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On the Set of 2 − Common Consequent of Primitive Digraphs with Exact *d* Vertices Having Loop

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ABSTRACT

Let d and n are positive integers, $n \ge 2, 1 \le d \le n$. In this paper we obtain that the set of the 2 − common consequent of primitive digraphs of order n with exact d vertices having loop is $\left\{1, 2, \cdots, n - \left\lceil \frac{d}{2} \right\rceil \right\}$.

Keywords: Boolean Matrix; Common Consequent; Primitive Digraph

1. Introduction

Let $V = \{a_1, \cdots, a_n\}$ be a finite set of order n, $G = (V, E)$ be a digraph. Elements of V are referred as vertices and those of E as arcs. The arc of E from vertex u to vertex v is denoted by (u, v). Let A be a $n \times n$ matrix over the Boolean algebra $\{0,1\}$. If the adjacency matrix of G is A, where $A = (m_{ij})$, $m_{ij} = 1$, if $(a_i, a_j) \in E$ and $m_{ij} = 0$ otherwise, then A is Boolean matrix. G is called adjoint digraph of A.

The map: $A \leftrightarrow G$ is isomorphism.

Let G^l be a digraph corresponding to the A^l, and

$$a_i G^l = \left\{ a_j \in V \middle| (a_i, a_j) \in E(G^l) \right\},$$

where $l > 0$ is an integer.

In 1983, Š. Schwarz [1] introduced a concept of the common consequent as follows.

Definition 1.1 Let G be a digraph. We say that a pair of vertices $(a_i, a_j), a_i \ne a_j$, has a common consequent $(c.c.)$ if there is an integer $l > 0$ such that

$$a_i G^l \cap a_j G^l \ne \phi \tag{1}$$

If (a_i, a_j) have a $c.c.$ then the least integer $l > 0$ for which (1) holds is denoted by $L_G(a_i, a_j)$.

Definition 1.2 Let G be a digraph. The generalized vertex exponent of G, denoted by $\exp_G(1)$, is the least integer $l > 0$ such that

$$\bigcap_{i=1}^{n} a_i G^l \ne \phi \tag{2}$$

In 1996, Bolian Liu [2] extends the common consequent to the $k -$common consequent $(k - c.c.)$ as follows.

Definition 1.3 Let G be a digraph. We say that a group of vertices

$$\{a_{i_1}, \cdots, a_{i_k}\} \subseteq V = \{a_1, \cdots, a_n\},$$
$$2 \le k \le n, a_{i_s} \ne a_{i_t}, s \ne t,$$

has a $k -$common consequent $(k - c.c.)$, if there is an integer $l > 0$ such that

$$\bigcap_{j=1}^{k} a_{i_j} G^l \ne \phi \tag{3}$$

If $\{a_{i_1}, \cdots, a_{i_k}\}$ have a $k - c.c.$, then the least integer $l > 0$ for which (3) holds is denoted by $L_G(a_{i_1}, \cdots, a_{i_k})$.

If there is at least one group $\{a_{i_1}, \cdots, a_{i_k}\}$ for which $L_G(a_{i_1}, \cdots, a_{i_k})$ exists, we define

$$L_G(k) = \max L_G(a_{i_1}, \cdots, a_{i_k}),$$

where $\{a_{i_1}, \cdots, a_{i_k}\}$ runs through all groups with k elements for which $L_G(a_{i_1}, \cdots, a_{i_k})$ exists. If there is no group $\{a_{i_1}, \cdots, a_{i_k}\}$ for which $L_G(a_{i_1}, \cdots, a_{i_k})$ exists, we define $L_G(k) = 0$. $L_G(k)$ is called $k - c.c.$ of G.

A digraph G is said to be strongly connected if there exists a path from u to v for all $u, v \in V(G)$. A digraph G is said to be primitive if there exists a positive integer p such that there is a walk of length p from u to v for all $u, v \in V(G)$. The smallest such p is called the primitive exponent of G.

A digraph G is primitive iff G is strongly connected and the greatest common divisor of all cycle lengths of G is 1.

Let $V = \{a_1, \cdots, a_n\}$ and $P_n(d)$ be the set of all primitive digraphs of order n with exact d vertices having loop. It is obvious that if $G \in P_n(d)$, then

$L_G\left(a_{i_1},\cdots,a_{i_k}\right)$ exists for any group $\left\{a_{i_1},\cdots,a_{i_k}\right\}$, $2\le k\le n$. We define

$$L(n,d,k)=\max\left\{L_G(k)\middle|G\in P_n(d)\right\}.$$

The properties of primitive digraphs and its $k-c.c.$ see [3-5]. In this paper we obtain that the set of the 2 − common consequent of primitive digraphs of order n with exact *d* vertices having loop is

$$\left\{1,2,\cdots,n-\left\lceil\frac{d}{2}\right\rceil\right\},$$

where n and d are positive integers, $n\ge 2,\ 1\le d\le n$, $\lceil a\rceil$ is the least integer greater or equal to *a*.

2. Preliminaries

It is easy to see that $L(n,d,k)$ exists by [1].

Lemma 2.1 Let $G=G(V)$ be a primitive digraph of order $n(n\ge 2)$ and V_1 be a nonempty proper subset of V, then V_1G contains at least one element of V which is not contained in V_1.

Lemma 2.2 Let $G=G(V)$ be a primitive digraph of order $n(n\ge 3)$ and $a_i\in V$, where vertex a_i with having a loop, $2\le k\le n$, then $\left|a_iG^{k-1}\right|\ge k$.

Proof: Since vertex a_i has a loop, hence $a_i\in a_iG$, and $\left|a_iG^{k-1}\right|\ge k$ by lemma 2.1.

The follow lemma is obvious.

Lemma 2.3 [2] If $2\le k_1\le k_2\le n$, G is a primitive digraph, then $L_G(k_1)\le L_G(k_2)$.

Lemma 2.4 Let $V=\{a_1,\cdots,a_n\}$,

$E=$

$\left\{(a_i,a_i),(a_j,a_{j+1}),(a_n,a_1)\middle|i=1,\cdots,d,\ j=1,\cdots,n-1\right\}$,

$G_0=(V,E)$,

where n,d,k are integers and $1\le d\le n$, $2\le k\le n$, then

$$L_{G_0}(2)=n-\left\lceil\frac{d}{2}\right\rceil \text{ and } L_{G_0}(n)=n-1.$$

Proof: First of all, It is obvious that G_0 is belong to $P_n(d)$.

Let $V_1=\{a_1,\cdots,a_d\},V_2=\{a_{d+1},a_{d+2},\cdots,a_n\}$, then V_1 is a set in which every vertex have a loop, For all $u,v\in V,u\ne v$.

Case 1 $u,v\in V_1$.

There exists a walk of length less than or equal to $n-\left\lceil\frac{n}{2}\right\rceil$ form u to v (or from v to u), and $n-\left\lceil\frac{n}{2}\right\rceil\le n-\left\lceil\frac{d}{2}\right\rceil$, then $uG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\cap vG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\ne\phi$.

Case 2 $u,v\in V_2$.

There exists a walk of length less than or equal to $n-d$ form u to a_1 (and form v to a_1),

$$n-d\le n-\left\lceil\frac{d}{2}\right\rceil,$$

then

$$uG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\cap vG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\supseteq\{a_1\}\ne\phi.$$

Case 3 $u\in V_1,v\in V_2$.

There exist a walk of length less than or equal to $n-d$ form v to $x\in V$, by Lemma 2.2,

$$\begin{aligned}\left|uG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\right| &\ge n-\left\lceil\frac{d}{2}\right\rceil+1\cdot\left|vG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\right|\\ &\ge d-\left\lceil\frac{d}{2}\right\rceil+1\cdot\left|uG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\cap vG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\right|\\ &=\left|uG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\right|+\left|vG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\right|-\left|uG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\cup vG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\right|\\ &\ge n-\left\lceil\frac{d}{2}\right\rceil+1+d-\left\lceil\frac{d}{2}\right\rceil+1-n=d-2\left\lceil\frac{d}{2}\right\rceil+2>0,\end{aligned}$$

hence $uG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\cap vG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\ne\phi$.

So we have $uG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\cap vG_0^{n-\left\lceil\frac{d}{2}\right\rceil}\ne\phi$ for all $u,v\in V$. Note that if $l<n-\left\lceil\frac{d}{2}\right\rceil$, then $a_{\left\lceil\frac{d}{2}\right\rceil}G_0{}^{l}\cap a_{d+1}G_0{}^{l}=\phi$.

Hence $L_{G_0}(2)=n-\left\lceil\frac{d}{2}\right\rceil$

Let u be arbitrary vertex belong to V, then there exists a walk of length less than or equal to $n-1$ form u to a_1, then $\bigcap_{i=1}^{n}a_iG_0^{n-1}\ne\phi$. It is easy to see that if $l<n-1$ then $\bigcap_{i=1}^{n}a_iG_0^{l}\ne\phi$.

Hence $L_{G_0}(n)=n-1$. The proof is now completed.

3. The Main Results

Theorem 3.1 Let $G\in P_n(d),n,d$ be integers,

$$n\ge 3,\ 1\le d\le n,$$

then $L(n,2,d)=n-\left\lceil\frac{d}{2}\right\rceil$

Proof: Let $V=\{a_1,\cdots,a_n\}$ be set of vertices of G and V_1 be subset of V in which each vertex have a loop, $V_2=V-V_1$. for all $u,v\in V,u\ne v$.

Case 1 $u,v\in V_1$.

There exists a walk of length less than or equal to $n-\left\lceil \frac{n}{2} \right\rceil$ form u to v (or from v to u), and $n-\left\lceil \frac{n}{2} \right\rceil \le n-\left\lceil \frac{d}{2} \right\rceil$, then $uG^{n-\left\lceil \frac{d}{2} \right\rceil} \cap vG^{n-\left\lceil \frac{d}{2} \right\rceil} \ne \phi$.

Case 2 $u,v \in V_2$.

Suppose that there be a walk of length equal to $n-\left\lceil \frac{d}{2} \right\rceil$ of $u: uu_1u_2\cdots u_{s-1}vx_1x_2\cdots x_p$, and there be a walk of length equal to $n-\left\lceil \frac{d}{2} \right\rceil$ of

$$v: vv_1v_2\cdots v_{t-1}uy_1y_2\cdots y_q,$$

where $s+p=t+q=n-\left\lceil \frac{d}{2} \right\rceil$.

Let $X=\{x_1x_2,\cdots,x_p\}$ and $Y=\{y_1y_2,\cdots,y_q\}$. If there be one vertex of X or Y belong to V_1, then

$$uG^{n-\left\lceil \frac{d}{2} \right\rceil} \cap vG^{n-\left\lceil \frac{d}{2} \right\rceil} \ne \phi.$$

Otherwise, $V-\{u,u_1,u_2,\cdots,u_{s-1},v,x_1,x_2,\cdots,x_p\}$ and $V-\{v,v_1,v_2,\cdots,v_{t-1},u,y_1,y_2,\cdots,y_q\}$ contains at most $\left\lceil \frac{d}{2} \right\rceil$ element of V_1. In other word, $\{u,u_1,u_2,\cdots,u_{s-1}\}$ contains at least $\left\lceil \frac{d}{2} \right\rceil$ element of V_1. Note that G is strongly connected, $u \ne v$. There exists a walk of length less than or equal to $n-\left\lceil \frac{d}{2} \right\rceil$ from v to one vertex of $\{u_1,u_2,\cdots,u_{s-1}\}$ which belong to V_1. Therefore

$$uG^{n-\left\lceil \frac{d}{2} \right\rceil} \cap vG^{n-\left\lceil \frac{d}{2} \right\rceil} \ne \phi.$$

Case 3 $u \in V_1, v \in V_2$.

There exist a walk of length less than or equal to $n-d$ form v to $x \in V_1$, by Lamma 2.2

$$\left|uG^{n-\left\lceil \frac{d}{2} \right\rceil}\right| \ge n-\left\lceil \frac{d}{2} \right\rceil+1 \cdot \left|vG^{n-\left\lceil \frac{d}{2} \right\rceil}\right|$$

$$\ge d-\left\lceil \frac{d}{2} \right\rceil+1 \cdot \left|uG^{n-\left\lceil \frac{d}{2} \right\rceil} \cap vG^{n-\left\lceil \frac{d}{2} \right\rceil}\right|$$

$$=\left|uG^{n-\left\lceil \frac{d}{2} \right\rceil}\right|+\left|vG^{n-\left\lceil \frac{d}{2} \right\rceil}\right|-\left|uG^{n-\left\lceil \frac{d}{2} \right\rceil} \cup vG^{n-\left\lceil \frac{d}{2} \right\rceil}\right|$$

$$\ge n-\left\lceil \frac{d}{2} \right\rceil+1+d-\left\lceil \frac{d}{2} \right\rceil+1-n=d-2\left\lceil \frac{d}{2} \right\rceil+2>0,$$

hence $uG^{n-\left\lceil \frac{d}{2} \right\rceil} \cap vG^{n-\left\lceil \frac{d}{2} \right\rceil} \ne \phi$.

So we have $uG^{n-\left\lceil \frac{d}{2} \right\rceil} \cap vG^{n-\left\lceil \frac{d}{2} \right\rceil} \ne \phi$ for all $u,v \in V$. Hence

$$L(n,2,d) \le n-\left\lceil \frac{d}{2} \right\rceil.$$

Note that

$$L_{G_0}(2)=n-\left\lceil \frac{d}{2} \right\rceil,$$

then

$$L(n,2,d)=n-\left\lceil \frac{d}{2} \right\rceil.$$

The proof is completed.

Corollary 3.2 Let $G \in P_n(d)$ and n,d be integers, $n \ge 3,\ 1 \le d \le n$, then $L(n,n,d)=n-1$.

Proof: Let V be a set of vertices of G and let u be an arbitrary vertex belong to V, then there exists a walk of length $n-1$ from u to x, where x having a loop. Hence

$$\bigcap_{i=1}^{n} a_i G^{n-1} \ne \phi, L(n,n,d)=n-1$$

Note that $L_{G_0}(n)=n-1$ by Lemma 2.4, hence

$$L(n,n,d)=n-1.$$

Applying Lemma 2.3, Theorem 2.1 and Theorem 2.2, we have conclusion.

Corollary 3.3 Let $G \in P_n(d), n,k,d$ and be integers, $n \ge 3,\ 2 \le k \le n,\ 1 \le d \le n$, then

$$n-\left\lceil \frac{d}{2} \right\rceil \le L(n,k,d) \le n-1.$$

Corollary 3.4 Let G be a primitive digraph of order n with girth $s(1 \le s \le n-1)$, then

$$L_G(2) \le s\left(n-\left\lceil \frac{s}{2} \right\rceil\right).$$

Proof: Since G is a primitive digraph of order n with girth s, then G^s is a primitive digraph of order n with exact s vertices having loop. By Theorem 3.1, we have

$$L_G(2) \le s\left(n-\left\lceil \frac{s}{2} \right\rceil\right).$$

Theorem 3.5 Let n and d be integers, $1 \le d \le n,\ n \ge 2$, then there exists $Q \in P_n(V,d)$ so that $L_Q(n,2)=r$ for arbitrary $r \in \left\{1,2,\cdots,n-\left\lceil \frac{d}{2} \right\rceil\right\}$.

Proof: Let $V=\{v_1,v_2,\cdots,v_n\}$.

We construct $Q \in P_n(V,d)$ so that $L_Q(n,2)=r$ for arbitrary $r \in \left\{1,2,\cdots,n-\left\lceil \frac{d}{2} \right\rceil\right\}$.

Case 1 $r = 1$.

Let

$$E_1 = \left\{(v_i, v_i), [v_j, v_1] \middle| i = 1,\cdots,d, j = 2,\cdots,n\right\},$$

$G(Q_1) = G(V, E_1)$. It is obvious that $Q_1 \in P_n(V,d)$ and $L_{Q_1}(n,2) = 1$.

Case 2 $2 \le r \le n - \left\lceil \frac{d}{2} \right\rceil$.

Case 2.1 $r > \left\lfloor \frac{d}{2} \right\rfloor$.

Let $m = r + \left\lceil \frac{d}{2} \right\rceil, 1 \le d \le n, 2 \le r \le n - \left\lceil \frac{d}{2} \right\rceil$,

Hence $d < m, 3 \le m \le n$.

Let

$$E_2 = \{(v_i, v_i), (v_j, v_{j+1}), (v_{m-1}, v_k), (v_k, v_1) | i = 1,\cdots,d, j = 1,2,\cdots,m-2, k = m, m+1,\cdots,n\}$$

$$G(Q_2) = G(V, E_2).$$

obviously,

$$Q_2 \in P_n(V,d).$$

Let $V_1 = \{v_1, v_2, \cdots, v_m\}$, then V_1 is the set of vertices which is in cycle lengths of m. Let $V_2 = V - V_1$, arbitrary vertex $u, v \in V, u \ne v$. If $u, v \in V_1$ or $u \in V_1, v \in V_2$, by Lemma 2.4,

$$uQ_2^{m-\left\lceil \frac{d}{2} \right\rceil} \cap vQ_2^{m-\left\lceil \frac{d}{2} \right\rceil} \ne \phi.$$

If $u, v \in V_2$, then $uQ_2^{m-\left\lceil \frac{d}{2} \right\rceil} \cap vQ_2^{m-\left\lceil \frac{d}{2} \right\rceil} \supset \{v_1\} \ne \phi$.

If $l < m - \left\lceil \frac{d}{2} \right\rceil$, then $v_{\left\lceil \frac{d}{2} \right\rceil} Q_2^l \cap v_{d+1} Q_2^l = \phi$.

Hence $L_{Q_2}(n,2) = r$.

Case 2.2 $r = \left\lfloor \frac{d}{2} \right\rfloor$.

Let $m = r + \left\lceil \frac{d}{2} \right\rceil$, then $m = d$.

Let

$$E_3 = \{(v_i, v_i), (v_j, v_{j+1}), [v_{m-1}, v_k], (v_k, v_1)(v_m, v_1) | i = 1,\cdots, d, j = 1,2,\cdots,m-1, k = m+1,\cdots,n\}$$

$$G(Q_3) = G(V, E_3).$$

It is obvious that $Q_3 \in P_n(V,d)$ and $L_{Q_3}(n,2) = r$.

Case 2.3 $r < \left\lfloor \frac{d}{2} \right\rfloor$.

Let $m = r + r = 2r$, then $m \le d$. Let

$$E_4 = \{(v_i, v_i), (v_j, v_{j+1}), [v_{m-1}, v_k], (v_k, v_1)(v_{m-1}, v_t)(v_m, v_1) | i = 1,\cdots,d, j = 1,2,\cdots,m-1, k = m+1,\cdots,n, s = d+1,\cdots,n, t = m+1,\cdots,d\}$$

$$G(Q_4) = G(V, E_4).$$

It is obvious that $Q_4 \in P_n(V,d)$ and $L_{Q_4}(n,2) = r$. The proof is now completed.

Remark 3.6 By Theorem 3.5, we obtain that the set of the 2 − common consequent of primitive digraphs of order n with exact d vertices having loop is

$$\left\{1,2,\cdots,n-\left\lceil \frac{d}{2} \right\rceil\right\}.$$

But, in Theorem 3.5, $Q \in P_n(V,d)$ is not unique.

Example.

Let

$$n = 7, d = 5, r = 3, V = \{1,2,3,4,5,6,7\},$$

$$E_5 = \{(i,i), (j, j+1), (6,1), (7,1), (5,7) | i = 1,2,\cdots, 5, j = 1,\cdots,5\}$$

$$E_6 = \{(i,i), (j, j+1), (6,1), (7,1), (5,7), (3,2) | i = 1,2,\cdots, 5, j = 1,\cdots,5\}$$

$$G(Q_i) = G(V, E_i), i = 5,6.$$

Obviously,

$$Q_i \in P_7(V,5), i = 5,6. \quad L_{Q_5}(7,2) = L_{Q_6}(7,2) = 3,$$

but $M_5 = M(Q_5)$ and $M_6 = M(Q_6)$ are not isomorphic digraph.

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