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# Symmetric Solutions of a Nonlinear Elliptic Problem with Neumann Boundary Condition

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## ABSTRACT

We show a result of symmetry for a big class of problems with condition of Neumann on the boundary in the case one dimensional. We use the method of reflection of Alexandrov and we show one application of this method and the maximum principle for elliptic operators in problems with conditions of Neumann. Some results of symmetry for elliptic problems with condition of Neumann on the boundary may be extended to elliptic operators more general than the Laplacian.

**Keywords:** Nonlinear Boundary Value Problems; Elliptic

## 1. Introduction

The maximum principle is one of the most used tools in the study of some differential equations of elliptic type. It is a generalization of the following well-known theorem of the elemental calculus "If *f* is a function of class $C^2$ in $[a,b]$ such that the second derivative is positive on ($a$, $b$) then the maximum value of *f* attains at the ends of $[a,b]$". It is important to point out that the maximum principle gives information about the global behavior of a function over a domain from the information of qualitative character in the boundary and without explicit knowledge of the same function. The maximum principle allows us, for example, to obtain uniqueness of solution of certain problems with conditions of the Dirichlet and Neumann type. Also it allows to obtain a priori estimates for solutions. These reasons make interesting the study of the maximum principle on several forms and its generalizations and the Hopf lemma. For example a geometric version of the maximum principle allows us to compare locally surfaces that coincide at a point. On the other hand, the maximum principle and the Alexandrov reflection principle in [1] have been used to prove symmetries with respect to some point, some plane, symmetries of domain and to determine asymptotic-symmetric behavior of the solutions of some elliptic problems. (See Serrin [2], Gidas, Ni and Nirenberg [3], Gidas, Ni and Nirenberg [4], Caffarelli, Gidas and Spruck [5], Berestycki and Nirenberg [6]). The first person in use this technic was Serrin. Serrin proved that: "If $u$ is a positive solution of the problem

$$\Delta u = -1 \quad \text{en} \quad \Omega$$

which is zero on the boundary and its outer normal derivative on the boundary is constant, then $\Omega$ is a ball and $u$ is radially symmetric with respect to the center of $\Omega$". Using the ideas of Serrin and a version of the maximum principle for functions that do not change of sign, Gidas Ni and Nirenberg proved that: "If $\Omega$ is a ball, $f \in C^1(\mathbb{R})$ and $u$ is a positive solution of the problem,

$$\Delta u + f(u) = 0 \quad \text{on} \quad \Omega$$

which is zero on the boundary, then $u$ is radially symmetric with respect to the center of the ball". Using the method of reflection and a version of maximum principle for thin domains Berestycki and Nirenberg made a generalization of this statement. Our proof shows that the technic used by Berestycki and Nirenberg for the study of symmetries of solutions of the elliptic problem with Dirichlet condition, can be applied in elliptic problems with Neumann conditions with nonlinear term $f(x,u(x))$.

## 2. Maximum Principle and Hopf Lemma

Our result is based on the well known maximum principle and on the Hopf lemma for the differential operator of the form (see [7-9])

$$L[u] = a(x)u'' + b(x)u' + c(x)u \tag{1}$$

where $x$ is in $(a,b)$. We suppose that the coefficients

 

$a(x), b(x)$ and $c(x)$ are bounded on $(a,b)$ and $a(x) > 0$, $c(x) \le 0$ for all $x \in (a,b)$.

**Theorem 2.1. (Maximum principle)**

Let $u \in C^2((a,b))$ be such that $L[u] > 0$. Then $u$ cannot attain its maximum value in $(a,b)$

**Lemma 2.2. (Hopf)**

Suppose $u \in C^2((a,b)) \cap C^0([a,b])$ satisfies

$$L[u] \ge 0 \quad \text{in} \quad (a,b).$$

Let $x_0 \in \partial(a,b)$ be such that

- $u$ is continuous at $x_0$,
- $u(x_0) > u(x)$ for all $x \in (a,b)$,
- $\frac{\partial u}{\partial \boldsymbol{\eta}}(x_0)$ existe.

Then $\frac{\partial u}{\partial \boldsymbol{\eta}}(x_0) > 0.$

## 3. Main Result

**Theorem 3.3.** Let $u \in C^2((-1,1)) \cup C^0([-1,1])$ be a solution of

$$\begin{cases} a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x,u(x)) \\ \text{on} \quad (-1,1), \\ u'(1) = -u'(-1), \end{cases}$$

where $a, c : [-1,1] \to \mathbb{R}$ are bounded functions and symmetric with respect to the origin such that $a(x) > 0$ and $c(x) \le 0$ for all $x \in [-1,1]$; $f \in C'(\mathbb{R} \times \mathbb{R}^+, \mathbb{R})$ is such that $f(x,t)$ is strictly increasing in $t$ for all $x \in [-1,1]$ and is symmetric to $x = 0$ for all $t \in \mathbb{R}^+$, and $b : [-1,1] \to \mathbb{R}$ is a bounded function and odd. Then $u$ is symmetric with respect to the origin.

**Proof:** Define the reflected function of $u$ in $[-1,1]$ by

$$v(x) = u(-x), \quad x \in [-1,1].$$

Hence, $v'(x) = -u'(-x), v''(x) = u''(-x)$. Then $v$ satisfies

$$\begin{cases} a(x)v''(x) + b(x)v'(x) + c(x)v(x) = f(x,v(x)) \\ \text{on} \quad (-1,1), \\ v'(1) = -v'(-1), \end{cases}$$

Define

$$w(x) = u(x) - v(x).$$

Then $w$ satisfies

$$\begin{cases} a(x)w''(x) + b(x)w'(x) + c(x)w(x) \\ = f(x,u(x)) - f(x,v(x)) \\ \text{on} \quad (-1,1) \\ w'(-1) = w'(1) = 0. \end{cases}$$

Since $w$ is continuous in $\overline{\Omega}$, there are $x_M, x_m \in \overline{\Omega}$ such that

$$w(x_m) = \min_{\Omega} w \quad \text{and} \quad w(x_M) = \max_{\Omega} w$$

Suppose that $x_M$ or $x_m \in (-1,1)$, then if $x_M \in (-1,1), w(x_M) \ge 0$ since $w(0) = 0$. Further $w''(x_M) \le 0, w'(x_M) = 0$. Therefore

$$f(x_M, u(x_M)) - f(x_M, v(x_M)) \le 0.$$

Since $f(.,t)$ is strictly increasing in $t$

$$w(x_M) \le 0.$$

Then

$$w(x_M) = 0.$$

Therefore

$$w(x) \le 0 \quad \text{for all} \quad x \in [-1,1].$$

If $x_m \in (-1,1)$, using a similar argue we demonstrate that $w \equiv 0$ on $[-1,1]$ and we obtain the same conclusion. Suppose that $x_m \in (-1,1)$, then $w(x_m) \le 0$. since $w(0) = 0$ Further $w''(x_m) \ge 0, w'(x_m) = 0$. Therefore

$$f(x_m, u(x_m)) - f(x_m, v(x_m)) \ge 0.$$

Since $f(.,t)$ is strictly increasing in $t$

$$w(x_m) \ge 0.$$

Then

$$w(x_m) = 0.$$

Therefore

$$w(x) \ge 0 \quad \text{for all} \quad x \in [-1,1].$$

We conclude

$$w \equiv 0 \quad \text{on} \quad [-1,1].$$

So $u$ is symmetric with respect to the origin.

We will prove that $x_m, x_M$ do not belong to $\partial[-1,1]$. Suppose now that $x_m, x_M \in \{-1,1\}$ and $w(x_m) < w(x) < w(x_M)$ for all $x \in (-1,1)$, then $w(x_M) > 0$ and $w(x_m) < 0$. If $x_m = -1$ and $x_M = 1$, then $w(x) \le 0$ in $(-1,\alpha)$ and $w(x) \ge 0$ in $(\beta,1)$, where $\alpha, \beta \in (-1,1)$ are such that $\alpha$ is the first zero of $w$ and $\beta$ is the last. Since $f(.,t)$ is strictly increasing in $t$, then

$$\begin{cases} a(x)w'' + b(x)w' + c(x)w \le 0 \quad \text{in} \quad (-1,\alpha), \\ w'(-1) = 0, \end{cases}$$

and

$$\begin{cases} a(x)w'' + b(x)w' + c(x)w \ge 0 \quad \text{in} \quad (\beta,1), \\ w'(1) = 0. \end{cases}$$

 

Applying maximum principle and Hopf lemma,

$$w'(-1) > 0, \quad w'(1) > 0,$$

since $w$ is not constant. Which contradicts the fact that

$$w'(-1) = w'(1) = 0.$$

Hence this case is impossible. It happens equally to $x_m = 1$ and $x_M = -1$. In conclusion we have that $w \equiv 0$ on $[-1,1]$ and therefore $u$ is symmetric with respect to $x = 0$.

## 4. Example

Taking

$$a(x) = 1, b(x) = x, c(x) = -1, f(x,u) = x^2 u$$

in Theorem 3.3, we have the following system

$$\begin{cases} u''(x) + xu'(x) - u(x) = x^2 u(x) \\ \text{on} \quad (-1,1), u'(1) = -u'(-1), \end{cases}$$

following the steps of the demonstration, it follows that $u$ is symmetric with respect to the origin.

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