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Positive-Definite Operator-Valued Kernels and Integral Representations

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ABSTRACT

A truncated trigonometric, operator-valued moment problem in section 3 of this note is solved. Let $\Gamma^s = \left\{\Gamma_n^s \in L(H), \Gamma_n^* = \Gamma_{-n}, \forall n \in Z^p, |n_i| \le s_i, 1 \le i \le p\right\}$ be a finite sequence of bounded operators, with $s = (s_1, \cdots, s_p) \in N^p, p \ge 1$ arbitrary, acting on a finite dimensional Hilbert space *H*. A necessary and sufficient condition on the positivity of an operator kernel for the existence of an atomic, positive, operator-valued measure E_Γ, with the property that for every $n \in Z^p$ with $|n_i| \le s_i, 1 \le i \le p$, the n^{th} moment of E_Γ coincides with the n^{th} term Γ_n^s of the sequence, is given. The connection between some positive definite operator-valued kernels and the Riesz-Herglotz integral representation of the analytic on the unit disc, operator-valued functions with positive real part in the class of operators in Section 4 of the note is studied.

Keywords: Unitary-Operator; Self-Adjoint Operator; Joint Spectral Measure of a Commuting Tuple of Operators; Spectral Projector; Complex Moments; Analytic Vectorial Functions

1. Introduction

About the scalar complex trigonometric moment problem we recall that: a sequence $\{t_n\}_{n\in Z}$ of complex numbers with $t_n = \overline{t_{-n}}$ is called positive semi-definite if for each $n \ge 0$, the Toeplitz matrix $T_n = (t_{i-j})_{i,j=0}^n$ is positive semi-definite. The problem of characterising the positive semi-definiteness of a sequence of complex numbers was completely solved by Carathéodory in [1], in the following theorem:

Theorem 1. *The Toeplitz matrix* $T_n = (t_{i-j})_{i,j=0}^n$ *is positive semi-definite and rank* $T_n = r$ *with* $1 \le r \le n+1$ *if and only if the matrix* T_{r-1} *is invertible and there exists* $\alpha_j \in T_1, j = 1,2,\cdots,r$ *with* $\alpha_j \ne \alpha_k$ *for* $j \ne k$ *and*

$$\rho_j > 0, j = 1,\cdots,r$$

such that

$$t_k = \sum_{j=1}^r \rho_j \alpha_j^k, \text{ for } k = 0,1,\cdots,n. \quad (1.1)$$

In the same paper [1], Charathéodory also proved that: if $1 \le r \le n$, then $\{\alpha_1,\cdots,\alpha_r\}$ are the roots of the polynomial

$$P(z) = \det\begin{pmatrix} t_0 & \overline{t_1} & \cdots & \overline{t_r} \\ t_1 & t_0 & \cdots & \overline{t_{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ t_{r-1} & t_{r-2} & \cdots & \overline{t_1} \\ 1 & z & \cdots & z^r \end{pmatrix}$$

which are all distinct and belong to T_1.

Another characterization of the positive semi-definiteness of a sequence of complex numbers was obtained by Herglotz in [2]. In [2], for $n \in Z$, the n^{th} moment of a finite measure μ on T_1 is defined by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\text{int}} d\mu(t) = \tilde{\mu}(n).$$

The following characterization of the positivity of a complex moment sequence is the main result in [2].

Theorem 2. *A sequence of complex numbers* $(t_n)_{n\in Z}$, $t_n = \overline{t_{-n}}$ *is positive semi-definite if and only if there exists a positive measure* μ *on the unit circle* T_1, *such that* $t_n = \tilde{\mu}(n)$ *for* $n \in Z$.

From Theorem 1 and Theorem 2, Charathéodory and Fejér in [3] deduce the following theorem:

Theorem 3. *Let* $(t_j)_{j=-n}^n$ *be given complex numbers. Then there exists a positive measure* μ *on* T_1, *such* *that*

$$\tilde{\mu}(j) = t_j, |j| \le n, \quad (1.2)$$

if and only if the Toeplitz matrix $T_n = (t_{i-j})_{i,j=0}^{n}$ *is positive semi-definite. Moreover, if* $1 \le rank\, T_n = r \le (n+1)$, *then there exists a positive measure* μ *supported on* r *points of the unit circle* T_1 *which satisfies* (1.2.)

Theorem 3 gives an answer to the scalar, truncated trigonometric moment problem.

Operator-valued truncated moment problems were studied in [4,5]. Regarding the truncated, trigonometric operator-valued moment problem, we recall that:

1) $E(\lambda)$, $-\pi \le \lambda \le \pi$ is called a spectral function if (a) each $E(\lambda)$ is a bounded, positive operator, (b) $E(\lambda) \le E(\mu)$ for $\lambda \le \mu$; it is orthogonal if each $E(\lambda)$ is an othogonal projection;

2) a finite sequence $\{A_0, \cdots, A_n\}$ of bounded operators on an arbitrary Hilbert space is called a trigonometric moment sequence if, there exists a spectral function $E(\lambda), (-\pi \le \lambda \le \pi)$ such that $A_k = \int_{-\pi}^{\pi} e^{ik\lambda} dE(\lambda)$ for every $k = 0, \cdots, n.$ In [4], the necessary and sufficient condition of representing a finite sequence of bounded operators on an arbitrary Hilbert space *H*, $\{A_k\}_{k=-n}^{n}$ with $A_n = A_{-n}^*, A_0 = Id_H$ as a trigonometric moment sequence is the positivity of the Toeplitz matrix $T_n = (A_{i-j})_{i,j=0}^{n}$ obtained with the given operators. The representing spectral function is obtained in [4] by generating an unitary operator, defined on the direct sum of $(n+1)$ copies of the Hilbert space H for obtaining an orthogonal spectral function and by applying Naimark's dilation theorem to get the representing spectral function from it. In [5], a multidimensional operator-valued truncated moment problem is solved. That is: given a sequence of bounded operators

$$\{\Gamma_n^s\}, n \in Z^p, |n_i| \le s_i, 1 \le i \le p,$$

acting on an arbitrary Hilbert space *H*, with

$$s = (s_1, \cdots, s_p) \in N^p, \Gamma_n^s = \Gamma_{-n}^{s*},$$

a necessary and sufficient condition for representing any such operator

$$\Gamma_m^s, m = (m_1, \cdots, m_p) \in Z^p, |m_i| \le s_i, 1 \le i \le p$$

as the m^{th} moment of a positive operator-valued measure is given. The necessary and sufficient condition in [5] for such a representation is again the positivity of the Toeplitz matrix

$$\{\Gamma_{l-m}\}_{l_i, m_i = -\left[\frac{s_i}{2}\right], i \in \overline{1,p}}^{\left[\frac{s_i+1}{2}\right]}$$

obtained with the given operators. The representing positive operator-valued measure, (spectral function), in [5] is obtained by applying Kolmogorov's decomposition positive kernels theorem.

Concerning the complex, operator-valued moment problem on a compact semialgebraic nonvoid set K, we recall that a sequence of bounded operators

$$\Gamma = \{\Gamma_{\alpha,\beta}\}_{\alpha,\beta \in Z_+^p},$$

acting on an arbitrary complex Hilbert spacea *H*, subject on the conditions $\Gamma_{\alpha,\beta}^* = \Gamma_{\beta,\alpha}$, $\Gamma_{0,0} = 1_H$ is called a K moment sequence if there exists an operator-valued positive measure F_Γ on K such that

$$\Gamma_{\alpha,\beta} = \int_K \bar{z}^\alpha z^\beta dF_\Gamma(z), \alpha, \beta \in Z_+^p.$$

A sequence of bounded operators $\{\Gamma_n\}_{n \in Z^p}$ with $\Gamma_n^* = \Gamma_{-n}, \forall n \in Z^p$ and $\Gamma_0 = Id_H$, acting on an arbitrary, complex, Hilbert space is called a trigonometric operator-valued moment sequence, if there exists a positive, operator-valued measure F_Γ on the p-dimensional complex torus T_1^p such that $\Gamma_\alpha = \int_{T_1^p} z^\alpha dF_\Gamma(z)$ for all $\alpha \in Z^p$. Some of the papers devoted to operator-valued moment problems are: [6-10], to quote only few of them. The operator-valued multidimensional complex moment problem is solved in [9] in the class of commuting multioperators that admit normal extension (subnormal operators) (Theorem 1.4.8., p. 188). In [9], Corollary 1.4.10., a necessary and sufficient condition for solving a trigonometric operator-valued moment problem is given. In [10], another proof of a quite similar necessary and sufficient existence condition on a sequence of bounded operators to admit an integral representation as trigonometric moment sequence with respect to some positive operator valued measure is given. In Section 4 of this note, we prove that the two existence conditions in [9,10] are equivalent.

The present note studies in Section 3 the representation measure of the truncated operator-valued moment problem in [5], only when the given operators act on a finite dimensional Hilbert space. In Proposition 3.1, Section 3, it is shown that the representing measure, in this case, is an atomic one. In Proposition 3.2, Section 3, the necessary and sufficient existence condition in Proposition 3.1 is stated also in terms of matrices.

In Section 4 of the note, is studied the connection between the problem of representing the terms of an operator sequence

$$\{\Gamma_n\}_{n \in Z}, \Gamma_n \in L(H), \Gamma_n^* = \Gamma_{-n}, \forall n \in Z, \Gamma_0 = Id_H$$

as moments of an operator valued, positive measure and the problem of Riesz-Herglotz type integral representation of some operator-valued, analytic function, with positive real part in the class of operators.

2. Preliminaries

Let $p \in N^*$ arbitrary,

$$s = (s_1, \cdots, s_p) \in N^p,$$

$$z = (z_1, \cdots, z_p) \in C^p, \overline{z} = (\overline{z_1}, \cdots, \overline{z_p}) \in C^p,$$

$$t = (t_1, \cdots, t_p) \in R^p$$

denote the complex, respectively the real variable in the complex, respectively real euclidian space. For

$$m = (m_1, \cdots, m_p) \in Z^p, q = (q_1, \cdots, q_p) \in N^p,$$

we denote

$$z^m = z_1^{m_1} \cdots z_p^{m_p}, z_i \neq 0, 1 \leq i \leq p,$$

$$\overline{z}^m = \overline{z}_1^{m_1} \cdots \overline{z}_p^{\ m_p}$$

and by $t^q = t_1^{q_1} \cdots t_p^{q_p}$. The sets:

$$T_1^p = \{(z_1, \cdots, z_p), |z_i| = 1 \text{ for all } 1 \leq i \leq p\}$$

represent the torus in C^p and $D = \{z \in C, |z| < 1\}$ the unit disc in C; if

$$(z_1, \cdots, z_p) \in T_1^p$$

and

$$m_i \leq 0, z_i^{m_i} = \overline{z_i}^{-m_i}.$$

For $s = (s_1, \cdots, s_p) \in N^p$, we denote with $\left[\frac{s_i}{2}\right]$ the integer part of the number $\frac{s_i}{2}$. The addition and subtraction in N^p, respectively in Z^p are considered on components. In the set $\{n \in Z^p, |n_i| \leq s_i, 1 \leq i \leq p\}$ the elements are treated in lexicographical order. If H is an arbitrary complex Hilbert space and

$$N = (N_1, \cdots, N_p) \in L^p(H)$$

a commuting multioperator, we denote by

$$N^m = N_1^{m_1} \circ \cdots \circ N_p^{m_p}$$

for all $m \in N^p$ and, as usual, $L(H)$ is the algebra of bounded operators on H; also δ_{ij} denotes the Kronecker symbol for $i, j \in Z$. Let

$$\Gamma^s = \{\Gamma_n^s\}_{n \in Z^p} = \{\Gamma_n^s \in L(H), |n_i| \leq s_i, \forall 1 \leq i \leq p\}$$

be a sequence of bounded operators on H subject to the conditions $\Gamma_{-n}^s = \Gamma_n^{s*}$ for all

$$n \in Z^p, |n_i| \leq s_i, \forall 1 \leq i \leq p$$

and $\Gamma_0^s = Id_H$. For such a finite sequence of operators, in [5], a necessary and sufficient condition for the existance of a a positive Borel operator-valued measure F_Γ on $Bor(T_1^p)$, such that the representations

$$\Gamma_n^s = \int_{T_1^p} z^n \mathrm{d}F_\Gamma(z), \forall n = (n_1, \cdots, n_p) \in Z^p, \quad |n_i| \leq s_i, 1 \leq i \leq p \tag{2.1.}$$

hold, it is given. Such a measure is called a representing measure for $\{\Gamma_n^s\}_{n \in Z^p}$.

In Section 3 of this note, in Proposition 3.1, we give a necessary and sufficient condition for the existence of an atomic representing measure of a truncated, operator-valued moment problem as in (2.1.) in case that the operators $\{\Gamma_n^s\}_{n \in Z^p}$. act on a finite dimensional Hilbert space. In Proposition 3.2 of this note, the necessary and sufficient existence condition for the representing measure in (2.1.) is reformulated in terms of matrices.

In section 4, Proposition 4.2, we establish a Riesz-Herglotz formula for representing an analytic, operator-valued function on D, with real positive part in the class of operators. The obtained, representation formula for such functions is the same as in the scalar case [11, 12]. In this case, the representing measure is a positive operator-valued measure. The proof of Proposition 4.1 in this note is based on the characterization on an operator-sequence $\{\Gamma_n\}_{n \in Z^p}$ to be a trigonometric, operator-valued moment sequence in [9]. The represented analytic, operator-valued function is the function which has as the Taylor' s coefficients the operators $\{\Gamma_k\}_{k \in N}$.

3. An Operator-Valued Truncated Trigonometric Moment Problem on Finite Dimensional Spaces

Let $s = (s_1, \cdots, s_p) \in N^p$ be arbitrary and consider the set

$$I = \otimes_1^p \left\{-\left[\frac{s_i}{2}\right], \left(-\left[\frac{s_i}{2}\right] + 1\right), \cdots, \left[\frac{s_i + 1}{2}\right]\right\}$$

with the lexicographical order ($\otimes$ represents the cartesian product of the mentioned sets), H a finite dimensional Hilbert space with

$$\dim_C H = r \quad \text{and} \quad d = \left[\prod_{j=1}^p (s_j + 1)\right] \cdot r.$$

Proposition 3.1. *Let*

$$\Gamma^s = \{\Gamma_n^s\}_{n \in Z^p} = \{\Gamma_n^s \in L(H), |n_i| \leq s_i, \forall 1 \leq i \leq p\}$$

be a sequence of bounded operators on H, *with* $\Gamma_{-n}^s = \Gamma_n^{s*}$ *for all* $n \in Z^p, |n_i| \leq s_i, \forall 1 \leq i \leq p, \Gamma_0^s = Id_H$. *The following assertions are equivalent*:

(i) $\sum_{m,n\in I}\left\langle \Gamma^s_{n-m}x_n, x_m\right\rangle_H \geq 0$ for all sequences $\{x_n\}_{n\in I}$ in H.

(ii) There exists the multisequence

$$\left\{\left(\lambda^1_{i_1},\lambda^2_{i_2},\cdots,\lambda^p_{i_p}\right)\in T_1^{\,p},\left(i_1,\cdots,i_p\right)\in\{1,\cdots,d\}^p\right\}$$

of d^p points and the bounded, positive operators, $F^{12\cdots p}_{i_1i_2\cdots i_p}$ such that

$$\Gamma^s_{n_1,\cdots,n_p} = \sum_{\left(i_1,\cdots,i_p\right)\in\{1,\cdots,d\}^p}\left(\lambda^1_{i_1}\right)^{n_1}\left(\lambda^2_{i_2}\right)^{n_2}\cdots\left(\lambda^p_{i_p}\right)^{n_p}F^{12\cdots p}_{i_1i_2\cdots i_p} \quad (3.1)$$

for all $n=\left(n_1,\cdots,n_p\right)\in Z^p, |n_i|\leq s_i, 1\leq i\leq p.$

(iii) *There exists a positive atomic operator-valued measure* F_Γ *on* $Bor\left(T_1^{\,p}\right)$ *such that*:

$$\Gamma^s_n = \int_{T_1^p} z^n \mathrm{d}F_\Gamma(z), \forall n=\left(n_1,\cdots,n_p\right)\in Z^p,$$

with $|n_i|\leq s_i, 1\leq i\leq p.$

Proof. $(i)\Rightarrow(ii)$. On the set

$$I=\otimes_1^p\left\{-\left[\frac{s_i}{2}\right],-\left[\frac{s_i}{2}+1\right],\cdots,\left[\frac{s_i+1}{2}\right]\right\}$$

we have the lexicographical order. The finite sequence of operators $\Gamma^s=\left\{\Gamma^s_n\right\}_{n\in Z^p,|n_i|\leq s_i}$ is considered double indexed *i.e.* $\Gamma^s_{n-m}=\Gamma(n,m)^s$; with this assumption, from (i), Γ^s can be viewed as an operator-valued kernel

$$\Gamma^s: I\times I\to L(H); \Gamma(n,m)^s=\Gamma^s_{n-m}.$$

Let $F=\{f: I\to H\}$ the C-vector space of functions defined on I with values in the finite dimensional Hilbert space H. With the aid of Γ^s, we can introduce on F the non-negative hermitian product:

$$\langle f,g\rangle_{\Gamma^s}=\sum_{i,j\in I}\left\langle\Gamma(i,j)f(i),g(j)\right\rangle_H;$$

according to (i), we have the positivity condition:

$$\langle f,f\rangle_{\Gamma^s}=\sum_{i,j\in I_1}\left\langle\Gamma(i,j)f(i),f(j)\right\rangle_H\geq 0.$$

The matrix associated to this kernel is a Toeplitz matrix of the form:

$$\Gamma^s_1=\begin{pmatrix}\Gamma^s_{0\cdots0} & \Gamma^s_{0\cdots-1} & \cdots & \Gamma^s_{-s_1-s_2\cdots-s_p}\\ \Gamma^s_{0\cdots1} & \cdots & \cdots & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ \Gamma^s_{s_1s_2\cdots s_p} & \cdots & \cdots & \Gamma^s_{0\cdots0}\end{pmatrix}.$$

From Kolmogorov's theorem, there exists the Hilbert space (essentially unique) K, obtained as the separate completeness of the C vector space of functions F with respect to the usual norm generated on the set of cosets of Cauchy sequences, (*i.e.* $F/'\sim'$), by the non-negative kernel Γ^s, respectively the space $K=\overline{F}/'\sim'^{\|\|}$ (when H is finite dimensional, the Hilbert space $K=\left\{\hat{f}, f\in F\right\}$). From the same theorem, there also exists the sequence of operators $\{h_m\}_{m\in I}\in L(H,K)$ such that $\Gamma^s(n,m)=h^*_mh_n$ for all $m,n\in I$. In this particular case for F, we have

$$K=\overline{F}/'\sim'^{\|\|\Gamma}=V_{m\in I_1}Ranh_mx$$

where $Ranh_mx$ denotes the range of the operators h_mx and $VRanh_mx$ denotes the closed linear span of the sets $Ranh_mx$, $x\in H$. The operators $h_m: H\to K$ are:

$$h_m(x)(j)=\prod_{k=1}^p\delta_{m_kj_k}x$$

with $m=\left(m_1,\cdots,m_p\right), j=\left(j_1,\cdots,j_p\right)\in I, x\in H$ and δ_{ij} the Kronecker symbol. Also, from the construction of K, we have $K=\overline{F}^{\|\|\Gamma}=V_{m\in Z^p}Ranh_mx$, where $Ranh_mx$ denotes the range of the operators h_mx and $VRanh_mx$ denotes the closed linear span of the sets $Ranh_mx$.

Let us consider the subsets

$$I_{1i}=\left\{-\left[\frac{s_1}{2}\right],\cdots,\left[\frac{s_1+1}{2}\right]\right\}\times\cdots$$
$$\times\left\{-\left[\frac{s_i}{2}\right],\cdots,\left[\frac{s_i-1}{2}\right]\right\}\times\cdots$$
$$\times\left\{-\left[\frac{s_p}{2}\right],\cdots,\left[\frac{s_p+1}{2}\right]\right\}\subset I,$$

$$I_{2i}=\left\{-\left[\frac{s_1}{2}\right],\cdots,\left[\frac{s_1+1}{2}\right]\right\}\times\cdots$$
$$\times\left\{-\left[\frac{s_i}{2}\right]+1,\cdots,\left[\frac{s_i+1}{2}\right]\right\}\times\cdots$$
$$\times\left\{-\left[\frac{s_p}{2}\right],\cdots,\left[\frac{s_p+1}{2}\right]\right\}\subset I,$$

the subspaces in K, $K_{1i}=V_{m\in I_{1i}}Ranh_mx$, $K_{2i}=V_{m\in I_{2i}}Ranh_mx$ and the operators $A_i: K_{1i}\to K_{2i}$ defined by the formula

$$A_i\left(\sum_{n\in I_{1i}}h_nx_n\right)=\sum_{n\in I_{1i}}h_{n+e_i}x_n$$

for any $1\leq i\leq p$ with $e_i=\left(\delta_{ik}\right)_{k=1}^p$ the standard basis in C^p. From the definition of A_i, since h_m are linear for all $m\in I_{1i}$, the same is true for the operators A_i for all $1\leq i\leq p$. For an arbitrary

$$y\in K_{1i}, y=\sum_{n\in I_{i1}}h_nx_n, \{x_n\}_{n\in I_{i1}}\subset H,$$

we have:

$$\langle A_i y, A_i y\rangle_{\Gamma^s} = \left\langle A_i \sum_{n\in I_{i1}} h_n x_n, A_i \sum_{m\in I_{i1}} h_m x_m \right\rangle_{\Gamma^s}$$
$$= \left\langle \sum_{n\in I_{i1}} h_{n+e_i} x_n, \sum_{m\in I_{i2}} h_{m+e_i} x_m \right\rangle_{\Gamma^s} = \left\langle \sum_{n,m\in I_{i1}} h^*_{m+e_i} h_{n+e_i} x_n, x_m \right\rangle_{\Gamma^s}$$
$$= \left\langle \sum_{n,m\in I_{i1}} \Gamma^s_{n-m} x_n, x_m \right\rangle_{\Gamma^s} = \left\langle \sum_{n,m\in I_{i1}} h^*_m h_n x_n, x_m \right\rangle_{\Gamma^s}$$
$$= \left\langle \sum_{n\in I_{i1}} h_n x_n, \sum_{m\in I_{i1}} h_m x_m \right\rangle_{\Gamma^s} = \|y\|^2_{\Gamma^s}$$

for all $1 \le i \le p$. We extend A_i to K_{i1} preserving the above definition and boundedness condition; the extensions $A_i : K_{i1} \to K_{i2}, 1 \le i \le p$ are denoted with the same letter A_i In case that

$$\left\{ h_{k_1\cdots k_{i-1}\left[\frac{s_i+1}{2}\right]k_{i+1}\cdots k_p} e_j \text{ and } h_{k_1\cdots k_{i-1}\left(-\left[\frac{s_i}{2}\right]\right)k_{i+1}\cdots k_p} e_j, \quad 1 \le j \le r, 1 \le i \le p \right\}$$

are *C*-linear independent operators with respect to the kernel Γ, and from above, the operators A_i are partial isometries, defined on linear closed subspaces $K_{i1} \subset K$ with values in $K_{i2} \subset K$, with equal deficiency indices. In this case, A_i admit an unitary extension on the whole space K for all $1 \le i \le p.$ Let us denote the extensions of these operators to K with the same letter A_i. The adjoints of A_i are defined by

$$A_i^*\left(\sum\nolimits_{n\in I_{i2}} h_n x_n\right) = \sum\nolimits_{n\in I_{i2}} h_{n-e_i} x_n$$

for all $1 \le i \le p.$ Obviously, for the extended operators $A_i^* A_i = A_i A_i^* = Id_K, 1 \le i \le p.$

In the same time, $A_i A_j x = A_j A_i x$ for all $x \in K_{1i} \cap K_{2i}$ and all $1 \le i, j \le p$; we preserve the commuting relations for the extended operators. When H is a finite dimensional Hilbert space with a basis $\{e_j\}_{j=1}^r$, the same is true for the obtained Hilbert space K All the vectors $\{h_m e_j, m \in I \subset Z^p, m \text{ arbitrary fixed}, j \in \{1,\cdots,r\}\}$ are *C*-linear independent in K with respect to the kernel Γ. Indeed, if

$$\sum\nolimits_{p,q\in I} \left\langle \Gamma_{p-q}\left(\alpha h_m e_i + \beta h_m e_j\right)(p), \left(\alpha h_m e_i + \beta h_m e_j\right)(q)\right\rangle_H = 0$$

equivalent with $|\alpha|^2 + |\beta|^2 + 2\Re\left(\alpha\overline{\beta}\langle e_i, e_j\rangle_H\right) = 0$, this equality implies $\alpha = \beta = 0.$ We consider that all the vectors $\{h_m e_j, m \in I, 1 \le j \le r\}$ are *C*-linear independent in K with respect to the kernel $\Gamma.$ We have then, $\dim_C K = |I| \cdot r = d$.

A basis in K is

$$B = \left\{h_m e_j, m \in I \subset Z^p, j \in \overline{1,r}\right\}.$$

Let $A_i : K_{1i} \to K_{2i}$ be the defined isometries, with

$$K_{1i} = VRan_{m\in I_{1i}, j\in\overline{1,r}} h_m x$$

and

$$K_{2i} = VRan_{m\in I_{2i}} h_m x;$$

for

$$E_{1i} = VRan_{m\in I\subset Z^p, m_i = \left[\frac{s_i+1}{2}\right]} h_m x$$

and

$$F_{2i} = VRan_{m\in I\subset z^p, m_i = -\left[\frac{s_i}{2}\right]} h_m x.$$

We have $K = K_{1i} \oplus E_{1i}$ and also $K = F_{2i} \oplus K_{2i}.$ We consider $\widetilde{E_{1i}}$ the orthonormal algebraic complement of the space K_{1i} in K, respectively $\widetilde{F_{2i}}$ the orthonormal complement of K_{2i}. When

$$q = \left[\prod\nolimits_{k=1,k\ne i}^p (s_k+1)\cdot r\right]$$

for $p \ne 1$ and $q = r$ when $p = 1$; we have

$$\dim_C \widetilde{E_{1i}} = \dim_C \widetilde{F_{2i}} = q.$$

Let $\{u_1^i,\cdots,u_q^i\}$ be an orthonormal basis in E_{1i}, respectively $\{v_1^i,\cdots,v_q^i\}$ an orthonormal basis in $\widetilde{F_{2i}}$ We extend the partial isometries $A_i : K_{1i} \to K_{2i}$, $1 \le i \le p$ to the whole spaces K in the following way:

$$A_i\left(u_j^i\right) = v_j^i, \forall j \in \overline{1,q}, i \in \overline{1,p}.$$

Because

$$\left\langle A_i u_j^i, A_i u_j^i\right\rangle_K = \left\langle v_j^i, v_j^i\right\rangle_K = 1 = \left\langle u_j^i, u_j^i\right\rangle_K,$$

and

$$\left\langle A_i u_j^i, v_k^i\right\rangle_K = \left\langle v_j^i, v_k^i\right\rangle_K = \delta_{jk} = \left\langle u_j^i, A_i^* v_k^i\right\rangle_K = \left\langle u_j^i, u_k^i\right\rangle_K,$$

it results that also the extensions are isometries and $A_i^* = A_i^{-1}$; that is $A_i : K \to K$ are unitary operators for all $1 \le i \le p$; (the extended operators are denoted with the same letters). The commuting relations $A_i A_j = A_j A_i$ are also preserved $1 \le i \le p$ In the above conditions, the commuting multioperator $(A_1,\cdots,A_p)$ consisting of unitary operators on K admits joint spectral measure, whose joint spectrum $\sigma(A_1,\cdots,A_p) \subset T_1^p$. Considering the construction of K, we obtain $A_i^* h_0 = h_{-e_i}$ and by induction $A_i^n = h_{ne_i}$ for all

$$n \in \left\{1,\cdots,\left[\frac{s_i+1}{2}\right]\right\}, 1 \le i \le p.$$

Because on the finite dimensional space K, all the operators $A_i \in L(K)$ are unitary and compact one, their spectrum $\sigma(A_i)\subset T_1$ consists only of the $A_i's$ principal values. The principal values are the roots of the characteristic polynomials associated with the matrix of A_i in suitable basis in K, for all $1\le i\le p$. The characteristic polynomials of A_i are all complex variable polynomials of the same degree

$$d=\left[\prod_{i=1}^{p}(s_i+1)\cdot r\right]=\dim K$$

with the roots $\{\lambda_j^i\}_{j\in\overline{1,d}}, |\lambda_j^i|=1, \forall 1\le i\le p, j\in\overline{1,d}$.

Let $\{P_j^i\}_{j\in\overline{1,d}}, 1\le i\le p$, be the family of the spectral projectors associated with the families of the principal values $\{\lambda_j^i\}_{j\in\overline{1,d}}$ that is $P_j^i=E^i(\{\lambda_j^i\})$ with E^i the spectral measures of $A_i, 1\le i\le p$. From the definition of P_j^i, we have $P_j^i\circ P_q^i=0, (P_j^i)^2=P_j^i$ for all $j\ne q, j,q\in\{1,\cdots,d\}$ and $A_i=\sum_{j=1}^{d}\lambda_j^i\cdot P_j^i$. Because $A_i\circ A_j=A_j\circ A_i$, we have also

$$E^i\circ E^j=E^j\circ E^i, 1\le i,j\le p.$$

Consequently, for $m=(m_1,\cdots,m_p)\in Z^p$, we have obtain:

$$A^m=A_1^{m_1}\circ\cdots\circ A_p^{m_p}=\left[\sum_{j=1}^{d}(\lambda_j^1)^{m_1}P_j^1\right]\circ\cdots\circ\left[\sum_{j=1}^{d}(\lambda_j^p)^{m_p}P_j^p\right].$$

From Kolmogorov's decomposition theorem for $m,n\in I\subset Z^p$, we have

$$\begin{aligned}\Gamma_{n-m}^s&=\Gamma^s(n,m)=h_m^*h_n=h_0^*A^{m*}A^nh_0=h_0^*A_1^{m_1*}\cdots A_p^{m_p*}A_1^{n_1}\cdots A_p^{n_p}h_0\\&=h_0^*\left[\sum_{(i_1,\cdots,i_p)\in\{1,\cdots,d\}^p}\overline{\lambda_{i_1}^1}^{m_1}\cdots\overline{\lambda_{i_p}^p}^{m_p}P_{i_1}^1\cdots P_{i_p}^p\right]\circ\left[\sum_{(s_1,\cdots,s_p)\in\{1,\cdots,d\}^p}\lambda_{s_1}^{1n_1}\cdots\lambda_{s_p}^{pn_p}P_{s_1}^1\cdots P_{s_p}^p\right]h_0\\&=h_0^*\left[\sum_{(i_1,\cdots,i_p)\in\{1,\cdots,d\}^p}\lambda_{i_1}^{1n_1-m_1}\cdots\lambda_{i_p}^{pn_p-m_p}P_{i_1}^1\cdots P_{i_p}^p\right]h_0=\sum_{(i_1,\cdots,i_p)\in\{1,\cdots,d\}^p}\lambda_{i_1}^{1n_1-m_1}\cdots\lambda_{i_p}^{pn_p-m_p}h_0^*P_{i_1}^1\cdots P_{i_p}^ph_0\\&=\sum_{(i_1,\cdots,i_p)\in\{1,\cdots,d\}^p}\lambda_{i_1}^{1n_1-m_1}\cdots\lambda_{i_p}^{pn_p-m_p}F_{i_1i_2\cdots i_p}^{12\cdots p}\end{aligned}$$

with $F_{i_1i_2\cdots i_p}^{12\cdots p}=h_0^*\circ P_{i_1}^1\circ\cdots\circ P_{i_p}^p\circ h_0$ positive operators. That is:

$$\Gamma_n^s=\sum_{(i_1,\cdots,i_p)\in\{1,\cdots,d\}^p}\lambda_{i_1}^{1n_1}\cdots\lambda_{i_p}^{pn_p}F_{i_1i_2\cdots i_p}^{12\cdots p}, \forall n=(n_1,\cdots,n_p)\in Z^p, |n_i|\le s_i, 1\le i\le p \tag{3.2.}$$

(*i.e.* assertion (ii))

$(ii)\Rightarrow(iii)$. Let $F_\Gamma=\sum_{(i_1,i_2,\cdots,i_p)\in\{1,\cdots,d\}^p}F_{i_1i_2\cdots i_p}^{12\cdots p}$ be a positive, atomic operator-valued measure on T_1^p. From $(ii)(3.2.)$ we have:

$$\Gamma_n^s=\int_{T_1^p}z^n\mathrm{d}F_\Gamma(z), \forall n\in Z^p, \text{with } |n_i|\le s_i, 1\le i\le p$$

(*i.e.* assertion (*iii*)).

$(iii)\Rightarrow(i)$. If

$$\Gamma_n^s=\int_{T_1^p}z^n\mathrm{d}F_\Gamma(z), \forall n\in Z^p, \text{with } |n_i|\le s_i, 1\le i\le p$$

and F_Γ is a positive operator-valued measure, we have:

$$\begin{aligned}\sum_{m,n\in I}\langle\Gamma_{n-m}^sx_n,x_m\rangle_H&=\sum_{n,m\in I}\left\langle\int_{T_1^p}z^{n-m}\mathrm{d}F_\Gamma(z)x_n,x_m\right\rangle_H\\&=\int_{T_1^p}\left\langle\sum_{n\in I}z^n\mathrm{d}F_\Gamma^{\frac12}(z),\sum_{m\in I}z^m\mathrm{d}F_\Gamma^{\frac12}(z)\right\rangle_H\\&=\int_{T_1^p}\mathrm{d}\left\|\sum_{n\in I}z^nF_\Gamma^{\frac12}(z)\right\|^2\ge0\end{aligned}$$

that is (i).

Proposition 3.1, in case *H* a finite dimensional space, statements $(i)\Leftrightarrow(ii)$ implies also a similar, straightforward characterization, as in the scalar case [6]:

Proposition 3.2. *When*

$$m=\prod_{j=1}^{p}(s_j+1) \text{ and } \{\Gamma_n^s\}_{n\in Z^p, |n_i|\le s_i},$$

operators acting on a finite dimensional space H *with* $\dim_C=r$, *are as in Proposition* 1, *the Toeplitz matrix*

$$T_m=(\Gamma_n^s)_{n\in Z^p, |n_i|\le s_i, 1\le i\le p}=\Gamma_1^s=\begin{pmatrix}\Gamma_{0\cdots0}^s&\Gamma_{0\cdots-1}^s&\cdots&\Gamma_{-s_1-s_2\cdots-s_p}^s\\\Gamma_{0\cdots1}^s&\cdots&\cdots&\vdots\\\vdots&\ddots&\ddots&\vdots\\\Gamma_{s_1s_2\cdots s_p}^s&\cdots&\cdots&\Gamma_{0\cdots0}^s\end{pmatrix},$$

is positive semidefinite if and only if it can be factorized as $T_m=RDR^*$ *with*

$$R=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_k^1 \\ \vdots & \ddots & \ddots & \vdots \\ \left(\lambda_1^2\right)^{s_2}\cdot\left(\lambda_1^1\right)^{s_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \left(\lambda_1^p\right)^{s_p}\cdots\left(\lambda_1^2\right)^{s_2}\left(\lambda_1^1\right)^{s_1} & \left(\lambda_1^p\right)^{s_p}\cdots\left(\lambda_1^2\right)^{s_2}\left(\lambda_2^1\right)^{s_1} & \cdots & \left(\lambda_k^p\right)^{s_p}\cdots\left(\lambda_k^1\right)^{s_1} \end{pmatrix},$$

$R\in M\left(m,d^p\right)(C), d=\left[\prod_{j=1}^{p}\left(s_j+1\right)r\right]$ and D *the diagonal matrix*

$$D=\begin{pmatrix} F_{1\cdots1}^{12\cdots p} & 0 & \cdots & 0 \\ 0 & F_{11\cdots2}^{12\cdots p} & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & F_{1\cdots1k}^{12\cdots p} & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & F_{dd\cdots d}^{12\cdots p} \end{pmatrix},$$

$D\in M\left(d^p,d^p\right)$ *with entries the positive operators* $\left\{F_{i_1\cdots i_p}^{12\cdots p}\right\}_{i_1,\cdots,i_p\in\{1,\cdots,d\}}$ *on the principal diagonal.*

4. A Riesz-Herglotz Formula for Operator-Valued, Analytic Functions on the Unit Disk

Remark 4.1. *Let* $\left\{\Gamma_\alpha\right\}_{\alpha\in Z^p}$ *be a sequence of bounded operators, acting on an arbitrary, separable, complex Hilbert space* H, *such that* $\Gamma_\alpha^*=\Gamma_\alpha$ *for all* $\alpha\in Z^p$ *and* $\Gamma_0=Id_H$. *The following statements are equivalent*:

(*a*) $\sum_{\alpha,\beta=0}^{n}\Gamma_{\alpha-\beta}\xi_\alpha\overline{\xi_\beta}\ge 0$ *for all* $n\in N^p$ *and all sequences of complex numbers* $\left\{\xi_\alpha\right\}_{\alpha\in N^p}$ *with only finite nonzero terms.*

(*b*) *There exists a positive, operator-valued measure* F_Γ *on* $T_1^p\subset C^p$ *such that*

$$\Gamma_\alpha=\int_{T_1^p}z^\alpha \mathrm{d}F_\Gamma(z),\forall\alpha\in Z^p.$$

(*c*) *The operator kernel* $\left\{\Gamma_\alpha\right\}_{\alpha\in Z^p}$ *is positive semidefinite on* H, *that is it satisfies*

$$\sum_{\alpha,\beta=-n}^{n}\left\langle\Gamma_{\alpha-\beta}x_\alpha,x_\beta\right\rangle_H\ge 0$$

for all $n\in N^p$, *all sequences of vectors* $\left\{x_\alpha\right\}_{\alpha\in[-n,n]}\in H$ *and all* $n\in N^p$.

Proof. (*a*) $\Leftrightarrow$ (*b*) was solved in [9], Corollary 1.4.10.

(*b*) $\Rightarrow$ (*c*) represents the sufficient condition in Proposition 1, [10].

(*c*) $\Rightarrow$ (*a*). Let $\left\{f_\alpha\right\}_\alpha\subset H$, with $f_\alpha=\xi_\alpha x$ for an arbitrary $x\in H$. From (*c*), it results

$$\sum_{\alpha,\beta=-n}^{n}\left\langle\Gamma_{\alpha-\beta}x,x\right\rangle_H\xi_\alpha\overline{\xi_\beta}\ge 0;$$

that is the operator kernel satisfies

$$\sum_{\alpha,\beta=-n}^{n}\Gamma_{\alpha-\beta}\xi_\alpha\overline{\xi_\beta}\ge 0\Rightarrow\sum_{\alpha',\beta'=0}^{2n}\Gamma_{\alpha'-\beta'}\xi_{\alpha'}\overline{\xi_{\beta'}}\ge 0$$

(that is statement (*a*)).

Because the trigonometric polynomials are uniformly dense in the space of the continuous functions on T_1^p, it results that the representing measure of the operator moment sequence is unique.

For the proof of the following Proposition 4.2, we recall some observations.

A bounded monotonic sequence of positive non-negative operators converges in the strong operator topology to a non-negative operator (pp. 233, [11]). Due to this remark, if $f:I\subset R\to L(H), f(\theta)\ge 0,\forall\theta\in I$ is a continuous, positive operator-valued function on the compact set $I\subset R$, we define the Riemann integral of the function f with respect to the Lebesgue measure $\mathrm{d}\theta$. The definition are the usual one in the class of positive operators. That is: the limits of the riemannian sums associated to the function f, arbitrary divisions Δ of I and arbitrary intermediar points $\left\{\xi_n\right\}_n\in\Delta$ exists (are limits of bounded monotonic sequence of non-negative operators), and from the continuity assumption of f on the compact set I, are all the same. We denote the common limits, as usual with $\int_I f(\theta)\mathrm{d}\theta$. We apply this natural construction in the proof of the following result.

Proposition 4.2. *Let* $f:D\to L(H)$ *be an analytic, vectorial function, with values in the set of bounded operators on a complex, separable Hilbert space* H. *The following statements are equivalent*:

(*a*) $\Re f(z)\ge 0,\forall z\in D$ *and* $\Re f(0)=Id_H$.

(*b*) (*Riesz-Herglotz formula*) *There exists a positive operator-valued measure* F_f *on* $[-\pi,\pi]$ *with*

$$\int_{-\pi}^{\pi}F_f(\theta)\mathrm{d}\theta=Id_H$$

and an operator $C\in L(H), C^*=C$ *such that*:

$$f(z)=\mathrm{i}C+\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\mathrm{d}F_f(\theta),\forall z\in D.$$

The proof follows quite the similar steps as the proof of the Riesz-Herglotz formula for analytic, scalar functions with real positive part ([11,12].)

Proof. (*a*) $\Rightarrow$ (*b*) Let

$$f(z) = \Re f(0) + \mathrm{i}\Im f(0) + \sum_{n=1}^{\infty} \Gamma_n z^n$$

be the Taylor expansion of f, $\forall z \in D$ with

$\Re f(0) = Id_H, \Gamma_n \in L(H), \forall n \in N$ and $\overline{\lim}_n \sqrt[n]{\|\Gamma_n\|} \le 1.$

We define $\Gamma_{-n} = \Gamma_n^*$ for all $n \ge 1$. In this case , we obtain for all $z \in D$,

$$\overline{f(z)} = \Re f(0) - \mathrm{i}\Im f(0) + \sum_{n=1}^{\infty} \Gamma_n^* \overline{z}^{-n}.$$

If we consider $0 < r < 1$ arbitrary and $z = r\mathrm{e}^{\mathrm{i}\theta}, \theta \in [-\pi, \pi]$, the previous equality becomes

$$f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) + \overline{f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)} = 2\Re f(0) + \sum_{n=1}^{\infty} \Gamma_n r^n \mathrm{e}^{\mathrm{i}n\theta} + \sum_{-\infty}^{n=-1} \Gamma_n r^{-n} \mathrm{e}^{\mathrm{i}n\theta}.$$

As a consequence of the orthogonality of the system of functions $\left\{\mathrm{e}^{\mathrm{i}k\theta}\right\}_{k \in Z}$ with respect to the usual scalar product defined on $L^2([-\pi,\pi], \mathrm{d}\theta)$, from the the previous remark and f's uniform convergent expansions, for all sequences $\{\xi_n\}_n \in C$ and all $n \in N$ we obtain:

$$\begin{aligned}
&\lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) + \overline{f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)} \right] \left| \xi_0 + \xi_1 \mathrm{e}^{\mathrm{i}\theta} + \cdots + \xi_n \mathrm{e}^{\mathrm{i}n\theta} \right|^2 \mathrm{d}\theta \\
&= \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[2\Re f(0) + \sum_{k=1}^{\infty} \left(r^k \mathrm{e}^{\mathrm{i}k\theta}\right)\Gamma_k + \sum_{-\infty}^{k=-1} \left(r^{-k}\mathrm{e}^{\mathrm{i}k\theta}\right)\Gamma_k \right] \left| \xi_0 + \xi_1 \mathrm{e}^{\mathrm{i}\theta} + \cdots + \xi_n \mathrm{e}^{\mathrm{i}n\theta} \right|^2 \mathrm{d}\theta \\
&= \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[2\Re f(0) + \sum_{k=1}^{\infty} \left(r^k \mathrm{e}^{\mathrm{i}k\theta}\Gamma_k\right) + \sum_{-\infty}^{k=-1} \left(r^{-k}\mathrm{e}^{\mathrm{i}k\theta}\Gamma_k\right) \right] \left(\sum_{p,q=0}^{n} \left(\mathrm{e}^{\mathrm{i}(p-q)\theta}\right) \right) \mathrm{d}\theta \\
&= \left[2\Re f(0)\right] \sum_{p=0}^{n} \left|\xi_p\right|^2 + \sum_{p,q=0,\, p\ne q}^{n} \Gamma_{p-q} \xi_p \overline{\xi_q} \ge 0.
\end{aligned}$$

We normalize this relation by dividing it with 2 and obtain, for $\tilde{\Gamma}_n = \frac{\Gamma_n}{2}, \widetilde{\Gamma_n^*} = \widetilde{\Gamma_{-n}}, n \in N$, the following inequalities:

$$\sum_{p,q=0}^{n} \tilde{\Gamma}_{p-q} \xi_p \overline{\xi_q} \ge 0$$

for all sequences $\{\xi_k\}_{k=0}^n \subset C$ and all arbitrary $n \in N$, with

$$\tilde{\Gamma}_n \in L(H), \tilde{\Gamma}_n^* = \tilde{\Gamma}_{-n}$$

and $\tilde{\Gamma}_0 = Id_H = \Re f(0)$.

In the above conditions from Theorem 1.4.8, [9], there exists a positive operator-valued measure F_1 on T_1 such that

$$\tilde{\Gamma}_{p-q} = \tilde{\Gamma}_{p,q} = \int_{T_1} \overline{z}^p z^q \mathrm{d}F_1(z), p, q \in N \text{ and } \int_{T_1} \mathrm{d}F_1(z) = \tilde{\Gamma}_0 = Id_H = \Re f(0).$$

For $q = 0$ and $p \in N$ we have $\tilde{\Gamma}_{p-0} = \int_{T_1} \overline{z}^p \mathrm{d}F_1(z)$.

Let the homeomorphism $\psi : [-\pi, \pi] \to T_1, \psi(\theta) = \mathrm{e}^{\mathrm{i}\theta}$ and the positive operator-valued measure

$$F_2 = F_1 \circ \psi, F_2 : Bor([-\pi, \pi]) \to A(H).$$

Accordingly to this measure we obtain the representations:

$$\tilde{\Gamma}_p = \int_{T_1} \overline{z}^p \mathrm{d}F_1(z) = \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}p\theta} \mathrm{d}F_2(\theta), \forall p \in N$$

and

$$\int_{-\pi}^{\pi} \mathrm{d}F_2(\theta) = Id_H = \Re f(0).$$

Assured by the integral representations of the operators $\tilde{\Gamma}_\alpha$ we have:

$$\begin{aligned}
f(z) &= f(0) + \frac{1}{2}\left[\sum_{n=0}^{\infty} \Gamma_n z^n + \sum_{n=1}^{\infty} \Gamma_n z^n \right] - \frac{\Gamma_0}{2} = f(0) - \Re f(0) + \frac{1}{2}\left[\sum_{n=0}^{\infty} 2\tilde{\Gamma}_n z^n + \sum_{n=1}^{\infty} 2\tilde{\Gamma}_n z^n \right] \\
&= \mathrm{i}\Im f(0) + \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}n\theta} z^n \mathrm{d}F_2(\theta) + \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n+1)\theta} z^{n+1} \mathrm{d}F_2(\theta) = \mathrm{i}\Im f(0) + \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} \mathrm{d}F_2(\theta) \\
&= \mathrm{i}C + \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} \mathrm{d}F_2(\theta), \forall z \in D, \text{with } C = \frac{f(0) - f(0)^*}{2\mathrm{i}}.
\end{aligned}$$

$(b)\Rightarrow(a)$ $\frac{f(z)+f(z)^*}{2}=\int_{-\pi}^{\pi}\frac{2\left(1-|z|^2\right)}{\left|\left(e^{i\theta}-z\right)\right|^2}dF_2(\theta)\geq 0,$

f is analytic on D, $f(0)=iC+\int_{-\pi}^{\pi}dF_2(\theta)$ and $\Re f(0)=\int_{-\pi}^{\pi}dF_2(\theta)=Id_H$.

For the operator-valued analytic functions on D we can state the same characterization theorem as in the the scalar case (Theorem 3.3, [11],) that is:

Theorem 4.3. *Let* $\Gamma=\{\Gamma_n\}_{n\in Z}$ *be a sequence of bounded operators acting on an arbitrary, separable, complex Hilbert space* H, *subject to the conditions* $\Gamma_n^*=\Gamma_{-n}$ *for all* $n\in N$, $\Gamma_0=Id_H$. *The following statements are equivalent*:

(*a*) *There exists an unique, positive, operator-valued measure* F_Γ *on* T_1 *such that*:

$$\Gamma_n=\int_{T_1}z^n dF_\Gamma(z),\forall n\in Z.$$

(*b*) *The Toeplitz matrix* $\{\Gamma_{n-m}\}_{n,m=0}^{+\infty}$ *is positive semi-definite*.

(*c*) *There exists an analytic vectorial function* $F:D\to L(H),\Re F(z)\geq 0$ *for all* $z\in D$ *and*

$$F(z)=Id_H+iC+2\sum_{n=1}^{\infty}\Gamma_n z^n$$

for some $C\in L(H)$ *with* $C^*=C$.

(*d*) *There exists a separable, Hilbert space* K, *an operator* $h_0:H\to K$ *and an unitary operator* $U\in L(K)$, *such that* $\Gamma_n=h_0^*U^n h_0,\forall n\in Z$ *and* $h_0^*h_0=Id_H$.

Proof. $(a)\Leftrightarrow(b)$ was solved in [9], Th.1.4.8., p. 188. We sketch the proof of implication $(a)\Leftrightarrow(b)$.

$$\sum_{n,m=0}^{p}\Gamma_{n-m}\xi_n\overline{\xi_m}dF_\Gamma(z)=\sum_{n,m=0}^{p}\int_{T_1}z^n\overline{z}^m$$
$$=\int_{T_1}d<F_\Gamma^{\frac{1}{2}}(z)\sum_n z^n\xi_n,F_\Gamma^{\frac{1}{2}}\sum_m z^n\xi_m\geq\int_{T_1}d\left\|F_\Gamma^{\frac{1}{2}}(z)\sum_n z^n\xi_n\right\|^2\geq 0,p\in N,\text{arbitrary}.$$

$(a)\Rightarrow(c)$ As in above Proposition 4.2, there exists a positive operator-valued measure $F_2:[-\pi,\pi]\to L(H)$ such that $\Gamma_n=\int_{-\pi}^{\pi}e^{in\theta}dF_2(\theta),\forall n\in N.$ In this case, for the function $F(z)=Id_H+iC+2\sum_{n=1}^{\infty}\Gamma_n z^n$, we have

$$\|\Gamma_n\|^{\frac{1}{n}}=\sup\|\Gamma_n x\|^{\frac{1}{n}}=\sup_{\|x\|=1}\left\|\int_{T_1}z^n dF_\Gamma(z)x\right\|^{\frac{2}{2n}}=\sup_{\|x\|=1}\int_{T_1}\left|z^n\right|^2 d\left\langle F_\Gamma(z)x,x\right\rangle^{\frac{1}{n}}\leq 1;$$

that is F is analytic on D. Also from (*a*), we have:

$$F(z)=Id_H+iC+\left(\sum_{n=0}^{\infty}\Gamma_n z^n+\sum_{n=1}^{\infty}\Gamma_n z^n\Gamma_n\right)-\Gamma_0=iC+\sum_{n=0}^{\infty}\int_{-\pi}^{\pi}e^{-in\theta}z^n dF_2(\theta)+\int_{-\pi}^{\pi}e^{-in\theta}z^n\frac{z}{e^{i\theta}}dF_2(\theta)$$
$$=iC+\int_{-\pi}^{\pi}\frac{z+e^{i\theta}}{e^{i\theta}-z}dF_2(\theta).$$

From the above representation, it results:

$$2\Re F(z)=F(z)+\overline{F(z)}=2Id_H+\int_{-\pi}^{\pi}\frac{z+e^{i\theta}}{e^{i\theta}-z}dF_2(\theta)=2Id_H+\int_{-\pi}^{\pi}\frac{2}{\left|e^{i\theta}-z\right|^2}dF_2(\theta)\geq 0,\forall z\in D.$$

(c) $\Rightarrow$ (a) As the same proof in Proposition 4.2, we have

$$\lim_{r\to 1,r\in R}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[F\left(re^{i\theta}\right)+\overline{F\left(re^{i\theta}\right)}\right]\left|\xi_0+e^{i\theta}\xi_1+\cdots+\xi_n e^{in\theta}\right|^2 d\theta$$
$$=\lim_{r\to 1,r\in R}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[2Id_H+2\sum_{n=-\infty,n\neq 0}^{+\infty}\Gamma_n r^n e^{in\theta}\right]\sum_{p,q=0}^{n}e^{i(p-q)\theta}\xi_p\overline{\xi_q}d\theta=2\sum_{p,q=0}^{n}\Gamma_{p-q}\xi_p\overline{\xi_q}d\theta$$
$$=2\sum_{p,q}^{n}\Gamma_{q-p}\xi_p\overline{\xi_q}\geq 0$$

for arbitrary $n \in N$. From this inequality, it results that there exist the representations $\Gamma_q = \int_{T_1} z^q \mathrm{d}F_1(z), \forall q \in Z$ with F_1 a positive operator valued measure on T_1 ([9], Th. 1.3.2), this is (*a*).

The equivalence, $(b) \Leftrightarrow (d)$. From remark 4.1.we have $(b) \Leftrightarrow (c)$ ((*c*) from Remark 4.1.). The equivalence $(c) \Leftrightarrow (d)$ is the main result in [10], Proposition 1. p. 116. From [10], Proposition 1, (condition (*c*) in Remark 4.1.) assured the existence of a Hilbert space K, an operator $h_0 : H \to K$ and an unitary operator $U \in L(K)$ such that $\Gamma_n = h_0^* U^n h_0, \forall n \in Z$, that is (*d*); (the Hilbert Space K, the unitary operator U are obtained by applyng Kolmogorov's decomposition theorem on positive semidefinite kernels.) Conversely $(d) \Rightarrow (a)$ is immediately.

5. Conclusion

We give a necessary and sufficient condition on a finite sequence of bounded operators, acting on a finite dimensional Hilbert space, to admit an integral representation as complex moment sequence with respect to an atomic, positive, operator-valued measure. We also established a Riesz-Herglotz representation formula for operator-valued, analytic functions on the unit disc, with real positive part in the class of operators.

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