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Integral Inequalities of Hermite-Hadamard Type for *r*-Convex Functions

Lingxiong Han, Guofeng Liu
College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, China
Email: hlx2980@163.com

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ABSTRACT

The main aim of this present note is to establish three new Hermite-Hadamard type integral inequalities for *r*-convex functions. The three new Hermite-Hadamard type integral inequalities for *r*-convex functions improve the result of original one by Hölder's integral inequality, Stolarsky mean and convexity of function.

Keywords: Hermite-Hadamard Integral Inequality; r-Convex Function; Logarithmic Mean; Stolarsky Mean

1. Introduction

The inequalities

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which discovered by C. Hermite and Hadamard for all convex functions $f:[a,b]\to(-\infty,+\infty)$ are known in the literature as Hermite-Hadamard inequalities.

We note that Hermite-Hadamard inequalities may be regarded as a refinement of the concept of convexity and they follows easily from Jenson's inequality. Hermite-Hadamard inequalities for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [1-6].

Let $f^p(x), g^q(x)$ be integrable functions on

$$[a,b], p,q>0, \frac{1}{p}+\frac{1}{q}=1,$$

then the well known Hölder's integral inequality is given as

$$\int_a^b f(x)g(x)\,\mathrm{d}x \le \left(\int_a^b f^p(x)\,\mathrm{d}x\right)^{1/p}\cdot\left(\int_a^b g^q(x)\,\mathrm{d}x\right)^{1/q}. \tag{1.2}$$

The following definition is well known in the literature.

Definition 1.1. Suppose

$$f: I\subseteq(-\infty,\infty)\to(-\infty,\infty).$$

If following inequality holds

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y) \tag{1.3}$$

for any $x,y\in I$ and $t\in[0,1]$, then we say f is convex function on I.

In [1], C. E. M. Pearce, J. Pecaric and V. Simic introduced the definition of r-convex function and studied the inequalities of Hermite-Hadamard type for r-convex functions.

Definition 1.2. ([1]) A function

$$f:[a,b]\subseteq[0,\infty)\to(0,\infty)$$

is said to be r-convex function on $[a,b]$, if

$$f(tx+(1-t)y) \le \begin{cases} \left[tf^r(x)+(1-t)f^r(y)\right]^{1/r}, & \text{if } r\ne 0,\\ f^t(x)f^{1-t}(y), & \text{if } r=0.\end{cases} \tag{1.4}$$

holds for any $x,y\in[a,b]$ and $t\in[0,1]$.

We have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions.

The integral power mean M_p (see [2]) of a positive function f on $[a,b]$ is a functional given by

$$M_p(f) = \begin{cases} \left(\frac{1}{b-a}\int_a^b f^p(t)\,\mathrm{d}t\right)^{1/p}, & p\ne 0,\\ \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,\mathrm{d}t\right), & p=0.\end{cases} \tag{1.5}$$

The Stolarsky mean $E(a,b;r,s)$ (see [7]) of two positive numbers a,b is given by

$$E(a,b;r,s)=\begin{cases}\left(\frac{r}{s}\cdot\frac{a^s-b^s}{a^r-b^r}\right)^{1/(s-r)}, & rs(s-r)(b-a)\neq 0,\\ \left(\frac{1}{r}\cdot\frac{a^r-b^r}{\ln a-\ln b}\right)^{1/r}, & s=0,r(b-a)\neq 0,\\ \mathrm{e}^{-\frac{1}{r}}\left(\frac{a^{a^r}}{b^{b^r}}\right)^{1/(a^r-b^r)}, & r=s,r(b-a)\neq 0,\\ \sqrt{ab}, & r=s=0,a\neq b,\\ a, & a=b.\end{cases} \tag{1.6}$$

In [2], following theorem is given.

Theorem 1.1. ([2]) Let $f(x)$ be a positive r-convex function on $[a,b]$ and $G:[0,1]\to(-\infty,+\infty)$ is defined by

$$G(t)=\begin{cases}\left\{\frac{1}{b-a}\int_a^b\left[\frac{x-a}{b-a}f^r(tb+(1-t)x)+\frac{b-x}{b-a}f^r(ta+(1-t)x)\right]^{p/r}\mathrm{d}x\right\}^{1/p}, & r\neq 0,p\neq 0,\\ \left\{\frac{1}{b-a}\int_a^b\left[f^{\frac{x-a}{b-a}}(tb+(1-t)x)f^{\frac{b-x}{b-a}}(ta+(1-t)x)\right]^{p}\mathrm{d}x\right\}^{1/p}, & r=0,p\neq 0,\\ \exp\left\{\frac{1}{b-a}\int_a^b\ln\left[\frac{x-a}{b-a}f^r(tb+(1-t)x)+\frac{b-x}{b-a}f^r(ta+(1-t)x)\right]^{1/r}\mathrm{d}x\right\}^{1/p}, & r\neq 0,p=0,\\ \exp\left\{\frac{1}{b-a}\int_a^b\left[f^{\frac{x-a}{b-a}}(tb+(1-t)x)f^{\frac{b-x}{b-a}}(ta+(1-t)x)\right]\mathrm{d}x\right\}, & r=p=0.\end{cases} \tag{1.7}$$

Then

(i) $G(t)$ is monotonically increasing on $[0,1]$;

(ii) $G(0)=M_p(f), G(1)=E(f(a),f(b);r,p+r)$.

In [4], following theorems are given.

Theorem 1.2. ([3]) Let $f:[a,b]\to(0,\infty)$ be r-convex function on $[a,b]$ with $a<b$. Then the following inequality holds for $0<r\le 1$,

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\le\left(\frac{r}{r+1}\right)^{1/r}\left[f^r(a)+f^r(b)\right]^{1/r}.$$

Theorem 1.3. ([3]) Let f, $g:[a,b]\to(0,\infty)$ be r_1-convex and r_2-convex functions respectively on $[a,b]$ with $a<b$ Then the following inequality holds for $0<r_1,r_2\le 2$,

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le\frac{1}{2}\left(\frac{r_1}{r_1+2}\right)^{2/r_1}\cdot\left(f^{r_1}(a)+f^{r_1}(b)\right)^{2/r_1}+\frac{1}{2}\left(\frac{r_2}{r_2+2}\right)^{2/r_2}\left(g^{r_2}(a)+g^{r_2}(b)\right)^{2/r_2}.$$

Theorem 1.4. ([3]) Let f, $g:[a,b]\to(0,\infty)$ be r_1-convex and r_2-convex functions respectively on $[a,b]$ with $a<b$ Then the following inequality holds

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le\left(\frac{f^{r_1}(a)+f^{r_1}(b)}{2}\right)^{1/r_1}\left(\frac{g^{r_2}(a)+g^{r_2}(b)}{2}\right)^{1/r_2}$$

for $r_1>1$ and $\frac{1}{r_1}+\frac{1}{r_2}=1$.

2. Main Results

In this paper we obtain some new Hermite-Hadamard type integral inequalities for r-convex functions and improve the results of Theorems 1.2-1.4.

The following are extensions of Hermite-Hadamard type inequality:

Theorem 2.1. Let $f:[a,b]\subseteq[0,\infty)\to(0,\infty)$ be r-convex function on $[a,b]$ with $a<b$, $r\in(-\infty,+\infty)$. Then

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\le E(f(a),f(b);r,r+1). \tag{2.1}$$

Proof. Let $x = ta + (1-t)b,\ 0 \le t \le 1$, then

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \int_0^1 f\big(ta + (1-t)b\big)\,\mathrm{d}t.$$

If $f(a) = f(b)$, by the r-convexity of f, we have

$$f\big(ta + (1-t)b\big) \le f(a) = f(b)$$

for any $0 \le t \le 1$. So the conclusion is valid.

If $f(a) \ne f(b)$, we have to discuss three cases as following:

Case 1. If $r = -1$, we have

$$f\big(ta + (1-t)b\big) \le \left[tf^{-1}(a) + (1-t)f^{-1}(b)\right]^{-1}$$

for any $0 \le t \le 1$. Hence, we obtain

$$\int_0^1 f\big(ta + (1-t)b\big)\,\mathrm{d}t \le \int_0^1 \left[tf^{-1}(a) + (1-t)f^{-1}(b)\right]^{-1}\mathrm{d}t = \frac{\ln f^{-1}(a) - \ln f^{-1}(b)}{f^{-1}(a) - f^{-1}(b)} = E\big(f(a), f(b); -1, 0\big).$$

Case 2. If $r = 0$, we have

$$f\big(ta + (1-t)b\big) \le f^{t}(a) f^{1-t}(b)$$

for any $0 \le t \le 1$. Hence, we obtain

$$\int_0^1 f\big(ta + (1-t)b\big)\,\mathrm{d}t \le \int_0^1 f^{t}(a) f^{1-t}(b)\,\mathrm{d}t = \frac{f(a) - f(b)}{\ln f(a) - \ln f(b)} = E\big(f(a), f(b); 0, 1\big).$$

Case 3. If $r \ne 0, r \ne -1$, we have

$$f\big(ta + (1-t)b\big) \le \left[tf^{r}(a) + (1-t)f^{r}(b)\right]^{1/r}$$

for any $0 \le t \le 1$. Hence, we get

$$\int_0^1 f\big(ta + (1-t)b\big)\,\mathrm{d}t \le \int_0^1 \left[tf^{r}(a) + (1-t)f^{r}(b)\right]^{1/r}\mathrm{d}t = \frac{r}{r+1}\,\frac{f^{r+1}(a) - f^{r+1}(b)}{f^{r}(a) - f^{r}(b)}$$
$$= E\big(f(a), f(b); r, r+1\big).$$

The proof of Theorem 2.1 is complete.

Corollary 2.1.1. If $r = 1$ in Theorem 2.1, we have

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a) + f(b)}{2}. \tag{2.2}$$

Theorem 2.2. Let

$$f, g : [a,b] \subseteq [0,\infty) \to (0,\infty)$$

be r_1-convex and r_2-convex functions respectively on $[a,b]$ with $a < b$, $r_1, r_2 \in (-\infty, +\infty)$. Then the following inequality holds

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le \left[E\left(f^{p}(a), f^{p}(b); \frac{r_1}{p}, \frac{r_1}{p}+1\right)\right]^{1/p} \times \left[E\left(g^{q}(a), g^{q}(b); \frac{r_2}{q}, \frac{r_2}{q}+1\right)\right]^{1/q} \tag{2.3}$$

for any $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $x = ta + (1-t)b,\ 0 \le t \le 1$, then we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x = \int_0^1 f\big(ta + (1-t)b\big)\,g\big(ta + (1-t)b\big)\,\mathrm{d}t.$$

If $f(a) \ne f(b), g(a) \ne g(b)$, then

1) when $r_1 r_2 \ne 0$, by the r_1-convexity and r_2-convexity of functions f, g respectively, we have

$$f\big(ta + (1-t)b\big) \le \left[tf^{r_1}(a) + (1-t)f^{r_1}(b)\right]^{1/r_1}$$

and

$$g\big(ta + (1-t)b\big) \le \left[tg^{r_2}(a) + (1-t)g^{r_2}(b)\right]^{1/r_2}$$

for any $0 \le t \le 1$. So we obtain

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x = \int_0^1 f\big(ta + (1-t)b\big)\,g\big(ta + (1-t)b\big)\,\mathrm{d}t$$
$$\le \int_0^1 \left[tf^{r_1}(a) + (1-t)f^{r_1}(b)\right]^{1/r_1} \cdot \left[tg^{r_2}(a) + (1-t)g^{r_2}(b)\right]^{1/r_2}\mathrm{d}t.$$

By the Hölder's integral inequality and Theorem 2.1, we have

$$\int_0^1\left[tf^{r_1}(a)+(1-t)f^{r_1}(b)\right]^{1/r_1}\cdot\left[tg^{r_2}(a)+(1-t)g^{r_2}(b)\right]^{1/r_2}\mathrm{d}t$$
$$\le\left(\int_0^1\left[tf^{r_1}(a)+(1-t)f^{r_1}(b)\right]^{p/r_1}\mathrm{d}t\right)^{1/p}\cdot\int_0^1\left(\left[tg^{r_2}(a)+(1-t)g^{r_2}(b)\right]^{q/r_2}\mathrm{d}t\right)^{1/q}$$
$$=\int_0^1\left(\left[t\left[f^p(a)\right]^{r_1/p}+(1-t)\left[f^p(b)\right]^{r_1/p}\right]^{p/r_1}\mathrm{d}t\right)^{1/p}\cdot\int_0^1\left(\left[t\left[g^q(a)\right]^{r_2/q}+(1-t)\left[g^q(b)\right]^{r_2/q}\right]^{q/r_2}\mathrm{d}t\right)^{1/q}$$
$$=\left[E\left(f^p(a),f^p(b);\frac{r_1}{p},\frac{r_1}{p}+1\right)\right]^{1/p}\times\left[E\left(g^q(a),g^q(b);\frac{r_2}{q},\frac{r_2}{q}+1\right)\right]^{1/q}.$$

2) when $r_1r_2=0$, we just prove for $r_1=0,r_2\neq0$ which is similar to $r_1=r_2=0$ and $r_1\neq0,r_2=0$. By the Hölder's integral inequality, Theorem 2.1 and r_1-convexity and r_2-convexity of functions f,g respectively, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x$$
$$=\int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\mathrm{d}t$$
$$\le\int_0^1 f^t(a)f^{1-t}(b)\left[tg^{r_2}(a)+(1-t)g^{r_2}(b)\right]^{1/r_2}\mathrm{d}t$$
$$\le\left(\int_0^1 f^{pt}(a)f^{p(1-t)}(b)\mathrm{d}t\right)^{1/p}$$
$$\cdot\left(\int_0^1\left[tg^{r_2}(a)+(1-t)g^{r_2}(b)\right]^{q/r_2}\mathrm{d}t\right)^{1/q}$$
$$=\left[E\left(f^p(a),f^p(b);0,1\right)\right]^{1/p}$$
$$\cdot\left[E\left(g^q(a),g^q(b);\frac{r_2}{q},\frac{r_2}{q}+1\right)\right]^{1/q}.$$

If $f(a)=f(b)$ or $g(a)=g(b)$, by Theorem 2.1 we obtain the conclusion, which the proof of Theorem 2.2 is completed.

Corollary 2.2.1. Under the conditions of Theorem 2.2, if $\frac{1}{r_1}+\frac{1}{r_2}=1$ for any $r_1,r_2>0$, then we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le\left(\frac{f^{r_1}(a)+f^{r_1}(b)}{2}\right)^{1/r_1}\left(\frac{g^{r_2}(a)+g^{r_2}(b)}{2}\right)^{1/r_2}. \tag{2.4}$$

In particular, if $r_1=r_2=2$, then we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le\left(\frac{f^2(a)+f^2(b)}{2}\right)\cdot\left(\frac{g^2(a)+g^2(b)}{2}\right).$$

If $f(a)=f(b)$, $g(a)=g(b)$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le f(a)g(a)$$

Corollary 2.2.2. Under conditions of Theorem 2.2, if $f(x)=g(x)$ and $r_1=r_2$ then we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le E^2\left(f(a),f(b);r_1,2+r_1\right). \tag{2.5}$$

In particular, if $r_1=r_2=0$, then we have

$$\frac{1}{b-a}\int_a^b f^2(x)\mathrm{d}x\le\frac{f^2(a)+f^2(b)}{2}$$

Theorem 2.3. Let $f,g:[a,b]\subseteq[0,\infty)\to(0,\infty)$, $fg\in L[a,b]$, $r_1,r_2\in(-\infty,+\infty)$ and $f^p(x),g^q(x)$ be r_1-convex and r_2-convex functions respectively on $[a,b]$ with $a<b.$ Then the following inequality holds

$$\frac{1}{b-a}\int_a^b f^2(x)\mathrm{d}x\le\left[E\left(f^p(a),f^p(b);r_1+r_2+1\right)\right]\cdot\left[E\left(g^q(a),g^q(b)\right);r_1+r_2+1\right]^{1/q} \tag{2.6}$$

for any $p,q>0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $x=ta+(1-t)b,\ 0\le t\le1$, then we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x=\int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\mathrm{d}t.$$

By the Hölder's integral inequality, Theorem 2.1 and r_1-convexity and r_2-convexity of function $f^p(x),g^q(x)$ respectively, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x$$
$$=\int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\mathrm{d}t$$
$$\le\left(\int_0^1 f^p\left(ta+(1-t)b\right)\mathrm{d}t\right)^{1/p}$$
$$\cdot\left(\int_0^1 g^q\left(ta+(1-t)b\right)\mathrm{d}t\right)^{1/q}$$
$$\le\left[E\left(f^p(a),f^p(b);r_1,r_1+1\right)\right]^{1/p}$$
$$\cdot\left[E\left(g^q(a),g^q(b);r_2,r_2+1\right)\right]^{1/q}$$

This completed the proof of Theorem 2.3.

Corollary 2.3.1. Under the conditions of Theorem 2.3, if $p = q = 2$ and $r_1 = r_2 = \frac{1}{2}$, then we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \sqrt{\frac{f^2(a)+f(a)f(b)+f^2(b)}{3}} \cdot \sqrt{\frac{g^2(a)+g(a)g(b)+g^2(b)}{3}}. \tag{2.7}$$

In particular, if $f(a) = f(b), g(a) = g(b)$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le f(a)g(a)$$

In this paper, we obtained three new Hermite-Hadamard type integral inequalities for r-convex functions, which improved the results of Theorems 1.2-1.4 by Hölder's integral inequality, Stolarsky mean and convexity of function. The special case of new Hermite-Hadamard type integral inequalities is classical Hermite-Hadamard type integral inequality. So it improved the classical one.

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