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Some Results on (1,2*n* – 1)-Odd Factors

Man Liu[1], Qingzhi Yu[1], Shuling Wang[1], Changhua Huang[2]
[1]Department of Basic Course, Air Force Logistics College, Xuzhou, China
[2]Aerial Four Station Department, Air Force Logistics College, Xuzhou, China
Email: liuman8866@163.com

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ABSTRACT

Let G be a graph. If there exists a spanning subgraph F such that $d_F(x) \in \{1,3,\cdots,2n-1\}$, then F is called to be $(1,2n-1)$-odd factor of G. Some sufficient and necessary conditions are given for $G-U$ to have $(1,2n-1)$-odd factor where U is any subset of $V(G)$ such that $|U|=k$.

Keywords: Claw Free Graphs; $(1,2n-1)$-Odd Factor; Factor-Criticality

1. Introduction

We consider finite undirected graph without loops and multiple edges .Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Given $x \in V(G)$, the set of vertexes adjacent to x is said to be the neighborhood of x, denoted by $N_G(x)$, and $d_G(x)=|N_G(x)|$ is called the degree of x, If there exists a spanning subgraph F such that $d_F(x) \in \{1,3,5,\cdots,f(x)\}$, then F is called a $(1,f)$-odd factor of G, especially, if for every $x \in V(G)$ such that $f(x)=2n-1$, then it is called $(1,2n-1)$-odd factor. especially, $(1,2n-1)$-odd factor is 1-factor when $n = 1$. For a subset $S \subset V(G)$, let $G-S$ denote the subgraph obtained from G by deleting all the vertexes of S together with the edges incident with the vertexes of $S \cdot o(G-S)$ denotes the number of odd components of $G-S$. The sufficient and necessary condition for graph to have $(1,f)$-odd factor was given in paper [1] Ryjacek [2] introduced one kind of new closure operation: let G be a graph, $x \in V(G)$, if the subgraph induced by $N_G(x)$ is not complete graph, we consider the following operation: jointing every pair of nonadjacent vertex in $N_G(x)$ makes $G[N_G(x)]$ to be a complete graph. The operation is called local completely at point x. If the subgraph induced by $N_G(x)$ is *k*-vertex connected, then vertex x is called local k-vertex connected graph G.

Favaron gave the concept of *k*-factor critical in paper [3]. If $|V(G)| \ge k+2$, and for any $T \subseteq V(G)$, $|T|=k$, $G-T$ is perfect matching ,then we call the graph G to be *k*-factor critical. Of course, 0-factor critical graph is perfect matching. Favaron popularized a series of the properties of perfect matching to *k*-factor critical, at the same time the sufficient and necessary conditions were given for the graph to be *k*-factor critical, more results in factor critical graphs were referred to [4,5].

For $(1,f)$-odd factor, Chen Ci-ping [6] gave a sufficient condition for a matching with exactly k edges extended to $(1,2n-1)$-odd factor. Teng Cong generalized some results on *k*- to $(1,f)$-odd factor, and proved that the connected graph G exists $(1,f)$-odd factor with *k*-extended, then for the any edge e of G, $G-e$ exists $(1,f)$-odd factor [7] with $(k-1)$-extended. If there exists $(1,f)$-odd factor of G with *k*-extended, then there exists $(1,f)$-odd factor with $(k-1)$-extended, and G is $(k+1)$-connected [8]. We will popularize some results of *k*-factor critical to $(1,2n-1)$-odd factor, and gain several sufficient and necessary conditions for $G-U$ to have $(1,2n-1)$-odd factor for any subset U of $V(G)$ such that $|U|=k$.

2. Main Results

We start with some lemmas as following.

Lemma 1 The sufficient and necessary condition for a graph G to have $(1,2n-1)$-odd factor after cutting off any k vertexes is

$$o(G-B) \le (2n-1)(|B|-k)(\forall |B| \ge k)$$

Proof For set U with any k vertexes, $G'=G-U$ has $(1,2n-1)$-odd factor, next we will prove

$$o(G-B) \le (2n-1)(|B|-k)(\forall |B| \ge k)$$

For any $B \subseteq V(G)$ and $|B| \ge k$, let $B = U \cup B'$, where $|U|=k$. Since $G'=G-U$ has a $(1,2n-1)$-odd factor, by the sufficient and necessary condition for graph with $(1,2n-1)$-odd factor we have

$$o(G'-B')\le(2n-1)|B'|.$$

Noting that $G'-B'=G-B$,

Therefore

$$o(G-B)=o(G'-B')\le(2n-1)|B'|$$
$$=(2n-1)(|B|-k)$$

For any $B\subseteq V(G)$ and $|B|\ge k$ we have

$$o(G-B)\le(2n-1)(|B|-k),$$

the following that the set U with any k vertexes, $G'=G-U$ has $(1,2n-1)$-odd factor, *i.e.*, for any $B'\subseteq V(G')$, there $o(G'-B')\le(2n-1)|B'|$.

Noting that $B=U\bigcup B'$, of course $|B|\ge k$.

By

$$o(G-B)\le(2n-1)(|B|-k),$$

and $G'-B'=G-B$, we have

$$o(G'-B')=o(G-B)\le(2n-1)(|B|-k)$$
$$=(2n-1)|B'|.$$

Lemma 2 [9] Connected claw free graphs of even order have 1-factor.

Lemma 3 Connected claw free graphs of even order have $(1,2n-1)$-odd factor.

Proof If $n=1$, by lemma 2, the conclusion is proved. Assume that $n\ge 2$.

By contradiction, we assume that G has no $(1,2n-1)$-odd factor, *i.e.*, $\exists S\subseteq V(G)$ such that

$$o(G-S)>(2n-1)|S|\ge 3|S|(n\ge 2).$$

then there exists $x\in S$ such that x connecting with three components of $G-S$ at least. If not, for $\forall x\in S$, x connects with two components of $G-S$ at most, consequently $o(G-S)\le 2|S|$, contradiction.

Theorem 1 Let G be graph with p order, x,y are a couple of nonadjacent vertexes and satisfy

$$d_G(x)+d_G(y)\ge p+k-1,$$

then the sufficient and necessary condition for G removing any k vertexes with $(1,2n-1)$-odd factor is that $G+xy$ getting rid of any k vertexes with $(1,2n-1)$-odd factor.

Proof The necessary condition is obvious, next we prove the sufficient condition.

By contradiction, let $G+xy$ remove any k vertexes with $(1,2n-1)$-odd factor, but there exist k vertexes after getting rid of the k vertexes of G without $(1,2n-1)$-odd factor. By lemma 1, there exists

$$B\subseteq V(G),|B|\ge k$$

such that

$$o(G-B)>(2n-1)(|B|-k),$$

and

$$o(G+xy-B)\le(2n-1)(|B|-k).$$

at the same time, by $o(G-B)+|B|\equiv p(\mathrm{mod}\,2)$ and $p\equiv k(\mathrm{mod}\,2)$,

Thereby $o(G-B)\ge(2n-1)(|B|-k)+2$.

Furthermore, by $o(G+xy-B)\ge o(G-B)-2$,

Consequently

$$(2n-1)(|B|-k)\ge o(G+xy-B)\ge o(G-B)-2$$
$$\ge(2n-1)(|B|-k)+2-2$$

Accordingly

$$o(G-B)=(2n-1)(|B|-k)+2$$

and

$$o(G+xy-B)=(2n-1)(|B|-k).$$

It shows that x,y are part of two odd components C_1,C_2 of $G-B$ respectively.

Thus

$$d_G(x)+d_G(y)\le|V(C_1)|-1+|V(C_2)|-1+2|B|.$$

On the other hand, by hypothesis

$$d_G(x)+d_G(y)\ge p+k-1\ge|B|+|V(C_1)|+|V(C_2)|$$
$$+(2n-1)(|B|-k)+k-1.$$

But

$$(2n-2)|B|>(2n-2)k-1.$$

Contradiction.

Theorem 2 Let $t(\le k+1)$ connected graph G be p order, x,y are a couple of any nonadjacent vertexes of G, and satisfy

$$|N_G(x)N_G(y)|\ge p-t+k-1,$$

then the sufficient and necessary condition for G removing any k vertexes with $(1,2n-1)$-odd factor is $G+xy$ getting rid of any k vertexes with $(1,2n-1)$-odd factor.

Proof G is a spanning subgraph of $G+xy$, so the necessary condition is obvious.

Next we prove the sufficient condition. We suppose $G+xy$ getting rid of any k vertexes with $(1,2n-1)$-odd factor, but G is not, *i.e.* there exist

$$B\subseteq V(G),|B|\ge k$$

such that

$$o(G-B)>(2n-1)(|B|-k).$$

Be similar to the discussion of theorem 1

$$o(G-B) > (2n-1)(|B|-k)+2$$

and

$$o(G+xy-B) = (2n-1)(|B|-k).$$

thereby x, y are part of two odd components C_1, C_2 of $G-B$ respectively.

Noting that

$$|N_G(x) \cup N_G(y)| \le |V(C_1)|-1+|V(C_2)|-1+|B| \quad (1)$$

By hypothesis

$$\begin{aligned} &|N_G(x) \cup N_G(y)| \\ &\ge p-t+k-1 \ge |V(C_1)|+|V(C_2)|+|B| \quad (2) \\ &+(2n-1)(|B|-k)-t+k-1 \end{aligned}$$

Combining (1) with (2)

$$-2 \ge (2n-1)(|B|-k)-t+k-1$$

Consequently

$$\frac{t-k-1}{2n-1}+k \ge |B| \ge k,$$

but $t \le k+1$.

Contradiction.

Theorem 3 Let G be claw free graphs, x be partial k connection point. G' be graph obtained by locally fully on G in x point, then for $U \subseteq V(G), |U| = k$, the sufficient and necessary condition for $G-U$ with $(1,2n-1)$-odd factor is $G'-U$ with $(1,2n-1)$-odd factor.

Proof G is a spanning subgraph of G', so the necessary condition is obvious.

Next we prove the sufficient condition. Let $G'-U$ have $(1,2n-1)$-odd factor, $G-U$ have no $(1,2n-1)$-odd factor. $G'-U$ has $(1,2n-1)$-odd factor,
$|V(G')| \equiv k(\mathrm{mod}\,2)$, so $|V(G)| \equiv k(\mathrm{mod}\,2)$.

On the other hand, G is claw free, so $G-U$ is claw free.

By lemma 2, lemma 3, $G-U$ has two odd components at least.

If $x \notin U$, let $x \in C_0$ (C_0 is branch of $G-U$). Now, $G-U$ has the same odd components as $G'-U$, therefore, $G-U$ has $(1,2n-1)$-odd factor. which is contradiction.

Next let $x \in U$, since $G'-U$ has not odd components, for any odd components of $G-U$,

$$N_G(x) \cap V(C) \ne \Phi$$

is complete.

Let x_1, x_2 be adjacent vertexes of x in two odd components of $G-U$ respectively.

Then x_1, x_2 is nonadjacent in the induced subgraph of $N_G(x)-(U-\{x\})$, which is contradiction to the fact that x is a locally k connected vertex, since

$$|U-\{x\}| \le k-1$$

The proof is complete.

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