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Hyers-Ulam Stability of a Generalized Second-Order Nonlinear Differential Equation

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ABSTRACT

In this paper we have established the stability of a generalized nonlinear second-order differential equation in the sense of Hyers and Ulam. We also have proved the Hyers-Ulam stability of Emden-Fowler type equation with initial conditions.

Keywords: Nonlinear Differential Equation; Hyers-Ulam Stability; Emden-Fowler; Second-Order

1. Introduction

In 1940 Ulam posed the basic problem of the stability of functional equations: Give conditions in order for a linear mapping near an approximately linear mapping to exist [1]. The problem for approximately additive mappings, on Banach spaces, was solved by Hyers [2]. The result obtained by Hyers was generalized by Rassias [3].

After then, many mathematicians have extensively investigated the stability problems of functional equations (see [4-6]).

Alsina and Ger [7] were the first mathematicians who investigated the Hyers-Ulam stability of the differential equation $g' = g$. They proved that if a differentiable function $y: I \to R$ satisfies $|y' - y| \le \varepsilon$ for all $t \in I$, then there exists a differentiable function $g: I \to R$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|g - y| \le 3\varepsilon$, for all $t \in I$. This result of alsina and Ger has been generalized by Takahasi *et al* [8] to the case of the complex Banach space valued differential equation $y' = \lambda y$.

Furthermore, the results of Hyers-Ulam stability of differential equations of first order were also generalized by Miura *et al.* [9], Jung [10] and Wang *et al.* [11].

Motivation of this study comes from the work of Li [12] where he established the stability of linear differential equation of second order in the sense of the Hyers and Ulam $y'' = \lambda y$. Li and Shen [13] proved the stability of nonhomogeneous linear differential equation of second order in the sense of the Hyers and Ulam $y'' + p(x)y' + q(x)y + r(x) = 0$, while Gavruta *et al.* [14] proved the Hyers-Ulam stability of the equation $y'' + \beta(x)y = 0$ with boundary and initial conditions.

The author in his study [15] estabilshed the Hyers-Ulam stability of the equations of the second order

$$z'' + p(x)z' + q(x)z = h(x)|z|^{\beta} e^{\left(\frac{\beta-1}{2}\right)\int p(x)\mathrm{d}x} \operatorname{sgn} z, \beta \in (0,1)$$

and

$$z'' + p(x)z' + (q(x) - \alpha(x))z = 0$$

with the initial conditions $z(x_0) = 0 = z'(x_0)$.

In this paper we investigate the Hyers-Ulam stability of the following nonlinear differential equation of second order

$$z'' - F(x, z(x)) = 0 \quad (1)$$

with the initial condition

$$z(a) = z'(a) = 0 \quad (2)$$

where

$$z \in C^2(I), I = [a,b], |F(x, z(x))| \le A|z|^{\alpha},$$
$$\alpha > 0, -\infty < a < b < \infty, -\infty < z < \infty, \text{ and } F(x,0) = 0.$$

Moreover we investigate the Hyers-Ulam stability of the Emden-Fowler nonlinear differential equation of second order

$$z'' - h(x)|z|^{\alpha} \operatorname{sgn} z = 0 \quad (3)$$

with the initial condition

$$z(a) = z'(a) = 0 \quad (4)$$

where

$$z \in C^2(I), I = [a,b], -\infty < a < b < \infty, \alpha \ne 1, \alpha > 0$$

and $h(x)$ is bounded in $\mathbb{R}$.

Definition 1.1 We will say that the Equation (1) has the Hyers-Ulam stability if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0, z \in C^2[a,b]$, if

$$\left|z'' - F\left(x, z(x)\right)\right| \le \varepsilon \tag{5}$$

with the initial condition (2), then there exists a solution $w(x) \in C^2[a,b]$ of the Equation (1), such that

$$\left|z(x) - w(x)\right| \le K\varepsilon .$$

Definition 1.2 We say that Equation (3) has the Hyers-Ulam stability with initial conditions (4) if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0, z \in C^2[a,b]$, if

$$\left|z'' - |z|^{\alpha} \operatorname{sgn} z\right| \le \varepsilon \tag{6}$$

and $z(a) = z'(a) = 0$, then there exists some $w \in C^2[a,b]$ satisfying $w'' - |w|^{\alpha} \operatorname{sgn} w = 0$ and $w(a) = w'(a) = 0$, such that $\left|z(x) - w(x)\right| \le K\varepsilon$.

2. Main Results on Hyers-Ulam Stability

Theorem 2.1 If $z \in C^2[a,b]$ is such that

$$|z(x)| \le |z'(x)| \quad \text{and} \quad A < \left(\max_{a\le x\le b} |z(x)|\right)^{1-\alpha},$$

then the Equation (1) is stable in the sense of Hyers and Ulam.

Proof. Let $\varepsilon > 0$ and z be a twice continuously differentiable real-valued function on $I = [a,b]$. We will show that there exists a function $w(x) \in C^2(I)$ satisfying Equation (1) such that

$$|z(x) - w(x)| \le k\varepsilon$$

where k is a constant that never depends on ε nor on $w(x)$. Since $z(x)$ is a continuous function on I, then it keep its sign on some interval $[a,x] \subseteq I$.

Without loss of generality assume that $z(x) > 0$ on $[a,x]$. Assume that $\varepsilon > 0, z \in C^2(I)$ satisfies the inequation (5) with the initial conditions (2) and that

$$M = \max_{a\le x\le b} |z(x)|.$$

From the inequality (5) we have

$$-\varepsilon \le z'' - F\left(x, z(x)\right) \le \varepsilon \tag{7}$$

Since $z(x) > 0$ on $[a,x] \subseteq I$ and $z(a) = 0$, then by Mean Value Theorem $z' > 0$ in (a,x). Multiplying the inequality (7) by $z' > 0$ and then integrating from a to x, we obtain

$$-2\varepsilon z \le z'^2(x) - 2\int_a^x F\left(x, z(x)\right) z' \mathrm{d}x \le 2\varepsilon z$$

Since $|z(x)| \le |z'(x)|$ we get that

$$z^2(x) \le 2\int_a^x F\left(x, z(x)\right) z' \mathrm{d}x + 2\varepsilon z \le AM^{\alpha-1} z^2 + 2\varepsilon z$$

Therefore

$$\max_{a\le x\le b} |z(x)| \le \frac{2\varepsilon}{1 - AM^{\alpha-1}}$$

Hence $|z(x)| \le k\varepsilon$, for all $x \ge a$. Obviously, $w_0(x) = 0$ satisfies the Equation (1) and the zero initial condition (2) such that

$$|z(x) - w_0(x)| \le k\varepsilon$$

Hence the Equation (1) has the Hyers-Ulam stability with initial condition (2).

Remark 2.1 Suppose that $z \in C^2(I)$ satisfies the inequality (5) with the initial condition (2). If

$$|z'(x)| \le |z(x)|$$

then, if

$$A > \left(\max_{a\le x\le b} |z(x)|\right)^{1-\alpha}$$

we can similarly show that the Equation (1) has the Hyers-Ulam stability with initial condition (2).

Theorem 2.2 Suppose that $z : [a,b] \to \mathbb{R}$ is a twice continuously differentiable function and $|z(x)| \le |z'(x)|$. If $B < \left(\max_{a\le x\le b} |z(x)|\right)^{1-\alpha}$ then the Equation (3) is stable in the sense of Hyers and Ulam.

Proof. Let $\varepsilon > 0$ and z be a twice continuously differentiable real-valued function on $I = [a,b]$. We will show that there exists a function $w(x) \in C^2(I)$ satisfying Equation (3) such that

$$|z(x) - w(x)| \le k\varepsilon$$

where k is a constant that never depends on ε nor on $w(x)$. Since $z(x)$ is a continuous function on I then it keeps its sign on some interval $[a,x] \subseteq I$. Without loss of generality assume that $z(x) > 0$ on $[a,x]$.

Suppose that $\varepsilon > 0, z \in C^2(I)$ satisfies the inequation (6) with the initial conditions (4) and that $M = \max_{a\le x\le b} |z(x)|$. We have

$$-\varepsilon \le z'' - h(x) z^{\alpha} \le \varepsilon \tag{8}$$

Since $z' > 0$ in (a,x) then, Multiplying the inequality (8) by z' and integrating, we obtain

$$-2\varepsilon z \le z'^2(x) - 2\int_a^x h(x) z^{\alpha} z' \mathrm{d}x \le 2\varepsilon z$$

By hypothesis $|z(x)| \le |z'(x)|$, so we get that

$$z^2(x) \le 2h(x^*)M^{\alpha-1}\int_a^x z\cdot z'\mathrm{d}x + 2\varepsilon z \le BM^{\alpha-1}z^2 + 2\varepsilon z$$

Therefore

$$\max_{a\le x\le b}|z(x)| \le \frac{2\varepsilon}{1-BM^{\alpha-1}}$$

Hence $|z(x)| \le k\varepsilon$, for all $x \ge x_0$. Clearly, the zero function $w_0(x)=0$ satisfies the Equation (1) and the zero initial condition (2) such that

$$|z(x)-w_0(x)| \le k\varepsilon$$

Hence the Equation (3) has the Hyers-Ulam stability with initial condition (4).

Remark 2.2 Suppose that $z\in C^2(I)$ satisfies the inequality (6) with the initial condition (4). If

$$|z'(x)| \le |z(x)| \quad \text{then, if} \quad B > \left(\max_{a\le x\le b}|z(x)|\right)^{1-\alpha}$$

we can similarly show that the Equation (3) has the Hyers-Ulam stability with initial condition (4).

Example 2.2 Consider the equation

$$z'' - \varepsilon^2 \sin\left(\frac{x}{\varepsilon^2+1}\right)z^{\frac{1}{2}} = 0 \qquad (9)$$

and the inequality

$$\left|z'' - \varepsilon^2 \sin\left(\frac{x}{\varepsilon^2+1}\right)z^{\frac{1}{2}}\right| \le \varepsilon \qquad (10)$$

where $0\le x\le 1$.

It should be noted that for a given $\varepsilon>0, z(x)=\varepsilon^2 x$, satisfies the inequation (10) and the conditions of the Theorem 2.2. Therefore the Equation (9) has the Hyers-Ulam stability.

3. A Special Case of Equation (3)

Consider the special case (when $\alpha=1$) of the Equation (3)

$$z'' - h(x)z = 0 \qquad (11)$$

with the initial conditions

$$z(a) = z'(a) = 0 \qquad (12)$$

and the inequation

$$|z'' - h(x)z| \le \varepsilon \qquad (13)$$

where $z\in C^2(I), I=[a,b], -\infty<a<b<\infty$.

Theorem 3.1 Assume that $z:[a,b]\to\mathbb{R}$ is a twice continuously differentiable function and $|z(x)|\le|z'(x)|$ Then, If $B<1$ the Equation (11) is stable in the sense of Hyers and Ulam.

Proof. Assume that $\varepsilon>0$ and that z is a twice continuously differentiable real-valued function on $I=[a,b]$. We will show that there exists a function $w(x)\in C^2(I)$ satisfying Equation (11) such that

$$|z(x)-w(x)| \le k\varepsilon$$

where k is a constant that never depends on ε nor on $w(x)$. Since $z(x)$ is a continuous function on I then it keeps its sign on some interval $[a,x]\subseteq I$. Without loss of generality assume that $z(x)>0$ on $[a,x]\subseteq I$. Suppose that $\varepsilon>0, z\in C^2(I)$ satisfies the inequation (13) with the initial conditions (12).

We have

$$-\varepsilon \le z'' - h(x)z \le \varepsilon \qquad (14)$$

Applying the Mean Value Theorem to the function $z(x)$ on the interval $[a,x]$, we find that $z'>0$ in (a,x). Multiplying the inequality (14) by $z'>0$ and then integrating we obtain

$$-2\varepsilon z \le z'^2(x) - 2\int_a^x h(x)z\cdot z'\mathrm{d}x \le 2\varepsilon z$$

If $|z(x)|\le|z'(x)|$, we obtain the inequality

$$z^2(x) \le 2h(x^*)\int_a^x z\cdot z'\mathrm{d}x + 2\varepsilon z \le Bz^2 + 2\varepsilon z$$

Therefore

$$\max_{a\le x\le b}|z(x)| \le \frac{2\varepsilon}{1-B}$$

Thus $|z(x)|\le k\varepsilon$, for all $x\ge x_0$. The zero solution $w_0(x)=0$ of the Equation (11) with the zero initial condition (12) such that

$$|z(x)-w_0(x)| \le k\varepsilon$$

Hence the Equation (11) has the Hyers-Ulam stability with initial condition (12).

Remark 3.1 Assume that $z\in C^2(I)$ satisfies the inequality (13) with the initial condition (12). If $|z'(x)|\le|z(x)|$ then, if $B>1$ we can similarly obtain the Hyers-Ulam stability criterion for the Equation (11) has with initial condition (12).

Remark 3.2 It should be noted that if $z<0$ on $(a,x]$ and $z(a)=0$, hence $z'<0$ on (a,x), then in the proofs of Theorem 2.1, 2.2 and 3.1, we can multiply by $z'<0$ the inequation (7) (and (8), (14)) to obtain the inequality

$$\varepsilon z' \le z'\cdot z'' - z'F(x,z(x)) \le -\varepsilon z'$$

Then we can apply the same argument used above to get sufficient criteria for the Hyers-Ulam stability for the Equations (1), (3) and (11).

Example 3.1 Consider the equation

$$z'' - \frac{1}{2}\cos\left(\frac{1}{x+1}\right)z = 0 \qquad (15)$$

and the inequality

$$\left|z''-\frac{1}{2}\cos\left(\frac{1}{x+1}\right)z\right|\le\varepsilon \tag{16}$$

where $0\le x\le 1$.

First it should be noted that for a given $\varepsilon>0$, $z(x)=\varepsilon x$, satisfies the inequation (16) and the conditions of the Theorem 3.1. Therefore the Equation (15) has the Hyers-Ulam stability.

4. An Additional Case On Hyers-Ulam Stability

In this section we consider the Hyers-Ulam stability of the following equation

$$z''=\phi(x,z(x)) \tag{17}$$

with the initial condition

$$z(a)=z'(a)=0 \tag{18}$$

where

$$z\in C^2(I),\ I=[a,b],-\infty<a<b<\infty$$

and $\phi(x,z(x))$ is continuous for $x\in I,\ z\in\mathbb{R}$ such that

$$\left|\phi(x,z(x))-\phi(x,w(x))\right|\le L|z-w|.$$

Using an argument similar to that used in [16], we can prove the following Theorem:

Theorem 4.1 Suppose that $z:[a,b]\to\mathbb{R}$ is a twice continuously differentiable function.

If $\frac{L(b-a)^2}{2}<1$, then the problem (17), (18) is stable in the Hyers-Ulam sense.

Proof. Let $\varepsilon>0$ and z be a twice continuously differentiable real-valued function on $I=[a,b]$ satisfying the inequality

$$-\varepsilon\le z''-\phi(x,z(x))\le\varepsilon \tag{19}$$

We will show that there exists a function

$$w(x)\in C^2(I)$$

Satisfying Equation (18) such that

$$|z(x)-w(x)|\le k\varepsilon$$

where k is a constant that doesn't depend on ε nor on $w(x)$. If we integrate the inequality (19) with respect to x, we should obtain

$$-\frac{\varepsilon(b-a)^2}{2}\le-\frac{\varepsilon(x-a)^2}{2}\le z-\int_a^x\int_a^t\phi(s,w(s))\mathrm{d}s \le\frac{\varepsilon(x-a)^2}{2}\le\frac{\varepsilon(b-a)^2}{2} \tag{20}$$

It is clear that $w(x)=\int_a^x\int_a^t\phi(s,w(s))\mathrm{d}s$ is a solution of the Equation (21)

$$w''=\phi(x,w(x)) \tag{21}$$

satisfying the zero initial condition

$$w(a)=w'(a)=0 \tag{22}$$

Now, let's estimate the difference

$$|z(x)-w(x)|\le\left|z-\int_a^x\int_a^t\phi(s,z(s))\mathrm{d}s\right| +\left|\int_a^x\int_a^t\phi(s,w(s))\mathrm{d}s-\int_a^x\int_a^t\phi(s,z(s))\mathrm{d}s\right|$$

Since the function $\phi(x,z(x))$ satisfies the Lipschitz condition, and from the inequality (20) we have

$$|z(x)-w(x)|\le\frac{\varepsilon(b-a)^2}{2}+L\frac{(b-a)^2}{2}|w(s)-z(s)|$$

From which it follows that

$$|z(x)-w(x)|\le\frac{\varepsilon(b-a)^2}{2-L(b-a)^2}$$

where $2>L(b-a)^2$. Hence the problem (17), (18) has the Hyers-Ulam stability.

Remark 4.1 Notice that if $\phi(x,z(x))=h(x)|z|^\alpha$, satisfies Lipschitz condition in the region $x\in I,z\in\mathbb{R}$, then the Emden-Fowler nonlinear differential equation $z''-h(x)|z|^\alpha\operatorname{sgn}z=0$ is Hyers-Ulam stable with zero initial condition.

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