the presented method is 9 stages, we use a set of 9 free parameters. After optimizing the free parameters (coefficients), we concluded to a certain set of values of them. This set of values was found to outperform other representatives in a wide range of relevant problems.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Derivation of the Reduction Formula of Sixth Order and Seven Stages Runge-Kutta Method for the Solution of an Ordinary Differential Equation

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Abstract

This paper is describing in detail the way we define the equations which give the formulas in the methods Runge-Kutta 6th order 7 stages with the incorporated control step size in the numerical solution of Ordinary Differential Equations (ODE). The purpose of the present work is to construct a system of nonlinear equations and then by solving this system to find the values of all set parameters and finally the reduction formula of the Runge-Kutta (6,7) method (6th order and 7 stages) for the solution of an Ordinary Differential Equation (ODE). Since the system of high order conditions required to be solved is complicated, all coefficients are found with respect to 7 free parameters. These free parameters, as well as some others in addition, are adjusted in such a way to furnish more efficient R-K methods. We use the MATLAB software to solve several of the created subsystems for the comparison of our results which have been solved analytically. Some examples for five different choices of the arbitrary values of the systems are presented in this paper.

Keywords

Initial Value Problem, Runge-Kutta Methods, Ordinary Differential Equations

1. Introduction

A system of ordinary differential equations of the form:

$$y' = f(x, y), y(x_0) = y_0 \tag{1}$$

with $x_0 \in \mathbb{R}$, $y, y' \in \mathbb{R}^m$ and $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is called Initial Value Prob-

lem. Some related references are cited in the bibliography part [1] [2].

Carl David Tolmé Runge [3] and Martin Wilhelm Kutta [4] introduced the methods bearing their names almost in the turning of the 19th century. These methods are very practical and popular one-step methods for solving Ordinary Differential Equations.

The reduction formula [5] of the method Runge-Kutta for the ODE $y'(x)=f(x,y)$, $y(x_0)=y_0$ is given by the relation $y_{n+1}=y_n+\sum_{i=1}^{\nu} w_i K_i$ (2) with w_i as weighting factors, ν is the number of stages and $K_i=hf\left(x_n+c_i h, y_n+\sum_{j=1}^{i-1}\alpha_{ij}K_i\right)$ for $i=1,2,\cdots,\nu$ and $j=1,2,\cdots,i-1$ (3) with h the step size of this method. The values of w_i and expressions K_i will be determined by the values of the unknown variables α_{ij} and c_i. In every R-K method the relations $\sum_{i=1} w_i=1$ and $c_i=\sum_{j=1}^{i-1}\alpha_{ij}$ for $i=2,3,\cdots,\nu$ (4) must be valid.

High order R-K methods have been presented by Butcher, Shanks, Lawson, Fehlberg, Feagin, Hairer, Curtis and others.

The motivation for this work is to create high order R-K (6,7) methods in a simple and practical way. In the present work after the Introduction follow the paragraph with the Presentation of the R-K method 6th order and 7 stages, the choices of the arbitrary parameters (5 choices), the detailed presentation and the solutions of the respective systems that result, as well as the tables with the found values of the coefficients of the R-K method (6,7) and the summary expressions of K_i. Finally, the Conclusions follow.

2. Presentation of the Method

From BUTCHER'S TABLE I (**Table 1**) [6] the equations of the nonlinear system result, where the first column represent the **order of the method**, the second column the **symbolism** of the function $f(x,y)$ and its derivatives, and the third column and so on the number of **coefficients**, for the study of Runge-Kutta methods [6], of the equations of the nonlinear system. The values of w_i, c_i and α_{ij} will be found by the solving this system as well as the K_i and the reduction formula for the solution of the ordinary differential equation.

The equations of the nonlinear system in their general form are: [7]

$$\sum_{\kappa=1}^{\omega} w_\kappa = 1 \tag{5}$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa = \frac{1}{2} \tag{6}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 1} = \frac{1}{6} \tag{7}$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa^2 = \frac{1}{3} \tag{8}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1}\alpha_{\kappa\lambda}P_{\lambda 1}\right) = \frac{1}{24} \tag{9}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 2} = \frac{1}{12} \tag{10}$$

Table 1. The resulted Butcher's table.

(I)	F	1	1	1
(II)	$\{f\}$	1	1	2
(III)	$\{_2f\}_2$	1	2	6
	$\{f^2\}$	1	1	3
(IV)	$\{_3f\}_3$	1	6	24
	$\{_2f^2\}_2$	1	3	12
	$\{\{f\}f\}$	3	6	8
	$\{f^3\}$	1	1	4
(V)	$\{_4f\}_4$	1	24	120
	$\{_3f^2\}_3$	1	12	60
	$\{_2\{f\}f\}_2$	3	24	40
	$\{_2f^3\}_2$	1	4	20
	$\{\{_2f\}_2f\}$	4	24	30
	$\{\{f^2\}f\}$	4	12	15
	$\{\{f\}^2\}$	3	12	20
	$\{\{f\}f^2\}$	6	12	10
	$\{f^4\}$	1	1	5
(VI)	$\{_5f\}_5$	1	120	720
	$\{_4f^2\}_4$	1	60	360
	$\{_3\{f\}f\}_3$	3	120	240
	$\{_3f^3\}_3$	1	20	120
	$\{_2\{_2f\}_2f\}_2$	4	120	180
	$\{_3\{f^2\}f\}_3$	4	60	90
	$\{_2\{f\}^2\}_2$	3	60	120
	$\{_2\{f\}f^2\}_2$	6	60	60
	$\{_2f^4\}_2$	1	5	30
	$\{\{_3f\}_3f\}$	5	120	144
	$\{\{_2f^2\}_2f\}$	5	60	72
	$\{\{\{f\}f\}f\}$	15	120	48
	$\{\{f^3\}f\}$	5	20	24
	$\{\{_2f\}_2\{f\}\}$	10	120	72
	$\{\{f^2\}\{f\}\}$	10	60	36
	$\{\{_2f\}_2f^2\}$	10	60	36
	$\{\{f^2\}f^2\}$	10	30	18
	$\{\{f\}^2f\}$	15	60	24
	$\{\{f\}f^3\}$	10	20	12
	$\{f^5\}$	1	1	6

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa P_{\kappa 1} = \frac{1}{8} \tag{11}$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa^3 = \frac{1}{4} \tag{12}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 1} \right) \right] = \frac{1}{120} \tag{13}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\omega} \alpha_{\kappa\lambda} P_{\lambda 2} \right) = \frac{1}{60} \tag{14}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda P_{\lambda 1} \right) = \frac{1}{40} \tag{15}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 3} = \frac{1}{20} \tag{16}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa c_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1} \right) = \frac{1}{30} \tag{17}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa P_{\kappa 2} = \frac{1}{15} \tag{18}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 1}^2 = \frac{1}{20} \tag{19}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa^2 P_{\kappa 1} = \frac{1}{10} \tag{20}$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa^4 = \frac{1}{5} \tag{21}$$

$$\sum_{\kappa=6}^{\omega} w_\kappa \left\{ \sum_{\lambda=5}^{\kappa-1} \alpha_{\kappa\lambda} \left[\sum_{\mu=4}^{\lambda-1} \alpha_{\lambda\mu} \left(\sum_{\nu=3}^{\mu-1} \alpha_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{720} \tag{22}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 2} \right) \right] = \frac{1}{360} \tag{23}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} c_\mu P_{\mu 1} \right) \right] = \frac{1}{240} \tag{24}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 2} \right) = \frac{1}{120} \tag{25}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 1} \right) \right] = \frac{1}{180} \tag{26}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda P_{\lambda 2} \right) = \frac{1}{90} \tag{27}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1}^2 \right) = \frac{1}{120} \tag{28}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda^2 P_{\lambda 1} \right) = \frac{1}{60} \tag{29}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 4} = \frac{1}{30} \tag{30}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa c_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 1} \right) \right] = \frac{1}{144} \tag{31}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa c_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 2}\right) = \frac{1}{72} \tag{32}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa c_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda P_{\lambda 1}\right) = \frac{1}{48} \tag{33}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa P_{\kappa 3} = \frac{1}{24} \tag{34}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa P_{\kappa 1} \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{72} \tag{35}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 1} P_{\kappa 2} = \frac{1}{36} \tag{36}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa c_\kappa^2 \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{36} \tag{37}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa^2 P_{\kappa 2} = \frac{1}{18} \tag{38}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa P_{\kappa 1}^2 = \frac{1}{24} \tag{39}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa^3 P_{\kappa 1} = \frac{1}{12} \tag{40}$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa^5 = \frac{1}{6} \tag{41}$$

with

$$\omega = 7\,, \tag{42}$$

$$P_{\kappa\lambda} = \alpha_{\kappa 2} c_2^\lambda + \alpha_{\kappa 3} c_3^\lambda + \cdots + \alpha_{\kappa\kappa-1} c_{\kappa-1}^\lambda \text{ with } \kappa = 3,4,\cdots,7 \text{ and } \lambda = 1,2,3,4 \text{ [5]} \tag{43}$$

2.1. 1st Choice: $c_2 = c_4 = 1/3$, $c_3 = 2/3$, $c_5 = c_6 = 1/2$, $c_7 = 1$ [5]

From the system of (5), (6), (8), (12) και (41) results that:

$w_1 = \frac{11}{120}$ (44) $w_2 + w_4 = \frac{27}{40}$ (45) $w_3 = \frac{27}{40}$ (46) $w_5 + w_6 = -\frac{8}{15}$ (47) $w_7 = \frac{11}{120}$ (48) and setting $w_2 = 0$ (49) and $w_5 = w_6$ (50) we have $w_4 = \frac{27}{40}$ (51) and $w_5 = w_6 = -\frac{4}{15}$ (52) while the Equation (21) verified.

Since the above equations become somewhat lengthy, we introduce the following abbreviations: [7]

$$\alpha_{42} c_2 + a_{43} c_3 = \frac{\alpha_{42} + 2\alpha_{43}}{3} = \frac{S_4}{3} \tag{53}$$

$$\alpha_{52} c_2 + a_{53} c_3 + \alpha_{54} c_4 = \frac{\alpha_{52} + 2\alpha_{53} + \alpha_{54}}{3} = \frac{S_5}{3} \tag{54}$$

$$\alpha_{62} c_2 + a_{63} c_3 + \alpha_{64} c_4 + \alpha_{65} c_5 = \frac{2\alpha_{62} + 4\alpha_{63} + 2\alpha_{64} + 3\alpha_{65}}{6} = \frac{S_6}{6} \tag{55}$$

$$\alpha_{72} c_2 + a_{73} c_3 + \alpha_{74} c_4 + \alpha_{75} c_5 + \alpha_{76} c_6 = \frac{2\alpha_{72} + 4\alpha_{73} + 2\alpha_{74} + 3\alpha_{75} + 3\alpha_{76}}{6} = \frac{S_7}{6} \tag{56}$$

and

$$\alpha_{42}c_2^2 + a_{43}c_3^2 = \frac{\alpha_{42} + 4\alpha_{43}}{9} = \frac{S_4 + 2\alpha_{43}}{9} \tag{57}$$

$$\alpha_{52}c_2^2 + a_{53}c_3^2 + \alpha_{54}c_4^2 = \frac{S_5 + 2\alpha_{53}}{9} \tag{58}$$

$$\alpha_{62}c_2^2 + a_{63}c_3^2 + \alpha_{64}c_4^2 + \alpha_{65}c_5^2 = \frac{2S_6 + 8\alpha_{63} + 3\alpha_{65}}{36} \tag{59}$$

$$\alpha_{72}c_2^2 + a_{73}c_3^2 + \alpha_{74}c_4^2 + \alpha_{75}c_5^2 + a_{76}c_6^2 = \frac{2S_7 + 8\alpha_{73} + 3\alpha_{75} + 3\alpha_{76}}{36} \tag{60}$$

and

$$\alpha_{42}c_2^3 + a_{43}c_3^3 = \frac{S_4 + 6\alpha_{43}}{27} \tag{61}$$

$$\alpha_{52}c_2^3 + a_{53}c_3^3 + a_{54}c_4^3 = \frac{S_5 + 6\alpha_{53}}{27} \tag{62}$$

$$\alpha_{62}c_2^3 + a_{63}c_3^3 + \alpha_{64}c_4^3 + \alpha_{65}c_5^3 = \frac{4S_6 + 48\alpha_{63} + 15\alpha_{65}}{216} \tag{63}$$

$$\alpha_{72}c_2^3 + a_{73}c_3^3 + \alpha_{74}c_4^3 + \alpha_{75}c_5^3 + a_{76}c_6^3 = \frac{4S_7 + 48\alpha_{73} + 15\alpha_{75} + 15\alpha_{76}}{216} \tag{64}$$

and

$$\alpha_{42}c_2^4 + a_{43}c_3^4 = \frac{S_4 + 14\alpha_{43}}{81} \tag{65}$$

$$\alpha_{52}c_2^4 + a_{53}c_3^4 + \alpha_{54}c_4^4 = \frac{S_5 + 14\alpha_{53}}{81} \tag{66}$$

$$\alpha_{62}c_2^4 + a_{63}c_3^4 + \alpha_{64}c_4^4 + \alpha_{65}c_5^4 = \frac{8S_6 + 224\alpha_{63} + 57\alpha_{65}}{1296} \tag{67}$$

$$\alpha_{72}c_2^4 + a_{73}c_3^4 + \alpha_{74}c_4^4 + \alpha_{75}c_5^4 + a_{76}c_6^4 = \frac{8S_7 + 224\alpha_{73} + 57\alpha_{75} + 57\alpha_{76}}{1296} \tag{68}$$

Then we substitute the defined abbreviations in the original system, as well as the found values of c_2, c_3, c_4, c_5, c_6, c_7, w_2, w_3, w_4, w_5, w_6, w_7, and as a result the system is simplified. Also omitting the equations which are a linear combination of equations of the system the 30 × 15 system is obtained:

$$162\alpha_{32} + 162S_4 - 64S_5 - 32S_6 + 11S_7 = 120 \tag{69}$$

$$\begin{aligned} &162\alpha_{43}\alpha_{32} - 64\left(\alpha_{53}\alpha_{32} + \alpha_{54}S_4\right) - 64\left(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5\right) \\ &+11\left(2\alpha_{73}\alpha_{32} + 2\alpha_{74}S_4 + 2\alpha_{75}S_5 + \alpha_{76}S_6\right) = 30 \end{aligned} \tag{70}$$

$$648\alpha_{43} - 256\alpha_{53} - 32\left(8\alpha_{63} + 3\alpha_{65}\right) + 11\left(8\alpha_{73} + 3\alpha_{75} + 3\alpha_{76}\right) = 120 \tag{71}$$

$$108\alpha_{32} + 54S_4 - 32S_5 - 16S_6 + 11S_7 = 90 \tag{72}$$

$$\begin{aligned} &-32\alpha_{54}\alpha_{43}\alpha_{32} - 32\left[\alpha_{64}\alpha_{43}\alpha_{32} + \alpha_{65}\left(\alpha_{53}\alpha_{32} + \alpha_{54}S_4\right)\right] \\ &+11\left[\alpha_{74}\alpha_{43}\alpha_{32} + \alpha_{75}\left(\alpha_{53}\alpha_{32} + \alpha_{54}S_4\right) + \alpha_{76}\left(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5\right)\right] = 3 \end{aligned} \tag{73}$$

$$256\left(\alpha_{54}\alpha_{43}+\alpha_{64}\alpha_{43}+\alpha_{65}\alpha_{53}\right)-88\left(\alpha_{74}\alpha_{43}+\alpha_{75}\alpha_{53}+\alpha_{76}\alpha_{63}\right)-33\alpha_{76}\alpha_{65}=-12 \tag{74}$$

$$128\alpha_{54}S_4+128\,\alpha_{64}S_4+64\alpha_{65}S_5-44\alpha_{74}S_4-22\alpha_{75}S_5-11\alpha_{76}S_6=-12 \tag{75}$$

$$32\,\alpha_{65}-11\alpha_{75}-11\alpha_{76}=32 \tag{76}$$

$$54\,\alpha_{43}\alpha_{32}-16\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)-16\left(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5\right)=3 \tag{77}$$

$$108\,\alpha_{43}-32\,\alpha_{53}-32\,\alpha_{63}-12\,\alpha_{65}=3 \tag{78}$$

$$324\,\alpha_{32}^2+324\,S_4^2-128\,S_5^2-32\,S_6^2+11S_7^2=216 \tag{79}$$

$$72\,\alpha_{32}+18\,S_4-16\,S_5-8\,S_6+11S_7=72 \tag{80}$$

$$64\,\alpha_{65}\alpha_{54}\alpha_{43}\alpha_{32}-22\left\{\alpha_{75}\alpha_{54}\alpha_{43}\alpha_{32}+\alpha_{76}\left[\alpha_{64}\alpha_{43}\alpha_{32}+\alpha_{65}\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)\right]\right\}=-1 \tag{81}$$

$$32\,\alpha_{65}\alpha_{54}\alpha_{43}-11\left[\alpha_{75}\alpha_{54}\alpha_{43}+\alpha_{76}\left(\alpha_{64}\alpha_{43}+\alpha_{65}\alpha_{53}\right)\right]=0 \tag{82}$$

$$64\,\alpha_{65}\alpha_{54}S_4-22\,\alpha_{75}\alpha_{54}S_4-11\alpha_{76}\left(2\alpha_{64}S_4+\alpha_{65}S_5\right)=-3 \tag{83}$$

$$11\alpha_{76}\alpha_{65}=-8 \tag{84}$$

$$64\,\alpha_{65}\alpha_{53}\alpha_{32}-11\left[2\alpha_{75}\alpha_{53}\alpha_{32}+\alpha_{76}\left(2\alpha_{63}\alpha_{32}+\alpha_{65}S_5\right)\right]=-9 \tag{85}$$

$$648\,\alpha_{43}\alpha_{32}-256\,\alpha_{53}\alpha_{32}-128\left[2\alpha_{63}\alpha_{32}+\alpha_{65}\left(2\alpha_{53}+S_5\right)\right]+11\left[8\alpha_{73}\alpha_{32}+4\alpha_{75}\left(2\alpha_{53}+S_5\right)+\alpha_{76}\left(8\alpha_{63}+3\alpha_{65}+2S_6\right)\right]=144 \tag{86}$$

$$324\,\alpha_{43}\alpha_{32}^2-128\left(\alpha_{53}\alpha_{32}^2+\alpha_{54}S_4^2\right)-128\left(\alpha_{63}\alpha_{32}^2+\alpha_{64}S_4^2+\alpha_{65}S_5^2\right)+11\left(4\alpha_{73}\alpha_{32}^2+4\alpha_{74}S_4^2+4\alpha_{75}S_5^2+\alpha_{76}S_6^2\right)=36 \tag{87}$$

$$1944\,\alpha_{43}\alpha_{32}-768\,\alpha_{53}\alpha_{32}-768\,\alpha_{63}\alpha_{32}-320\,\alpha_{65}S_5+264\,a_{73}a_{32}+110\,\alpha_{75}S_5+55\,\alpha_{76}S_6=312 \tag{88}$$

$$11\left[\alpha_{74}\alpha_{43}\alpha_{32}+\alpha_{75}\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)+\alpha_{76}\left(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5\right)\right]=2 \tag{89}$$

$$-54\,\alpha_{43}\alpha_{32}+128\,\alpha_{54}\alpha_{43}+128\left(\alpha_{64}\alpha_{43}+\alpha_{65}\alpha_{53}\right)+11\left(2\alpha_{73}\alpha_{32}+2\alpha_{74}S_4+2\alpha_{75}S_5+2\alpha_{76}S_6\right)=18 \tag{90}$$

$$-216\,\alpha_{43}\alpha_{32}+88\,\alpha_{73}\alpha_{32}+44\,\alpha_{74}S_4+66\,\alpha_{75}S_5+33\,\alpha_{76}S_6=72 \tag{91}$$

$$-54\,\alpha_{43}+22\,\alpha_{65}+22\,\alpha_{73}=47 \tag{92}$$

$$324\,S_4\alpha_{43}\alpha_{32}-128\,S_5\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)-64\,S_6\left(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5\right)+11S_7\left(2\alpha_{73}\alpha_{32}+2\alpha_{74}S_4+2\alpha_{75}S_5+\alpha_{76}S_6\right)=60 \tag{93}$$

$$1296\,\alpha_{43}S_4-512\,\alpha_{53}S_5-\left(256\alpha_{63}+396\alpha_{65}\right)S_6+\left(88\alpha_{73}+33\alpha_{75}+33\alpha_{76}\right)S_7=288 \tag{94}$$

$$18\,\alpha_{43}\alpha_{32}=-1 \tag{95}$$

$$72\,\alpha_{43}-16\,\alpha_{53}-16\,\alpha_{63}-4\,\alpha_{65}=1 \tag{96}$$

$$108\,\alpha_{32}^2-108\,S_4^2+11S_7^2=144 \tag{97}$$

$$24\,\alpha_{32}-6\,S_4+11S_7=48 \tag{98}$$

From (69), (72), (80) και (98) result:

$$\alpha_{52} = \frac{2}{3} \tag{99}$$

$$S_4 = \frac{1}{6} \tag{100}$$

$$S_7 = 3 \tag{101}$$

and

$$4S_5 + 2S_6 = 3 \tag{102}$$

From (79) we find:

$$32S_5^2 + 8S_6^2 = 9 \tag{103}$$

so:

$$S_5 = \frac{3}{8} \tag{104}$$

$$S_6 = \frac{3}{4} \tag{105}$$

From (71), (76), (78), (92) and (96) result:

$$a_{43} = -\frac{1}{12} \tag{106}$$

$$a_{65} = \frac{1}{2} \tag{107}$$

$$a_{73} = \frac{63}{44} \tag{108}$$

and

$$a_{53} + a_{63} = -\frac{9}{16} \tag{109}$$

$$a_{75} + a_{76} = -\frac{16}{11} \tag{110}$$

From (84):

$$a_{76} = -\frac{16}{11} \tag{111}$$

and from (110):

$$a_{75} = 0 \tag{112}$$

while the Equations (95) and (97) are verified.

By substitution the found values from (75), (77) and (109) we find

$$a_{74} = \frac{18}{11} \tag{113}$$

and

$$a_{54} + a_{64} = -\frac{9}{8} \tag{114}$$

from (81):

$$4a_{53} + a_{54} = -\frac{9}{8} \tag{115}$$

and from (90):

$$192a_{53} - 32\left(a_{54} + a_{64}\right) = 0 \tag{116}$$

The solution of the system of Equations (109), (114), (115) and (116) is:

$$a_{53} = -\frac{3}{16} \tag{117}$$

$$a_{54} = -\frac{6}{16} \tag{118}$$

$$a_{63} = -\frac{6}{16} \tag{119}$$

and

$$a_{64} = -\frac{12}{16} \tag{120}$$

Using the defined abbreviations $S_4 = \alpha_{42} + 2\alpha_{43}$, $S_5 = \alpha_{52} + 2\alpha_{53} + \alpha_{54}$ $S_6 = 2\alpha_{62} + 4\alpha_{63} + 2\alpha_{64} + 3\alpha_{65}$ and $S_7 = 2\alpha_{72} + 4\alpha_{73} + 2\alpha_{74} + 3\alpha_{75} + 3\alpha_{76}$ result:

$$a_{42} = \frac{4}{12} \tag{121}$$

$$a_{52} = \frac{18}{16} \tag{122}$$

$$a_{62} = \frac{9}{8} \tag{123}$$

$$a_{72} = -\frac{9}{11} \tag{124}$$

From the relations $\sum_{j=1}^{i-1} \alpha_{ij} = c_i$, for $i = 2,3,4,5,6,7$ result:

$$a_{21} = \frac{1}{3} \tag{125}$$

$$a_{41} = \frac{1}{12} \tag{126}$$

$$a_{51} = -\frac{1}{16} \tag{127}$$

$$a_{61} = 0 \tag{128}$$

and

$$a_{71} = \frac{9}{44} \tag{129}$$

The remaining equations are verified.

Runge-Kutta methods usually presented in a so-called Butcher table (**Table 2**) [8].

Table 2. The so-called Butcher table.

c	A
	w^{T}

The table contains on the 1st column the coefficients c_i, the matrix A with the coefficients of a_{ij}, which appear in the Formulae of K_i, and w_i the coefficients in Formula of y_{i+1}.

In this type of table, we have $\boldsymbol{w}^{\mathrm{T}}, \boldsymbol{c} \in \mathbb{R}^{\mathrm{m}}$ while $\boldsymbol{A} \in \mathbb{R}^{m \times m}$. Then, the method shares m stages and in case that $c_1 = 0$ and $\boldsymbol{A}$ [5] is strictly lower triangular, it is evaluated explicitly.

Therefore we get **Table 3**:

with

$$K_1 = hf\left(x_n, y_n\right) \tag{130}$$

$$K_2 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1}{3}\right) \tag{131}$$

$$K_3 = hf\left(x_n + \frac{2h}{3}, y_n + \frac{2K_2}{3}\right) \tag{132}$$

$$K_4 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1 + 4K_2 - K_3}{12}\right) \tag{133}$$

$$K_5 = hf\left(x_n + \frac{h}{2}, y_n + \frac{-K_1 + 18K_2 - 3K_3 - 6K_4}{16}\right) \tag{134}$$

$$K_6 = hf\left(x_n + \frac{h}{2}, y_n + \frac{9K_2 - 3K_3 - 6K_4 + 4K_5}{8}\right) \tag{135}$$

$$K_7 = hf\left(x_n + h, y_n + \frac{9K_1 - 36K_2 + 63K_3 + 72K_4 - 64K_5}{44}\right) \tag{136}$$

and

$$y_{n+1} = y_n + \frac{11K_1 + 81K_3 + 81K_4 - 32K_5 - 32K_6 + 11K_7}{120} \tag{137}$$

2.2. 2nd Choice: $c_2 = c_3 = 1/4$, $c_4 = 2/4$, $c_5 = c_6 = 3/4$, $c_7 = 4/4$ [5]

From the system of (5), (6), (8), (12) και (41) results that:

$$w_1 = \frac{7}{90} \tag{138}$$

$$w_2 + w_3 = \frac{32}{90} \tag{139}$$

$$w_4 = \frac{12}{90} \tag{140}$$

$$w_5 + w_6 = \frac{32}{90} \tag{141}$$

Table 3. For choices values of arbitrary constants: $c_2 = c_4 = 1/3$, $c_3 = 2/3$, $c_5 = c_6 = 1/2$, $c_7 = 1$.

0							
1/3	1/3						
2/3	0	2/3					
1/3	1/12	4/12	−1/12				
1/2	−1/16	18/16	−3/16	−6/16			
1/2	0	9/8	−3/8	−6/8	4/8		
1	9/44	−36/44	63/44	72/44	0	−64/44	
	11/120	0	81/120	81/120	−32/120	−32/120	11/120

$w_7 = \frac{7}{90}$ (142) and setting $w_2 = 0$ (143) and $w_5 = w_6$ (144) we get $w_3 = \frac{32}{90}$ (145) and $w_5 = w_6 = \frac{16}{90}$ (146) while the Equation (21) verified.

Since the above equations become somewhat lengthy, we introduce the following abbreviations

$$\alpha_{42}c_2 + a_{43}c_3 = \frac{\alpha_{42} + \alpha_{43}}{4} = \frac{S_4}{4} \tag{147}$$

$$\alpha_{52}c_2 + a_{53}c_3 + \alpha_{54}c_4 = \frac{\alpha_{52} + \alpha_{53} + 2\alpha_{54}}{4} = \frac{S_5}{4} \tag{148}$$

$$\alpha_{62}c_2 + a_{63}c_3 + \alpha_{64}c_4 + \alpha_{65}c_5 = \frac{\alpha_{62} + \alpha_{63} + 2\alpha_{64} + 3\alpha_{65}}{4} = \frac{S_6}{4} \tag{149}$$

$$\alpha_{72}c_2 + a_{73}c_3 + \alpha_{74}c_4 + \alpha_{75}c_5 + \alpha_{76}c_6 = \frac{\alpha_{72} + \alpha_{73} + 2\alpha_{74} + 3\alpha_{75} + 3\alpha_{76}}{4} = \frac{S_7}{4} \tag{150}$$

and

$$\alpha_{42}c_2^2 + a_{43}c_3^2 = \frac{\alpha_{42} + \alpha_{43}}{16} = \frac{S_4}{16} \tag{151}$$

$$\alpha_{52}c_2^2 + a_{53}c_3^2 + \alpha_{54}c_4^2 = \frac{S_5 + 2\alpha_{54}}{16} \tag{152}$$

$$\alpha_{62}c_2^2 + a_{63}c_3^2 + \alpha_{64}c_4^2 + \alpha_{65}c_5^2 = \frac{S_6 + 2\alpha_{64} + 6\alpha_{65}}{16} \tag{153}$$

$$\alpha_{72}c_2^2 + a_{73}c_3^2 + \alpha_{74}c_4^2 + \alpha_{75}c_5^2 + a_{76}c_6^2 = \frac{S_7 + 2\alpha_{74} + 6\alpha_{75} + 6\alpha_{76}}{16} \tag{154}$$

and

$$\alpha_{42}c_2^3 + a_{43}c_3^3 = \frac{S_4}{64} \tag{155}$$

$$\alpha_{52}c_2^3 + a_{53}c_3^3 + \alpha_{54}c_4^3 = \frac{S_5 + 6\alpha_{54}}{64} \tag{156}$$

$$\alpha_{62}c_2^3 + a_{63}c_3^3 + \alpha_{64}c_4^3 + \alpha_{65}c_5^3 = \frac{S_6 + \alpha_{64} + 24\alpha_{65}}{64} \tag{157}$$

$$\alpha_{72}c_2^3 + a_{73}c_3^3 + \alpha_{74}c_4^3 + \alpha_{75}c_5^3 + a_{76}c_6^3 = \frac{S_7 + 6\alpha_{74} + 24\alpha_{75} + 24\alpha_{76}}{64} \tag{158}$$

and

$$\alpha_{42}c_2^4 + a_{43}c_3^4 = \frac{S_4}{256} \tag{159}$$

$$\alpha_{52}c_2^4 + a_{53}c_3^4 + \alpha_{54}c_4^4 = \frac{S_5 + 14\alpha_{54}}{256} \tag{160}$$

$$\alpha_{62}c_2^4 + a_{63}c_3^4 + \alpha_{64}c_4^4 + \alpha_{65}c_5^4 = \frac{S_6 + 4\alpha_{64} + 78\alpha_{65}}{256} \tag{161}$$

$$\alpha_{72}c_2^4 + a_{73}c_3^4 + \alpha_{74}c_4^4 + \alpha_{75}c_5^4 + a_{76}c_6^4 = \frac{S_7 + 14\alpha_{74} + 78\alpha_{75} + 78\alpha_{76}}{256} \tag{162}$$

Then we substitute the defined abbreviations in the original system, as well as the found values of c_2, c_3, c_4, c_5, c_6, c_7, w_2, w_3, w_4, w_5, w_6, w_7, and as a result the system is simplified. Also omitting the equations which are a linear combination of equations of the system and the 30×15 system is obtained:

$$32\alpha_{32} + 12S_4 + 16S_5 + 16S_6 + 7S_7 = 60 \tag{163}$$

$$\begin{aligned} &12\alpha_{43}\alpha_{32} + 16\left(\alpha_{53}\alpha_{32} + \alpha_{54}S_4\right) + 16\left(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5\right) \\ &+ 7\left(\alpha_{73}\alpha_{32} + \alpha_{74}S_4 + \alpha_{75}S_5 + \alpha_{76}S_6\right) = 15 \end{aligned} \tag{164}$$

$$16\left(2\alpha_{54}\right) + 16\left(2\alpha_{64} + 6\alpha_{65}\right) + 7\left(2\alpha_{74} + 6\alpha_{75} + 6\alpha_{76}\right) = 60 \tag{165}$$

$$32\alpha_{32} + 24S_4 + 48S_5 + 48S_6 + 28S_7 = 180 \tag{166}$$

$$\begin{aligned} &16\alpha_{54}\alpha_{43}\alpha_{32} + 16\left[\alpha_{64}\alpha_{43}\alpha_{32} + \alpha_{65}\left(\alpha_{53}\alpha_{32} + \alpha_{54}S_4\right)\right] \\ &+ 7\left[\alpha_{74}\alpha_{43}\alpha_{32} + \alpha_{75}\left(\alpha_{53}\alpha_{32} + \alpha_{54}S_4\right) + \alpha_{76}\left(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5\right)\right] = 3 \end{aligned} \tag{167}$$

$$16\alpha_{65}\left(2\alpha_{54}\right) + 7\left[\alpha_{75}\left(2\alpha_{54}\right) + \alpha_{76}\left(2\alpha_{64} + 6\alpha_{65}\right)\right] = 9 \tag{168}$$

$$16\alpha_{54}S_4 + 16\left(\alpha_{64}S_4 + 2\alpha_{65}S_5\right) + 7\left(\alpha_{74}S_4 + 2\alpha_{75}S_5 + 2\alpha_{76}S_6\right) = 21 \tag{169}$$

$$16\alpha_{65} + 7\left(\alpha_{75} + \alpha_{76}\right) = 8 \tag{170}$$

$$6\alpha_{43}\alpha_{32} + 4\left(\alpha_{53}\alpha_{32} + \alpha_{54}S_4\right) + 4\left(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5\right) = 3 \tag{171}$$

$$7\left(\alpha_{74} + 3\alpha_{75} + 3\alpha_{76}\right) = 12 \tag{172}$$

$$32\alpha_{32}^2 + 12S_4^2 + 16S_5^2 + 16S_6^2 + 7S_7^2 = 72 \tag{173}$$

$$2\alpha_{32} + 3S_4 + 9S_5 + 9S_6 + 7S_7 = 36 \tag{174}$$

$$32\alpha_{65}\alpha_{54}\alpha_{43}\alpha_{32} + 14\left\{\alpha_{75}\alpha_{54}\alpha_{43}\alpha_{32} + \alpha_{76}\left[\alpha_{64}\alpha_{43}\alpha_{32} + \alpha_{65}\left(\alpha_{53}\alpha_{32} + \alpha_{54}S_4\right)\right]\right\} = 1 \tag{175}$$

$$7\alpha_{76}\alpha_{65}2\alpha_{54} = 1 \tag{176}$$

$$16\alpha_{65}\alpha_{54}S_4 + 7\left[\alpha_{75}\alpha_{54}S_4 + \alpha_{76}\left(\alpha_{64}S_4 + 2\alpha_{65}S_5\right)\right] = 3 \tag{177}$$

$$7\alpha_{76}\alpha_{65} = 1 \tag{178}$$

$$64\alpha_{65}\alpha_{53}\alpha_{32} + 7\alpha_{75}\alpha_{53}\alpha_{32} + 7\left[\alpha_{76}\left(\alpha_{63}\alpha_{32} - \alpha_{65}S_5\right)\right] = -1 \tag{179}$$

$$16\left(6\alpha_{65}\right)\alpha_{54}+7\left(6\alpha_{75}\right)\alpha_{54}+7\left(6\alpha_{76}\right)\left(\alpha_{64}+3\alpha_{65}\right)=27 \tag{180}$$

$$\begin{aligned}&12\alpha_{43}\alpha_{32}^2+16\left(\alpha_{53}\alpha_{32}^2+\alpha_{54}S_4^2\right)+16\left(\alpha_{63}\alpha_{32}^2+\alpha_{64}S_4^2+\alpha_{65}S_5^2\right)\\&+7\left(\alpha_{73}\alpha_{32}^2+\alpha_{74}S_4^2+\alpha_{75}S_5^2+\alpha_{76}S_6^2\right)=12\end{aligned} \tag{181}$$

$$16\alpha_{54}S_4+16\alpha_{64}S_4+7\alpha_{74}S_4=3 \tag{182}$$

$$7\left[\alpha_{74}\alpha_{43}\alpha_{32}+\alpha_{75}\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)+\alpha_{76}\left(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5\right)\right]=1 \tag{183}$$

$$8\alpha_{65}\alpha_{54}=1 \tag{184}$$

$$7\left(\alpha_{74}S_4+2\alpha_{75}S_5+2\alpha_{76}S_6\right)=9 \tag{185}$$

$$64\alpha_{65}+7\alpha_{73}=20 \tag{186}$$

$$\begin{aligned}&12S_4\alpha_{43}\alpha_{32}+16S_5\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)+16S_6\left(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5\right)\\&+7S_7\left(\alpha_{73}\alpha_{32}+\alpha_{74}S_4+\alpha_{75}S_5+\alpha_{76}S_6\right)=20\end{aligned} \tag{187}$$

$$16S_5\alpha_{54}+16S_6\left(\alpha_{64}+3\alpha_{65}\right)+7S_7\left(\alpha_{74}+3\alpha_{75}+3\alpha_{76}\right)=42 \tag{188}$$

$$6\alpha_{43}\alpha_{32}=1 \tag{189}$$

$$8\alpha_{54}+8\left(\alpha_{64}+3\alpha_{65}\right)=9 \tag{190}$$

$$8\alpha_{32}^2+6S_4^2+12S_5^2+12S_6^2+7S_7^2=60 \tag{191}$$

$$2\alpha_{32}+6S_4+27S_5+27S_6+28S_7=120 \tag{192}$$

From (163), (166), (174) and (192) result:

$$\alpha_{32}=\frac{1}{8} \tag{193}$$

$$S_4=\frac{1}{2} \tag{194}$$

$$S_7=2 \tag{195}$$

and

$$S_5+S_6=\frac{9}{4} \tag{196}$$

From both Equations (173) and (191) we find

$$S_5^2+S_6^2=\frac{81}{32} \tag{197}$$

From the system $S_5^2+S_6^2=\frac{81}{32}$

and $S_5+S_6=\frac{9}{4}$ result:

$$S_5=S_6=\frac{9}{8} \tag{198}$$

From (176) and (178):

$$\alpha_{54}=\frac{1}{2} \tag{199}$$

From (184): $\alpha_{65}=\frac{1}{4}$ (200) from (178): $\alpha_{76}=\frac{4}{7}$ (201)

From (189): $\alpha_{43}=\frac{4}{3}$ (202) from (170): $\alpha_{75}=0$ (203)

From (172): $\alpha_{74}=0$ (204) from (182): $\alpha_{64}=-\frac{1}{8}$ (205) from (186): $\alpha_{73}=\frac{4}{7}$ (206)

From (171) and (189): $\alpha_{53}=0$ (207) and $\alpha_{63}=\frac{1}{4}$ (208)

The remaining equations are verified.

From $S_4=\frac{1}{2}$: $\alpha_{42}=-\frac{5}{6}$ (209) from $S_5=\frac{9}{8}$: $\alpha_{52}=\frac{1}{8}$ (210)

From $S_6=\frac{9}{8}$: $\alpha_{62}=\frac{3}{8}$ (211) from $S_7=2$: $\alpha_{72}=-\frac{2}{7}$ (212)

From the relations $\sum_{j=1}^{i-1}\alpha_{ij}=c_i$, $i=2,3,4,5,6,7$ result:

$$\alpha_{21}=\frac{1}{4} \tag{213}$$

$$\alpha_{31}=\frac{1}{8} \tag{214}$$

$$\alpha_{41}=0 \tag{215}$$

$$\alpha_{51}=\frac{1}{8} \tag{216}$$

$$\alpha_{61}=0 \tag{217}$$

and

$$\alpha_{71}=\frac{1}{7} \tag{218}$$

Therefore we get Table 4:

with

$$K_1=hf\left(x_n,y_n\right) \tag{219}$$

Table 4. For choices values of arbitrary constants: $c_2 = c_3 = 1/4$, $c_4 = 2/4$, $c_5 = c_6 = 3/4$, $c_7 = 4/4$.

0							
1/4	1/4						
1/4	1/8	1/8					
2/4	0	−5/6	8/6				
3/4	1/8	1/8	0	4/8			
3/4	0	3/8	2/8	−1/8	2/8		
4/4	1/7	−2/7	4/7	0	0	4/7	
	7/90	0	32/90	12/90	16/90	16/90	7/90

$$K_2 = hf\left(x_n + \frac{h}{4}, y_n + \frac{K_1}{4}\right) \tag{220}$$

$$K_3 = hf\left(x_n + \frac{h}{4}, y_n + \frac{K_1 + K_2}{8}\right) \tag{221}$$

$$K_4 = hf\left(x_n + \frac{2h}{4}, y_n + \frac{-5K_2 + 8K_3}{6}\right) \tag{222}$$

$$K_5 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{K_1 + K_2 + 4K_4}{8}\right) \tag{223}$$

$$K_6 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{3K_2 + 2K_3 - K_4 + 2K_5}{8}\right) \tag{224}$$

$$K_7 = hf\left(x_n + h, y_n + \frac{K_1 - 2K_2 + 4K_3 + 4K_6}{7}\right) \tag{225}$$

and

$$y_{n+1} = y_n + \frac{7K_1 + 32K_3 + 12K_4 + 16K_5 + 16K_6 + 7K_7}{90} \tag{226}$$

Working in a similar way we are presented three other choices:

2.3. 3rd Choice: $c_2 = c_4 = 4/12$, $c_3 = 8/12$, $c_5 = 3/12$, $c_6 = 9/12$, $c_7 = 12/12$ [5]

For 3rd choice we get Table 5:

with

$$K_1 = hf\left(x_n, y_n\right) \tag{227}$$

$$K_2 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1}{3}\right) \tag{228}$$

$$K_3 = hf\left(x_n + \frac{2h}{3}, y_n + \frac{2K_2}{3}\right) \tag{229}$$

$$K_4 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1 + 4K_2 - K_3}{12}\right) \tag{230}$$

Table 5. For choices values of arbitrary constants. $c_2 = c_4 = 4/12$, $c_3 = 8/12$, $c_5 = 3/12$, $c_6 = 9/12$, $c_7 = 12/12$.

0							
4/12	1/3						
8/12	0	2/3					
4/12	1/12	4/12	−1/12				
3/12	10/64	−9/64	0	15/64			
9/12	−3/64	−27/64	27/64	−45/64	96/64		
12/12	55/116	−81/145	−459/580	528/145	−384/145	128/145	
	29/360	0	27/200	27/200	64/225	64/225	29/360

$$K_5 = hf\left(x_n + \frac{h}{4}, y_n + \frac{10K_1 - 9K_2 + 15K_4}{64}\right) \tag{231}$$

$$K_6 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{-3K_1 - 27K_2 + 27K_3 - 45K_4 + 96K_5}{64}\right) \tag{232}$$

$$K_7 = hf\left(x_n + h, y_n + \frac{\frac{275K_1}{4} - 81K_2 - \frac{459K_3}{4} + 528K_4 - 384K_5 + 128K_6}{145}\right) \tag{233}$$

and

$$y_{n+1} = y_n + \frac{145K_1 + 243K_3 + 243K_4 + 512K_5 + 512K_6 + 145K_7}{1800} \tag{234}$$

2.4. 4th Choice: $c_2 = c_3 = 1/5$, $c_4 = 2/5$, $c_5 = 3/5$, $c_6 = 4/5$ και $c_7 = 5/5 = 1$ [5]

For 4th choice we get Table 6:

with

$$K_1 = hf\left(x_n, y_n\right) \tag{235}$$

$$K_2 = hf\left(x_n + \frac{h}{5}, y_n + \frac{K_1}{5}\right) \tag{236}$$

$$K_3 = hf\left(x_n + \frac{h}{5}, y_n + \frac{K_1 + K_2}{10}\right) \tag{237}$$

$$K_4 = hf\left(x_n + \frac{2h}{5}, y_n + \frac{-9K_2 + 25K_3}{40}\right) \tag{238}$$

$$K_5 = hf\left(x_n + \frac{3h}{5}, y_n + \frac{4K_1 - 17K_2 + 21K_3 + 16K_4}{40}\right) \tag{239}$$

$$K_6 = hf\left(x_n + \frac{4h}{5}, y_n + \frac{-8K_1 + 21K_2 + 35K_3 - 40K_4 + 40K_5}{60}\right) \tag{240}$$

$$K_7 = hf\left(x_n + h, y_n + \frac{66K_1 - 5K_2 - 165K_3 + 240K_4 - 120K_5 + 60K_6}{76}\right) \tag{241}$$

Table 6. For choices values of arbitrary constants: $c_2 = c_3 = 1/5$, $c_4 = 2/5$, $c_5 = 3/5$, $c_6 = 4/5$ και $c_7 = 5/5 = 1$.

0							
1/5	1/5						
1/5	1/10	1/10					
2/5	0	−9/40	25/40				
3/5	4/40	−17/40	21/40	16/40			
4/5	−8/60	21/60	35/60	−40/60	40/60		
1	66/76	−5/76	−165/76	60/19	−30/19	15/19	
	19/288	0	75/288	50/288	50/288	75/288	19/288

Table 7. For choices values of arbitrary constants: $c_2 = c_4 = 1/5$, $c_3 = 2/5$, $c_5 = 3/5$, $c_6 = 4/5$ και $c_7 = 5/5 = 1$.

0							
1/5	1/5						
2/5	0	2/5					
1/5	11/60	−4/60	5/60				
3/5	−3/20	4/20	3/20	8/20			
4/5	0	−12/15	−8/15	22/15	10/15		
1	51/76	140/76	225/76	−280/76	−120/76	60/76	
	19/288	0	50/288	75/288	50/288	75/288	19/288

and

$$y_{n+1} = y_n + \frac{19K_1 + 75K_3 + 50K_4 + 50K_5 + 75K_6 + 19K_7}{288} \tag{242}$$

2.5. 5th Choice: $c_2 = c_4 = 1/5$, $c_3 = 2/5$, $c_5 = 3/5$, $c_6 = 4/5$ και $c_7 = 5/5 = 1$ [5]

For 5th choice we get Table 7:

with

$$K_1 = hf\left(x_n, y_n\right) \tag{243}$$

$$K_2 = hf\left(x_n + \frac{h}{5}, y_n + \frac{K_1}{5}\right) \tag{244}$$

$$K_3 = hf\left(x_n + \frac{2h}{5}, y_n + \frac{2K_2}{5}\right) \tag{245}$$

$$K_4 = hf\left(x_n + \frac{h}{5}, y_n + \frac{11K_1 - 4K_2 + 5K_3}{60}\right) \tag{246}$$

$$K_5 = hf\left(x_n + \frac{3h}{5}, y_n + \frac{-3K_1 + 4K_2 + 3K_3 + 8K_4}{20}\right) \tag{247}$$

$$K_6 = hf\left(x_n + \frac{4h}{5}, y_n + \frac{-12K_2 - 8K_3 + 22K_4 + 10K_5}{15}\right) \tag{248}$$

$$K_7 = hf\left(x_n + h, y_n + \frac{51K_1 + 140K_2 + 225K_3 - 280K_4 - 120K_5 + 60K_6}{76}\right) \tag{249}$$

and

$$y_{n+1} = y_n + \frac{19K_1 + 50K_3 + 75K_4 + 50K_5 + 75K_6 + 19K_7}{288} \tag{250}$$

3. Conclusions

In the R-K methods it is $c_1 = 0$ and we choose $c_v = 1$, that is, in method (6,7) presented here, it is $c_7 = 1$. For methods up to 4th order the number of steps is

equal to the order of the method. For the higher order method, the number of steps exceeds its order. In the 6th order R-K method the steps are 7 and 8 or more. We look for the method with the fewest steps because increasing the steps also increases the number of parameters to be calculated. Because the system from which the parameters will be calculated is not linear, some "arbitrary" values [5] must be given to parameters to solve the system. We have given 5 choices as examples, to the values of the arbitrary constants and the solutions of the respective systems that result are made analytically.

The process of solving the system starts by giving values to c_i to calculate the w_i. The resulting system is a linear 6×7 system. To solve it we give the same value in two c_i eg $c_2 = c_3$ *or* $c_2 = c_4$ etc. while the values of c_i we choose to be in ascending order. As the variable c_2 does not occur often we prefer one of the two equals c_i to be c_2. In the choice *eg* $c_2 = c_3$ we consider $w_2 + w_3$ as one parameter and after it is found that $w_2 + w_3 = k$ we set $w_2 = 0$ and $w_3 = k$. In choices 2.1 and 2.2 the same values were given to two pairs of parameters c_i and we omitted one equation to make the system 5×5 and whose solution must verify the omitted equation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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A Kind of Time-Delayed COVID-19 Dynamical Model with Vaccination

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Abstract

In the paper, we study a kind of time-delayed novel coronavirus pneumonia dynamical model with vaccination. This model considers that people are vaccinated, and the human immune system has a series of processes, which need a certain time. We first obtain the disease-free equilibrium and the basic reproduction number R_0, and the system has a unique endemic equilibrium when $R_0 > 1$. Then we discuss the stability of the disease-free equilibrium and the endemic equilibrium with different delays τ. For $\tau = 0$, using the Lyapunov function approach, we obtained the stability of disease-free equilibrium and the endemic equilibrium, respectively. For any delay $\tau \neq 0$, using the Routh-Hurwitz Criteria, we obtained that the disease-free equilibrium is locally asymptotically stable. We also find the critical value τ_0 at the endemic equilibrium, and obtain the condition that the system has a Hopf bifurcation at the endemic equilibrium. Finally, with the suitable choices of the parameters, some numerical simulations are presented in order to verify the effectiveness of the obtained theoretical results.

Keywords

SVIQR Epidemic Model, the Time-Delayed COVID-19 Dynamical Model, Lyapunov Functional, Hopf Bifurcation, Stability

1. Introduction

In December 2019, a kind of new atypical viral pneumonia cases, which is caused by a novel coronavirus (2019-nCoV), appeared in Wuhan City, Hubei Province of China. Since then, the novel coronavirus pneumonia (COVID-2019) has rapidly spread to other provinces of China and other countries with the Spring Festival passenger flew. The number of suspected/confirmed cases has increased rapidly [1]. The outbreak of COVID-2019 has attracted the attention of the

countries all over the world. According to the cases of COVID-19 in Wuhan and Hubei, the number of COVID-19 is a significant upward trend [2] [3]. There is already abundant information on 2019-nCoV and its spread worldwide [4] [5], as well as the risks it poses to the population. Globally, as of 5:31 pm CET, 10 January 2022, there have been 305,914,601 confirmed cases of COVID-19, including 5,486,304 deaths, reported to World Health Organization (WHO) [6] [7]. Due to the sudden outbreak of COVID-2019, it is a new type of infectious disease, people have not found a new measure to analyze, prevent and treat it. Every outbreak of the coronavirus has caused huge economic losses to the affected countries. The epidemic of infectious diseases in history has seriously affected people's health and led to huge losses to the global economy [8] [9]. At the same time, the high mortality rate of the epidemic has also caused varying degrees panic. COVID-2019 spreads mainly through human contact. Therefore, governments throughout China and other countries have formulated policies for controlling populations to cut off the spread of the virus and to reduce the infection rate. At the same time, scientific medical treatment can effectively alleviate the epidemic situation and eventually terminate the epidemic situation after a period. However, the long-term population control has brought many inconveniences to the human daily life, and it has brought a great impact on the economy. Therefore, it is very important that the government makes the timely and effective treatment measures to eliminate COVID-2019. It is necessary to use traditional infectious disease dynamics models to analyze and predict the development of the epidemic. These can be achieved by establishing mathematical models that simulate the spread of the epidemic. Through the study of related mathematical models, the relevant parameters obtained through historical data inversion can measure the current situation of the epidemic in many ways, and numerical calculations from the model. Simulation can also predict the development trend of the epidemic, all of which provide effective reference value for proposing better and more scientific epidemic prevention and control schemes.

The classical mathematical models describing the spread of infectious diseases include SI, SIS, SIR, SIRS, SEIR, and SEIRS model, etc [10] [11] [12]. Its main idea is to divide the population into different compartments: susceptible(S), latent (E), infectious (I), and removed (R), and to establish the ordinary differential equation systems through the communication mechanism of one group transferring to another group, so as to reveal the law of the epidemic spread. These models (and their variants) are used to study the transmission of various infectious diseases such as measles, smallpox, rabies, Ebola virus, and population dynamics. After the SARS epidemic in 2003, the mathematical models of SARS, MERS and other coronavirus transmission laws were gradually enriched [13] [14].

Because most traditional models do not consider the effect of latency on propagation delay, even considering into the latent (E) in the SEIR models of COVID-19, it is only assumed that the latent E is weakly infectious, which cannot depict

the characteristic that 2019-nCoV can propagate in the latent period. In addition, the above model hardly considers the data delay caused by the time required for diagnosis. To sum up, a model of COVID-19 with time delay effect is needed to describe the novel crown pneumonia epidemic more appropriately. At the same time, with the development of the scientific medical treatment, there are many immune drugs. As of 9 January 2022, 9,126,987,353 vaccine doses have been administered [7]. The infectious disease dynamics model based on time-delayed dynamics system with vaccination proposed in this paper inverse the model parameters through public historical data. The parameter analysis shows the effectiveness of prevention and control measures of governments at all levels. Based on these parameters, we well simulated the development of the current epidemic and accurately predicted the future trend of the epidemic. The simulation results show that the epidemic situation can be controlled and gradually ended in a short time.

The arrangement of the paper is as follows. Firstly, using the SVIR model in [15] as our baseline model, we formulate the time-delayed COVID-19 dynamical model with vaccination, and show the model biologically reasonable and mathematically defined well in Section 2. We obtain the disease-free equilibrium, the endemic equilibrium, and a formula for the reproduction number, R_0, of the time-delayed COVID-19 dynamical model with vaccination by the next generation matrix [16] [17] in Section 3. In Section 4, when $R_0 > 1$, we prove that the model (5) has a unique positive equilibrium. Then we discuss the eigenvalues of the model at the equilibrium through Routh-Hurwitz criteria [18] for $R_0 > 1$ and $R_0 < 1$, respectively, so as to judge the stability of the disease-free equilibrium and the endemic equilibrium. When the delay $\tau = 0$, we construct Lyapunov function [19] to judge the global stability of the equilibrium. According to the conditions for the existence of Hopf bifurcation of delay differential equations [20] [21], we find that the system has Hopf bifurcation when $R_0 > 1$ and $\tau = \tau_0$. Finally, we carry out numerical simulation and get different simulation results according to different data to verify the correctness of the theoretical results in Section 5. We finally give brief discussions of our findings and our future research plan in Section 6.

2. The Model Formulation

2.1. The Building of the Delay Dynamical Model with Vaccination

From the study of infectious diseases, prevention is more important than treatment. The researchers have established mathematical models to show the dynamic behavior of infectious diseases based on the occurrence, transmission and development law of infectious diseases and related social factors. Theoretical analysis and numerical simulation are also carried out through the model [15] [22] [23] [24] [25]. It can show the development process of infectious diseases by simulating dynamic behavior, analyzing the causes and key factors of infectious diseases by studying the impact of prevention and control methods on the epi-

demic of infectious diseases, predict the epidemic law and development trend, seek for the development law of various infectious diseases, and find the optimal solution of disease prevention and control. These researchers also provide theoretical and quantitative basis for prevention as well as decision-making. What's more, vaccination is the most effective way to control the spread of the epidemic [26] [27] [28]. If people are vaccinated, the human immune system has a series of processes, which need a certain time [29] [30]. And if people can only have limited immune protection after vaccination [31], the vaccinators are not susceptible (S) and recovered (R) individuals, but belong to a new group. According to this situation, we build a time-delayed SVIQR model with incomplete vaccination based on the SVIR model established by Liu *et al.* [15].

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta S(t) I(t) - \mu S(t) - \rho S(t-\tau), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \rho S(t-\tau) - \sigma\beta V(t) I(t) - \mu V(t) - \alpha V(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta\left(S(t) + \sigma V(t)\right) I(t) - (\mu + \nu + \delta + \kappa) I(t), \\ \dfrac{\mathrm{d}Q(t)}{\mathrm{d}t} = \kappa I(t) - (\mu + \gamma) Q(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \alpha V(t) + \nu I(t) + \gamma Q(t) - \mu R(t), \end{cases} \tag{1}$$

with the initial conditions

$$S(\theta) = \varphi_1(\theta), V(\theta) = \varphi_2(\theta), I(\theta) = \varphi_3(\theta), Q(\theta) = \varphi_4(\theta), R(\theta) = \varphi_5(\theta),$$
$$\varphi_i(\theta) \in C[-\tau, 0), \varphi_i(\theta) \geq 0, \varphi_i(0) > 0, i = 1, 2, 3, 4, 5.$$

where $S(t), V(t), I(t), Q(t)$ and $R(t)$ are the density of the susceptible, vaccinated, infected, quarantined and recovered population at time t, respectively. τ is the time required for the vaccine to produce protection. Λ is the constant entering flux into the S-class, β is the transmission rates for individuals in S-class, μ is the constant natural death rate, α is complete vaccination rate, ρ is the vaccination rate, κ is isolation rate, δ is the death rate because of CODIV-19. $\frac{1}{\nu}$, $\frac{1}{\gamma}$ are the average time of the infectious period and quarantine, respectively. $\sigma\beta$ is the transmission rates for individuals in V-class, $0 \leq \sigma \leq 1$. When $\sigma = 0$ and $\sigma = 1$, it means that vaccination is completely effective and ineffective, respectively.

2.2. Positive Invariant Sets of the Model System

Below, we show that the system (1) is biologically reasonable and mathematically defined well. For the simplicity, we give the following notations:
$\mathbb{R}^5_+ = \{S(t) \geq 0\} \cup \{V(t) \geq 0\} \cup \{I(t) \geq 0\} \cup \{Q(t) \geq 0\} \cup \{R(t) \geq 0\}$ for the nonnegative orthant of space $\mathbb{R}^5$, and
$\mathrm{int}(\mathbb{R}^5_+) = \mathbb{R}^5_+ \setminus \{S(t) = V(t) = I(t) = Q(t) = R(t) = 0\}$ for the interior of the $\mathbb{R}^5_+$. From the system (1), we have

$$
\begin{aligned}
S(t) &= \exp\left(-\int_0^t (\mu+\beta I(\eta))\mathrm{d}\eta\right)\\
&\quad\times\left(S(0)+\int_0^t (\Lambda-\rho S(\eta-\tau))\exp\left(\int_\eta^t (\mu+\beta I(\xi))\mathrm{d}\xi\right)\mathrm{d}\eta\right),\\
V(t) &= \exp\left(-\int_0^t (\mu+\alpha+\sigma\beta I(\eta))\mathrm{d}\eta\right)\\
&\quad\times\left(V(0)+\int_0^t \rho S(\eta-\tau)\exp\left(\int_\eta^t (\mu+\alpha+\sigma\beta I(\xi))\mathrm{d}\xi\right)\mathrm{d}\eta\right),\\
I(t) &= I(0)\exp\left(-\int_0^t (\beta S(\eta)+\sigma V(\eta)-(\mu+\upsilon+\delta+\kappa))\mathrm{d}\eta\right),\\
Q(t) &= \exp(-(\mu+\gamma)t)\left(Q(0)+\int_0^t \kappa I(\eta)\exp((\mu+\gamma)(t-\eta))\mathrm{d}\eta\right),\\
R(t) &= \exp(-\mu t)\left(R(0)+\int_0^t (\alpha V(\eta)+\nu I(\eta)+\gamma Q(\eta))\exp(\mu(t-\eta))\mathrm{d}\eta\right).
\end{aligned}
\tag{2}
$$

If we suppose that $(S(0),V(0),I(0),Q(0),R(0))\in \mathrm{int}(\mathbb{R}_+^5)$, according to the third equation of (2), for all $t\geq 0$, $I(t)>0$. Because for any $t\geq 0$, $\Lambda-\rho S(t-\tau)>0$, hence, from other equation of (2), we have $S(t)>0$, $V(t)>0$, $Q(t)>0$, $R(t)>0$.

From the initial conditions, for all $t\in[-\tau,0)$, $S(t)>0$, $V(t)>0$, $I(t)>0$, $Q(t)>0$, $R(t)>0$, thus, $S(t)>0$, $V(t)>0$, $I(t)>0$, $Q(t)>0$, $R(t)>0$ for any $t\in[-\tau,+\infty)$.

In addition, we suppose that the total population $N(t)=S(t)+V(t)+I(t)+Q(t)+R(t)$. Thus, adding all the equations for $S(t),V(t),I(t),Q(t)$, and $R(t)$ in (1) together, we get

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t}=\Lambda-\mu N(t)-\delta I(t).$$

Because of $N(t)>I(t)>0$, we obtain

$$\Lambda-(\mu+\delta)N(t)\leq\frac{\mathrm{d}N(t)}{\mathrm{d}t}\leq\Lambda-\mu N(t).$$

Consequently, for any $t\in[0,+\infty)$,

$$
\begin{aligned}
&\frac{\Lambda}{\mu+\delta}+\left(N(0)-\frac{\Lambda}{\mu+\delta}\right)\exp(-(\mu+\delta)t)\leq N(t)\\
&\leq N^0+(N(0)-N^0)\exp(-\mu t),
\end{aligned}
\tag{3}
$$

where N^0 is the initial population, $N(0)$ is the population at time $t=0$, which yields that $N(0)\exp(-(\mu+\delta)t)\leq N(t)\leq N^0+|N(0)-N^0|$, for $t\in[0,T^+)$, where $T^+>0$. Thus, we obtain that $N(t)$ is bounded on $t\in[0,T^+)$, which implies that $T^+=+\infty$. Let $t\to+\infty$ in (3), we obtain

$$\frac{\Lambda}{\mu+\delta}\leq\liminf_{t\to+\infty}N(t)\leq\limsup_{t\to+\infty}N(t)\leq N^0.$$

For showing that the system (1) is biologically reasonable and mathematically defined well, we need to prove that the set

$$D=\left\{(S(t),V(t),I(t),Q(t),R(t))\in\mathrm{int}(\mathbb{R}_+^5):\frac{\Lambda}{\mu+\delta}\leq N(t)\leq N^0\right\}\tag{4}$$

is positively invariant for the system (1). Let $(S(0),V(0),I(0),Q(0),R(0))\in D$, then $S(t),V(t),I(t),Q(t)$, and $R(t)$ are defined on $[0,+\infty)$, with (3) holding for $N(t)$. Since $(S(0),V(0),I(0),Q(0),R(0))\in D$, it follows from (3) that $\frac{\Lambda}{\mu+\delta}\le N(t)\le N^0$ for all $t\in[0,+\infty)$. This means that D is a positively invariant set. Therefore, we have

Theorem 2.1. The set D is a positively invariant set for the model system (1).

Because the total population $N(t)=S(t)+V(t)+I(t)+Q(t)+R(t)$, the model system (1) is equivalent to the following model system,

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{\mathrm{d}t}=\Lambda-\beta S(t)I(t)-\mu S(t)-\rho S(t-\tau),\\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t}=\rho S(t-\tau)-\sigma\beta V(t)I(t)-\mu V(t)-\alpha V(t),\\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t}=\beta\big(S(t)+\sigma V(t)\big)I(t)-(\mu+\nu+\delta+\kappa)I(t),\\ \dfrac{\mathrm{d}Q(t)}{\mathrm{d}t}=\kappa I(t)-(\mu+\gamma)Q(t),\end{cases}\tag{5}$$

with the initial conditions and the positively invariant region Ω as follows,

$$\begin{aligned}&S(\theta)=\varphi_1(\theta),V(\theta)=\varphi_2(\theta),I(\theta)=\varphi_3(\theta),Q(\theta)=\varphi_4(\theta),\\&\varphi_i(\theta)\in C[-\tau,0),\varphi_i(\theta)\ge 0,\varphi_i(0)>0,i=1,2,3,4\\&\Omega=\left\{(S(t),V(t),I(t),Q(t))\in\operatorname{int}\left(\mathbb{R}^4_+\right):\frac{\Lambda}{\mu+\delta}\le N(t)\le N^0\right\}.\end{aligned}\tag{6}$$

3. The Equilibria of the Model System

When $t\to+\infty$, $S(t-\tau)=S(t)=S$, $V(t)=V$, $I(t)=I$, $Q(t)=Q$. Equilibrium points of system (5) are given by solutions $(S,V,I,Q)\in\mathbb{R}^4_+$ of

$$\begin{cases}\Lambda-\beta SI-\mu S-\rho S=0,\\ \rho S-\sigma\beta VI-\mu V-\alpha V=0,\\ \beta(S+\sigma V)I-(\mu+\nu+\delta+\kappa)I=0,\\ \kappa I-(\mu+\gamma)Q=0.\end{cases}\tag{7}$$

We easily obtain the disease-free equilibrium point $\boldsymbol{E}^0=(S_0,V_0,I_0,Q_0)$ of system (5) in the situation, where

$$S_0=\frac{\Lambda}{\rho+\mu},V_0=\frac{\rho\Lambda}{(\rho+\mu)(\mu+\alpha)},I_0=0,Q_0=0.$$

The basic reproduction number R_0 is known as the expected number of secondary cases produced by a typical infectious individual during its entire period of infectiousness in a completely susceptible population [17] [32] [33]. For system (5), we can find the new infected person matrix $\boldsymbol{\Phi}$ and the removed infected person matrix $\boldsymbol{\Psi}$, respectively:

$$\boldsymbol{\Phi}=\beta\big(S(t)+\sigma V(t)\big)I(t),\quad\boldsymbol{\Psi}=(\mu+\nu+\delta+\kappa)I(t),$$

and hence, the basic reproduction number R_0 is

$$R_0 = \rho\left(\mathbf{\Phi}\mathbf{\Psi}^{-1}\right)\Big|_{E^0} = \frac{\beta\left(S_0 + \sigma V_0\right)}{\mu+\nu+\delta+\kappa} = \frac{\Lambda\beta\left(\mu+\alpha+\sigma\rho\right)}{\left(\rho+\mu\right)\left(\mu+\alpha\right)\left(\mu+\nu+\delta+\kappa\right)}.$$

Let $c = \mu+\nu+\delta+\kappa$, the basic reproduction number can be written as

$$R_0 = \frac{\Lambda\beta\left(\mu+\alpha+\sigma\rho\right)}{c\left(\rho+\mu\right)\left(\mu+\alpha\right)}.$$

Theorem 3.1. Let $t \to +\infty$, then if $R_0 < 1$, the system (5) has no positive equilibria; if $R_0 > 1$, the system (5) has a unique positive equilibrium point $\boldsymbol{E}^* = \left(S^*, V^*, I^*, Q^*\right)$.

Proof. By the first equation of (7), we get $S = \frac{\Lambda}{\rho+\mu+\beta I}$. By the second equation of (7), we get $V = \frac{\rho S}{\mu+\alpha+\sigma\beta I} = \frac{\rho\Lambda}{\left(\rho+\mu+\beta I\right)\left(\mu+\alpha+\sigma\beta I\right)}$. By the fourth equation of (7), we get $Q = \frac{\lambda I}{\mu+\gamma}$. Then we substitute S, V and Q into the third equation of (7) and simplify it:

$$H\left(I\right) = h_1 I^2 + h_2 I + h_3 = 0, \tag{8}$$

where

$$\begin{aligned} h_1 &= -\sigma\beta^2 c < 0, \\ h_2 &= \beta\left(\Lambda\beta\sigma - c\left(\mu+\alpha+\sigma\left(\rho+\mu\right)\right)\right), \\ h_3 &= c\left(\mu+\alpha\right)\left(\rho+\mu\right)\left(R_0 - 1\right). \end{aligned} \tag{9}$$

If $R_0 < 1$, it is ease to prove that $h_2 < 0, h_3 < 0$. Hence the equation $H\left(I\right) = 0$ does not have positive roots, so the (5) has no positive equilibrium.

If $R_0 > 1$, then $h_3 > 0$. Hence the equation $H\left(I\right) = 0$ has a positive root and a negative root. We take the positive root as I^* and obtain the positive equilibrium point $\boldsymbol{E}^* = \left(S^*, V^*, I^*, Q^*\right)$ of the system (5), where

$$\begin{aligned} S^* &= \frac{\Lambda}{\rho+\mu+\beta I^*}, \\ V^* &= \frac{\rho\Lambda}{\left(\rho+\mu+\beta I^*\right)\left(\mu+\alpha+\sigma\beta I^*\right)}, \\ Q^* &= \frac{\lambda I^*}{\mu+\gamma}, \end{aligned} \tag{10}$$

and I^* is the unique positive root of the quadratic equation for $R_0 > 1$. □

4. The Stability of the Equilibria of the System (5)

Now, we research the stability of the equilibria of the system (5).

4.1. Global Stability of the Equilibria When $\tau = 0$

At first, we give the definition of the Lyapunov-candidate-function to research

the stability of the equilibria. Let $L:\mathbb{R}^n \to \mathbb{R}$ be a continuous scalar function, if L satisfies

$$L(\mathbf{0})=0, \quad L(\boldsymbol{x})>0, \quad \forall \boldsymbol{x} \in U \setminus \{\mathbf{0}\},$$

with U being a neighborhood region around $\boldsymbol{x}=\mathbf{0}$, then L is called a Lyapunov-candidate-function.

Next, we state the general Lyapunov stability theorem [34].

Lemma 4.1. Consider the following vector field

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \ \boldsymbol{x} \in \mathbb{R}^n. \tag{11}$$

Let $\boldsymbol{x}_0$ be a equilibrium point of (11) and let $L:U \to \mathbb{R}$ be a C^1 function defined on some neighborhood U of $\boldsymbol{x}_0$ such that

1) $L(\boldsymbol{x}_0)=0$ and $L(\boldsymbol{x})>$ if $\boldsymbol{x} \neq \boldsymbol{x}_0$,

2) $\dot{L}(\boldsymbol{x}) \leq 0$ in $U \setminus \{\boldsymbol{x}_0\}$,

then $\boldsymbol{x}_0$ is stable. Moreover, if

3) $\dot{L}(\boldsymbol{x})<0$ in $U \setminus \{\boldsymbol{x}_0\}$,

then $\boldsymbol{x}_0$ is asymptotically stable.

The proof and the geometric interpretation of the Lemma 4.1 can be found in the second chapter in reference [34].

Now, we prove the stability of the equilibrium points of the system (5).

Theorem 4.1. If $R_0<1$, $\tau=0$, then the disease-free equilibrium $\boldsymbol{E}^0$ of system (5) is globally asymptotically stable in Ω.

Proof. We transform the disease-free equilibrium $\boldsymbol{E}^0$ into the origin by transforming $X=S-S_0$, $Y=V-V_0$, $I=I$, $Q=Q$, then we get a new system (12) by the transformation.

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = -(\rho+\mu)X(t) - \beta S_0 I(t) - \beta X(t) I(t), \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \rho X(t) - \sigma\beta V_0 I(t) - \mu Y(t) - \alpha Y(t) - \sigma\beta Y(t) I(t), \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta(S_0+\sigma V_0) I(t) - cI(t) + \beta X(t) I(t) + \sigma\beta Y(t) I(t), \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = \kappa I(t) - (\mu+\gamma) Q(t). \end{cases} \tag{12}$$

The feasible region of system (12) is

$$\Omega_1 = \left\{ X > -S_0, Y > -V_0, I > 0, Q > 0, X+Y+I+Q \leq \frac{\Lambda}{\mu} - S_0 - V_0 \right\}.$$

We define a Lyapunov function L_1 as

$$L_1 = X^2 + \frac{\mu}{\rho} Y^2 + 2S_0 I.$$

The derivative of the Lyapunov function L_1 along the trajectories of system (12) is

$$\begin{aligned}\left.\frac{\mathrm{d}L_1}{\mathrm{d}t}\right|_{(12)} &= -2(\rho+\mu)X^2 - 2\beta S_0 XI - 2\beta X^2 I + 2S_0\left[\beta(S_0+\sigma V_0)-c\right]I \\ &\quad + 2\beta S_0 XI + 2\sigma\beta S_0 YI + 2\mu XY - 2\sigma\beta S_0 YI - \frac{2\mu^2}{\rho}Y^2 \\ &\quad - \frac{2\mu\alpha}{\rho}Y^2 - 2\frac{\mu}{\rho}\sigma\beta Y^2 I \\ &= -2\left[(\rho+\mu)X^2 - \mu XY + \frac{\mu^2}{\rho}Y^2\right] - 2S_0\left[c-\beta(S_0+\sigma V_0)\right]I \\ &\quad - 2\beta X^2 I - 2\frac{\mu}{\rho}\sigma\beta Y^2 I - \frac{2\mu\alpha}{\rho}Y^2\end{aligned} \tag{13}$$

If $R_0<1$, then $c-\beta(S_0+\sigma V_0)>0$. Because the arithmetic mean is larger than or equals to the geometric mean, we have $(\rho+\mu)X^2+\frac{\mu^2}{\rho}Y^2 \geq 2\sqrt{\frac{\rho+\mu}{\rho}}\mu XY \geq \mu XY$. For $\forall X,Y,I,Q\in\Omega_1$, there is $\left.\frac{\mathrm{d}L_1}{\mathrm{d}t}\right|_{(12)}<0$, so $\left.\frac{\mathrm{d}L_1}{\mathrm{d}t}\right|_{\text{Eq.(12)}}$ is negative in Ω_1. Hence, the disease-free equilibrium E^0 of system (5) is globally asymptotically stable. The proof is complete. □

Theorem 4.2. If $R_0>1$, $\tau=0$, then the endemic equilibrium $\boldsymbol{E}^*$ is globally asymptotically stable.

Proof. At first, we prove that the endemic equilibrium $\boldsymbol{E}^*$ is locally asymptotically stable.

If $R_0>1$, we obtain that the Jacobian matrix of the system (5) at the endemic equilibrium point $\boldsymbol{E}^*$ is

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{E}^*) &= \begin{pmatrix} -\mu-\rho\mathrm{e}^{-\lambda\tau}-\beta I^* & 0 & -\beta S^* & 0 \\ \rho\mathrm{e}^{-\lambda\tau} & -\mu-\alpha-\sigma\beta I^* & -\sigma\beta V^* & 0 \\ \beta I^* & \sigma\beta I^* & 0 & 0 \\ 0 & 0 & \kappa & -\mu-\gamma \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\Lambda}{S^*}+\rho\left(1-\mathrm{e}^{-\lambda\tau}\right) & 0 & -\beta S^* & 0 \\ \rho\mathrm{e}^{-\lambda\tau} & \frac{-\rho S^*}{V^*} & -\sigma\beta V^* & 0 \\ \beta I^* & \sigma\beta I^* & 0 & 0 \\ 0 & 0 & \kappa & -\mu-\gamma \end{pmatrix}\end{aligned} \tag{14}$$

When $\tau=0$, we get the characteristic equation of the matrix $\boldsymbol{J}(\boldsymbol{E}^*)$ is

$$\left(\lambda^3+a_1\lambda^2+a_2\lambda+a_3\right)(\lambda+\mu+\gamma)=0, \tag{15}$$

where

$$\begin{aligned} a_1 &= \frac{\Lambda}{S^*}+\frac{\rho S^*}{V^*}>0 \\ a_2 &= \sigma^2\beta^2 V^* I^* + \beta^2 S^* I^* + \frac{\rho\Lambda}{V^*}>0 \\ a_3 &= \sigma\beta^2\rho S^* I^* + \frac{\rho\beta S^{*2} I^*}{V^*} + \frac{\sigma^2\beta^2\Lambda V^* I^*}{S^*}>0 \end{aligned} \tag{16}$$

Obviously, the characteristic Equation (15) always has a negative real root $\lambda = -\mu - \gamma$. And the remaining roots are determined by the following equation,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{17}$$

By Routh-Hurwitz Criteria, the coefficients of the Equation (17) satisfy

$$H_1 = |a_1| > 0 \tag{18}$$

$$\begin{aligned} H_2 &= \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} = a_1 a_2 - a_3 \\ &= \frac{\rho\Lambda^2}{S^*V^*} + \beta^2 S^* I^* \left(\mu + \beta I^*\right) + \frac{\rho^2 \mu S^*}{V^*} \\ &\quad + \rho\beta^2 \left(1-\sigma\right)^2 S^* I^* + \rho\sigma\beta^2 S^* I^* \\ &> 0 \end{aligned} \tag{19}$$

$$H_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3 H_2 > 0 \tag{20}$$

therefore, the endemic equilibrium $\boldsymbol{E}^*$ is locally asymptotically stable.

Below, we prove that the endemic equilibrium $\boldsymbol{E}^*$ is globally asymptotically stable. When $\tau = 0$, delay differential equation system (5) becomes the ordinary differential equation system as follows,

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - \beta S(t) I(t) - \mu S(t) - \rho S(t) \\ \dfrac{\mathrm{d}V}{\mathrm{d}t} = \rho S(t) - \sigma\beta V(t) I(t) - \mu V(t) - \alpha V(t) \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta\left(S(t) + \sigma V(t)\right) I(t) - cI(t) \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = \kappa I(t) - \left(\mu + \gamma\right) Q(t). \end{cases} \tag{21}$$

We define a Lyapunov function L_2 as

$$L_2 = S - S^* - S^* \ln\frac{S}{S^*} + V - V^* - V^* \ln\frac{V}{V^*} + I - I^* - I^* \ln\frac{I}{I^*}.$$

The derivative of the Lyapunov function L_2 along the trajectories of system (21) is

$$\begin{aligned} \left.\frac{\mathrm{d}L_2}{\mathrm{d}t}\right|_{(21)} &= \Lambda - \mu S - \mu V - \alpha V - cI - \frac{\Lambda S^*}{S} + \left(\rho + \mu\right) S^* + \beta S^* I \\ &\quad - \beta I^* \left(S + \sigma V\right) + cI^* - \rho S \frac{V^*}{V} + \mu V^* + \alpha V^* + \sigma\beta V^* I \\ &= 2\Lambda + \rho S^* - \mu S - \rho S^* \frac{V}{V^*} - \left(\rho + \mu\right)\frac{S^{*2}}{S} - \beta I^* \frac{S^{*2}}{S} - \rho S \frac{V^*}{V} - \beta S I^* \\ &= 2\left[\left(\rho + \mu\right) S^* + \beta S^* I^*\right] + \rho S^* - \mu S - \rho S^* \frac{V}{V^*} - \beta I^* \frac{S^{*2}}{S} \\ &\quad - \left(\rho + \mu\right)\frac{S^{*2}}{S} - \rho S \frac{V^*}{V} - \beta S I^* \\ &= -\left(\mu S^* + \beta S^* I^*\right)\left(\frac{S^*}{S} + \frac{S}{S^*} - 2\right) - \rho S^* \left(\frac{S^*}{S} + \frac{V}{V^*} + \frac{SV^*}{S^*V} - 3\right). \end{aligned} \tag{22}$$

Because the arithmetic mean is larger than or equals to the geometric mean, we have

$$\frac{S^*}{S}+\frac{S}{S^*}-2\geq 0,$$

$$\frac{S^*}{S}+\frac{V}{V^*}+\frac{SV^*}{S^*V}-3\geq 0.$$

Hence for $\forall S,V,I,Q\in\Omega_1$, there is $\left.\frac{\mathrm{d}L_2}{\mathrm{d}t}\right|_{(21)}\leq 0$. So the endemic equilibrium E^* of system (5) is stable.

In addition, if $S^*\neq S, V^*\neq V$, then, $\left.\frac{\mathrm{d}L_2}{\mathrm{d}t}\right|_{(21)}<0$, hence, the endemic equilibrium $\boldsymbol{E}^*$ is globally asymptotically stable. The proof is complete. □

4.2. Local Stability of the Equilibria When $\tau\neq 0$

Theorem 4.3. For any time delay $\tau\neq 0$, the disease-free equilibrium $\boldsymbol{E}^0$ is locally asymptotically stable if $R_0<1$, and unstable if $R_0>1$.

Proof. We obtain that the Jacobian matrix (23) of the system (5) at the disease-free equilibrium $\boldsymbol{E}^0$ is

$$\boldsymbol{J}\left(\boldsymbol{E}_0\right)=\begin{pmatrix}-\mu-\rho\mathrm{e}^{-\lambda\tau} & 0 & -\beta S_0 & 0\\ \rho\mathrm{e}^{-\lambda\tau} & -\mu-\alpha & -\sigma\beta V_0 & 0\\ 0 & 0 & \beta\left(S_0+\sigma V_0\right)-c & 0\\ 0 & 0 & \kappa & -\mu-\gamma\end{pmatrix} \tag{23}$$

Hence, we get the characteristic equation for the above matrix $\boldsymbol{J}\left(\boldsymbol{E}_0\right)$ is

$$\left(\lambda+\mu+\rho\mathrm{e}^{-\lambda\tau}\right)\left(\lambda-\beta\left(S_0+\sigma V_0\right)+c\right)\left(\lambda+\mu+\alpha\right)\left(\lambda+\mu+\gamma\right)=0. \tag{24}$$

If $R_0<1$, then The characteristic Equation (24) always has three negative real roots

$$\lambda_1=\beta\left(S_0+\sigma V_0-c\right),\lambda_2=-\mu-\alpha,\lambda_3=-\mu-\gamma.$$

The other characteristic roots of (24) is determined by the following equation

$$\lambda+\mu+\rho\mathrm{e}^{-\lambda\tau}=0. \tag{25}$$

We suppose that a root of the Equation (24) is imaginary number λ_0. Through calculation, we get that its conjugate $\overline{\lambda_0}$ also satisfies the Equation (24). So the root of the Equation (24) must be a real number. Hence we obtain

$$\lambda=-\mu-\rho\mathrm{e}^{-\lambda\tau}\leq-\mu.$$

Therefore, the disease-free equilibrium $\boldsymbol{E}^0$ is locally asymptotically stable.

If $R_0>1$, then $\lambda=\beta\left(S_0+\sigma V_0\right)-c>0$. Hence the characteristic Equation (24) has at least one positive root. So the disease-free equilibrium $\boldsymbol{E}^0$ is unstable. The proof is complete. □

When $\tau\neq 0$, we get the characteristic equation about the matrix (14) is

$$\left(\lambda^3+b_1\lambda^2+b_2\lambda+b_3+\left(b_4\lambda^2+b_5\lambda+b_6\right)\mathrm{e}^{-\lambda\tau}\right)\left(\lambda+\mu+\gamma\right)=0, \tag{26}$$

where

$$\begin{aligned}
b_1 &= \frac{\Lambda}{S^*} + \frac{\rho S^*}{V^*} - \rho, \\
b_2 &= \sigma^2\beta^2 V^* I^* + \beta^2 S^* I^* + \frac{\rho\Lambda}{V^*} - \frac{\rho^2 S^*}{V^*}, \\
b_3 &= \frac{\rho\beta S^{*2} I^*}{V^*} + \frac{\sigma^2\beta^2\Lambda V^* I^*}{S^*} - \rho\sigma^2\beta^2 V^* I^*, \\
b_4 &= \rho, \\
b_5 &= \frac{\rho^2 S^*}{V^*}, \\
b_6 &= \rho\sigma\beta^2 S^* I^* + \rho\sigma^2\beta^2 V^* I^*.
\end{aligned} \tag{27}$$

The characteristic Equation (26) always has a negative real root $\lambda = -\mu - \gamma$. And the remaining roots are determined by the following equation,

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 + \left(b_4\lambda^2 + b_5\lambda + b_6\right)\mathrm{e}^{-\lambda\tau} = 0. \tag{28}$$

We assume that the pure imaginary number $\lambda = i\omega$ is the root of the Equation (28), and substitute $\lambda = i\omega$ into the equation,

$$\begin{aligned}
&-\omega^3 i - \left(b_4\left(\cos(\lambda\tau) - \sin(\lambda\tau)i\right) + b_1\right)\omega^2 + \left(b_5\left(\cos(\lambda\tau) - \sin(\lambda\tau)i\right) + b_2\right)\omega i \\
&+ b_6\left(\cos(\lambda\tau) - \sin(\lambda\tau)i\right) + b_3 = 0
\end{aligned} \tag{29}$$

We separate the real and imaginary parts of the Equation (29) and obtain the following system,

$$\begin{cases}
-b_1\omega^2 + b_3 = \left(b_4\omega^2 - b_6\right)\cos(\lambda\tau) - b_5\omega\sin(\lambda\tau), \\
-\omega^3 + b_2\omega = -\left(b_4\omega^2 - b_6\right)\sin(\lambda\tau) - b_5\omega\cos(\lambda\tau).
\end{cases} \tag{30}$$

This leads to

$$\left(-b_1\omega^2 + b_3\right)^2 + \left(-\omega^3 + b_2\omega\right)^2 = \left(b_4\omega^2 - b_6\right)^2 + b_5^2\omega^2. \tag{31}$$

We make $\varphi = \omega^2$, and then simplify the Equation (31)

$$G(\varphi) = \varphi^3 + g_1\varphi^2 + g_2\varphi + g_3 = 0, \tag{32}$$

where

$$\begin{aligned}
g_1 &= b_1^2 - b_4^2 - 2b_2, \\
g_2 &= b_2^2 + 2b_4b_6 - 2b_1b_3 - b_5^2, \\
g_3 &= b_3^2 - b_6^2.
\end{aligned} \tag{33}$$

Theorem 4.4. For any time delay $\tau \neq 0$, if $R_0 > 1$ and the coefficients in the $G(\varphi)$ meet the Routh-Hurwitz Criteria, then the endemic equilibrium $\boldsymbol{E}^*$ is locally asymptotically stable.

Proof. If the coefficients in the $G(\varphi)$ meet the Routh-Hurwitz Criteria, then there is no positive real root in the Equation (32). Hence, there is no pure imaginary root in the Equation (28). It is known from Theorem 4.2. that when $\tau = 0$, all roots of the Equation (28) are negative real parts. According to the

continuity of roots to parameters, for any time delay τ, all roots of the Equation (28) have strict negative real parts. Therefore the endemic equilibrium $\boldsymbol{E}^*$ is locally asymptotically stable. The proof is complete. □

Theorem 4.5. For any time delay $\tau \neq 0$, if $R_0 > 1$, then the endemic equilibrium $\boldsymbol{E}^*$ is locally asymptotically stable when $b_3 - b_6 < 0$ and $\tau \in [0, \tau_0)$; The system (5) undergoes a Hopf bifurcation at $\boldsymbol{E}^*$ when $\tau = \tau_k$; the endemic equilibrium $\boldsymbol{E}^*$ is unstable when $\tau > \tau_0$.

Proof. Because $b_3 + b_6 = \sigma\beta^2\rho S^* I^* + \dfrac{\rho\beta S^{*^2} I^*}{V^*} + \dfrac{\sigma^2\beta^2\Lambda V^* I^*}{S^*} > 0$, $g_3 < 0$ when $b_3 - b_6 < 0$. Hence $G(0) < 0$, $G(+\infty) > 0$. Because $G(\varphi)$ is a continuous function, at least one positive real number φ satisfies $G(\varphi) = 0$. Therefore the Equation (28) has a pair of pure imaginary roots $\pm i\omega_0$.

We get $\tau_k > 0$ from Equation (18), and the Equation (28) has a pair of pure imaginary roots $\pm i\omega_0$ when $\tau = \tau_k$, where

$$\begin{aligned}\tau_k = {} & \frac{1}{\omega_0}\arccos\frac{\left(-b_1\omega_0^2 + b_3\right)\left(b_4\omega_0^2 - b_6\right) - b_5\omega_0\left(-\omega_0^3 + b_2\omega_0\right)}{\left(b_4\omega_0^2 - b_6\right)^2 + \left(b_5\omega_0\right)^2} \\ & + \frac{2k\pi}{\omega_0}, (k = 0, 1, 2, \cdots).\end{aligned} \tag{34}$$

Taking the derivative of the Equation (28) with respect to τ,

$$\begin{aligned}&\left(3\lambda^2 + 2b_1\lambda + b_2\right)\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right) + \left(2b_4\lambda + b_5\right)\mathrm{e}^{-\lambda\tau}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right) \\ &+ \left(b_4\lambda^2 + b_5\lambda + b_6\right)\mathrm{e}^{-\lambda\tau}\left(-\lambda - \tau\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right) = 0,\end{aligned} \tag{35}$$

we obtain

$$\begin{aligned}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} &= \frac{\left(3\lambda^2 + 2b_1\lambda + b_2\right) + \left(2b_4\lambda + b_5\right)\mathrm{e}^{-\lambda\tau}}{\lambda\left(b_4\lambda^2 + b_5\lambda + b_6\right)\mathrm{e}^{-\lambda\tau}} - \frac{\tau}{\lambda} \\ &= \frac{3\lambda^2 + 2b_1\lambda + b_2}{-\lambda\left(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3\right)} + \frac{2b_4\lambda + b_5}{\lambda\left(b_4\lambda^2 + b_5\lambda + b_6\right)} - \frac{\tau}{\lambda}.\end{aligned} \tag{36}$$

Thus,

$$\begin{aligned}&\operatorname{sign}\left\{\frac{\mathrm{d}\left(Re\lambda\right)}{\mathrm{d}\tau}\bigg|_{\tau=\tau_k}\right\} = \operatorname{sign}\left\{Re\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\bigg|_{\lambda=i\omega_0}\right\} \\ &= \operatorname{sign}\left\{Re\left[\frac{3\lambda^2 + 2b_1\lambda + b_2}{-\lambda\left(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3\right)} + \frac{2b_4\lambda + b_5}{\lambda\left(b_4\lambda^2 + b_5\lambda + b_6\right)}\right]\bigg|_{\lambda=i\omega_0}\right\} \\ &= \operatorname{sign}\left\{\frac{3\omega_0^6 + 2\left(b_1^2 - 2b_2\right)\omega_0^4 + \left(b_2^2 - 2b_1b_3\right)\omega_0^2}{\left(-b_1\omega_0^3 + b_3\right)^2 + \left(-\omega_0^4 + b_2\omega_0^2\right)^2} + \frac{-2b_4^2\omega_0^4 + \left(2b_4b_6 - b_5^2\right)\omega_0^2}{\left(b_4\omega_0^3 - b_6\omega_0\right)^2 + b_5^2\omega_0^4}\right\}\end{aligned} \tag{37}$$

From (31), we get $\left(-b_1\omega_0^2 + b_3\right)^2 + \left(-\omega_0^3 + b_2\omega_0\right)^2 = \left(b_4\omega_0^2 - b_6\right)^2 + b_5^2\omega_0^2$, hence

$$\operatorname{sign}\left\{\left.\frac{\mathrm{d}(Re\lambda)}{\mathrm{d}\tau}\right|_{\tau=\tau_k}\right\}=\operatorname{sign}\left\{Re\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Big|_{\lambda=i\omega_0}\right\}=\operatorname{sign}\left\{\frac{G'\left(\omega_0^2\right)}{\left(b_4\omega_0^2-b_6\right)^2+b_5^2\omega_0^2}\right\}. \quad (38)$$

When $b_3-b_6<0$, $G'\left(\omega_0^2\right)>0$, so $\operatorname{sign}\left\{\frac{G'\left(\omega_0^2\right)}{\left(b_4\omega_0^2-b_6\right)^2+b_5^2\omega_0^2}\right\}=1$.

Hence $\left.\frac{\mathrm{d}(Re\lambda)}{\mathrm{d}\tau}\right|_{\tau=\tau_k}>0$.

When $\tau=\tau_0$, the Equation (28) has a pair of pure imaginary roots $\pm i\omega_0$, and the other roots are not integral multiples of $i\omega_0$. And $\left.\frac{\mathrm{d}(Re\lambda)}{\mathrm{d}\tau}\right|_{\tau=0}\neq 0$. Therefore System (5) undergoes a Hopf bifurcation at $\boldsymbol{E}^*$.

The Equation (28) does not have a pair of pure virtual roots when $\tau\in[0,\tau_0)$. According to the theorem 4.2 and the theorem 4.4, we obtain that the endemic equilibrium $\boldsymbol{E}^*$ is locally asymptotically stable.

Because $\left.\frac{\mathrm{d}(Re\lambda)}{\mathrm{d}\tau}\right|_{\tau=\tau_k}>0$, the Equation (28) has positive real number roots when $\tau>\tau_0$. Therefore, the endemic equilibrium $\boldsymbol{E}^*$ is unstable. The proof is complete. □

5. Numerical Simulation

In this section, we give some examples to imitate the theoretical results and to verify the correctness of the theoretical results.

Example 1. We choose a set parameters $\Lambda=1$, $\beta=0.2$, $\mu=0.05$, $\rho=0.8$, $\sigma=0.1$, $\nu=0.45$, $\kappa=0.3$, $\gamma=0.75$, $\alpha=0.28$, $\delta=0.01$, from which one can obtain $R_0=0.3609<1$. Therefore the disease-free equilibrium $\boldsymbol{E}^0$ of system (5) is globally asymptotically stable (The figures of the system (5) are like as Figure 1).

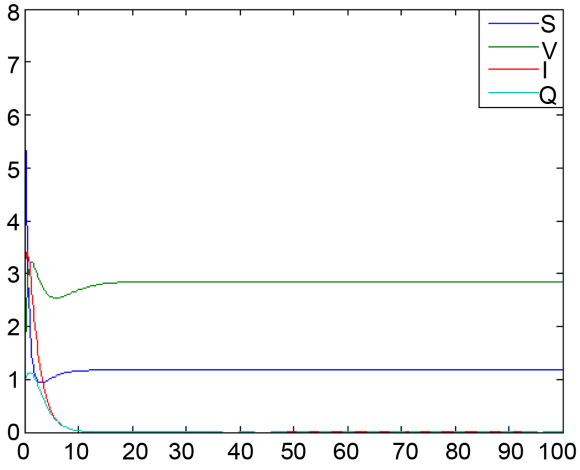

Figure 1. For $\tau=0$, the disease-free equilibrium $\boldsymbol{E}^0$ of system (5) is globally asymptotically stable.

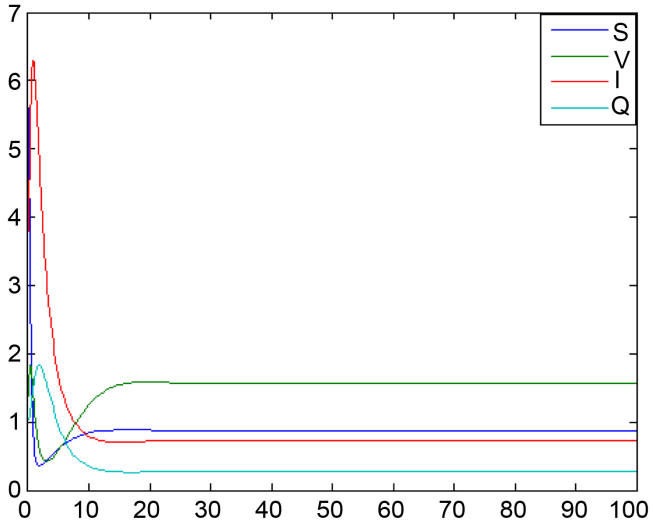

Figure 2. For $\tau = 0$, The endemic equilibrium $\boldsymbol{E}^*$ of system (5) is globally asymptotically stable.

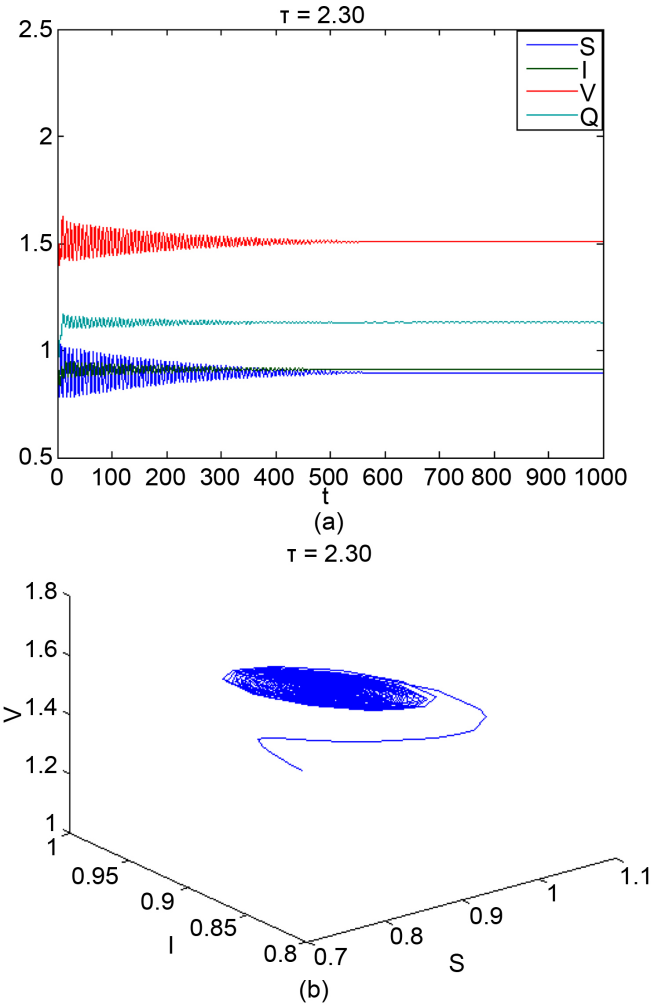

Figure 3. For $\tau = 2.30$, The endemic equilibrium $\boldsymbol{E}^*$ of system (5) is locally asymptotically stable.

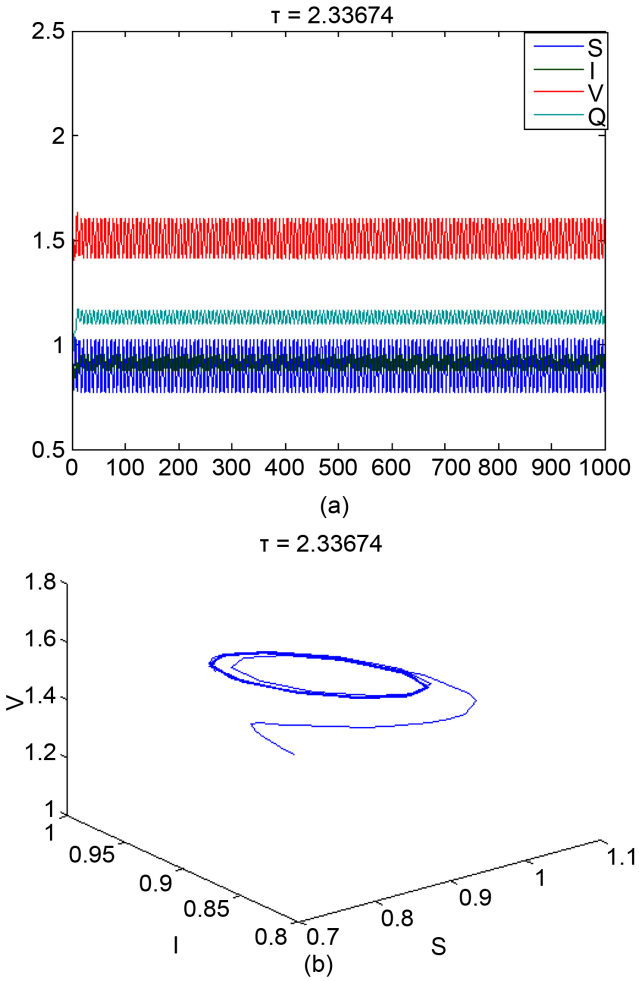

Figure 4. For $\tau = 2.33674$, (a) The endemic equilibrium $\boldsymbol{E}^*$ is periodic solution. (b) The system (5) undergoes a Hopf bifurcation at $\boldsymbol{E}^*$.

Example 2. We choose another set of parameters $\Lambda = 1$, $\beta = 0.4$, $\mu = 0.05$, $\rho = 0.8$, $\sigma = 0.4$, $\nu = 0.25$, $\kappa = 0.3$, $\gamma = 0.35$, $\alpha = 0.28$, $\delta = 0.01$, from which one can obtain $R_0 = 1.5195 > 1$, $\tau_0 = 2.33674$ and $b_3 - b_6 = -0.0003291 < 0$.

The endemic equilibrium $\boldsymbol{E}^*$ of System (5) is globally asymptotically stable when $\tau = 0$ (The figures of the System (5) are like as **Figure 2**).

The endemic equilibrium $\boldsymbol{E}^*$ is locally asymptotically stable when $b_3 - b_6 < 0$ and $\tau = 2.30 \in [0, \tau_0)$ (The figures of the System (5) are like as **Figure 3**).

The system (5) undergoes a Hopf bifurcation at $\boldsymbol{E}^*$ when $\tau = 2.52 \approx \tau_0$ (The figures of the system (5) are like as **Figure 4**).

6. Conclusion

In this paper, we construct a time-delayed COVID-19 system with vaccination to study the effect of time-delay τ on the stability of the equilibrium points. We

get the basic reproduction number, R_0, by the next generation matrix, and the system has a unique endemic equilibrium when $R_0 > 1$. Due to $\frac{\partial R_0}{\partial \rho} = \frac{\Lambda\beta\mu(\sigma-1)}{c(\mu+\alpha)(\rho+\mu)^2} < 0$, so we can reduce the basic reproduction number by increasing the vaccination rate. That is, if the vaccination rate is increased and COVID-19 is actively and effectively prevented, it will help to control the transmission of COVID-19.

For delay $\tau = 0$, using the Lyapunov function approach, we obtained that the disease-free equilibrium $\boldsymbol{E}^0$ and the endemic equilibrium $\boldsymbol{E}^*$ are globally asymptotically stable when $R_0 < 1$ and $R_0 > 1$, respectively. For any time delay $\tau \neq 0$, using the Routh-Hurwitz Criteria, we obtained that the disease-free equilibrium $\boldsymbol{E}^0$ is locally asymptotically stable when $R_0 < 1$. We also find the critical value τ_0 at the endemic equilibrium. The endemic equilibrium $\boldsymbol{E}^*$ is locally asymptotically stable when $\tau \in [0, \tau_0)$. The Hopf bifurcation appears in the system, and the endemic equilibrium forms periodic solutions when $\tau = \tau_0$. From a biological point of view, the time delay affects the spread of infectious diseases. Therefore, in the epidemic prevention and control measures, we should strengthen the development of vaccines to reduce the time to generate antibodies after vaccination, so that we can better control the spread of the epidemic.

Our analysis is still not complete. We have not been able to show the global stability of the infection-free equilibrium and the endemic equilibrium for any delay $\tau \neq 0$, which is planned in our future research.

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Ethics Approval

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Consent for Publication

Consent for publication.

Availability of Data and Material

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Code Availability

Software application or custom code.

Authors' Contributions

The main idea of this paper was proposed by the authors and the authors completed the final manuscript together.

Conflicts of Interest

The authors declare they have no financial interests.

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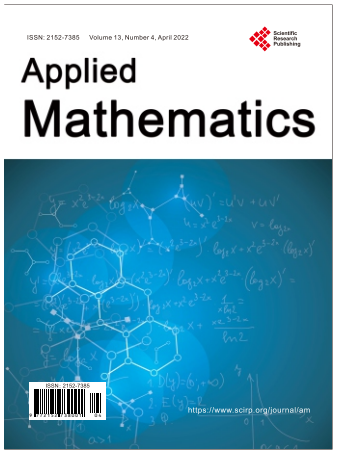

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