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Optimal Campaign in Leptospirosis Epidemic by Multiple Control Variables

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ABSTRACT

In this paper, we consider a leptospirosis epidemic model to implement optimal campaign by using multiple control variables. First, we show the existence of the control problem. Then we derive the conditions under which it is optimal to eradicate the leptospirosis infection and examine the impact of a possible educational/vaccination campaign using Pontryagin's Maximum Principle. We completely characterize the optimal control problem and compute the numerical solution of the optimality system using an iterative method. The results obtained from the numerical simulations of the model show that a possible educational/vaccination combined with effective treatment regime would reduce the spread of the leptospirosis infection appreciably.

Keywords: Mathematical Model; Optimal Control; Pontryagin's Maximum Principle; Numerical Results

1. Introduction

Leptospirosis disease is a globally zoonotic disease. The cause of the disease is bacteria which is called leptospira. Human as well as cattle are mostly infected from this disease. The human are infected by means of drinking the water in which a rat (dead) found, while cattle that drink this water are become infectious. The human whose urine is used by other animals and cattle are also infected, because the leptospirosis germs come out in urine. It is also reported that people belong to city are mostly infected from this disease and got liver infection. Leptospirosis is known by different names such Weil's disease, canicola fever, canefield fever, 7-day fever, nanukayami fever [1]. Weil is the first man who credited that described leptospirosis as a unique disease process in 1886, 30 years before Inada and his colleagues identified the causal organism. The symptoms of leptospirosis are high fever, headache, chills, muscle aches, conjunctivitis (red eyes), diarrhea, vomiting, and kidney or liver problems (which may also include jaundice), anemia and sometimes rash. Symptoms may last from a few days and up to several weeks. Some reports also show that deaths from this disease may occur but they are rare. For some cases, the infections can be mild and without obvious symptom [2-6]. Outbreaks of this disease depending on season which often linked to environmental factors involve animals, agricultural and occupational cycles [7].

The mathematical formulation and dynamical sketch of this infection has been studied by several authors see for example [8-12]. Pongsuumpun *et al.* [11] represents mathematical model and considered some real data for numerical simulation. A simple deterministic model for the spread of leptospirosis in Thailand can be found in [13]. In their work, they represented the rate of change for both rats and human population. The human population is further divided into two main groups Juveniles and adults. Zaman [14] considered the real data presented in [13] to study the dynamical behavior and role of optimal control theory. The dynamical interaction between leptospirosis infected vector and human population is studied by Zaman *et al.* [12]. In their work, they presented global dynamics and bifurcation analysis. They also showed the numerical simulations for different values of the interaction parameter.

In case of vector born diseases some authors focused on eradication of the disease, by targeting the vector population as a strategy for controlling the disease [15, 16] while some scientists studied the effect of vaccination on the dynamics of the disease [14]. These scientists believed that optimal control theory is a powerful mathematical tools which make the decision involving complex dynamical systems [17]. Optimal control method has been used to study dynamics of the disease see for example [17-19]. Very little has been done in the interaction between leptospirosis infected vector and human popula-

tion by applying multiple control variables to analyze and understand the dynamics of this infection in a community.

In this paper, we consider the basic model studied in [14] to incorporate some important epidemiological features and control functions. To control the spread of leptospirosis infection and the interaction of human with vector population, we use optimal theory to reduce the proportion of the infected human and infected vector until the disease cannot survive. At the long-term level of infected human by the interaction of infected vector which causes the spread of new infection. Therefore, if we can reduce the number of infected human further, so the disease does less well and will increase the recovered human. To do this, we introduce an educational/vaccination campaign by using three control variables. Our first control variable represents cover all cuts, water dry, full-cover boots, shoes and long sleeve shirts when handling animals, second control variable represents wash hands thoroughly on a regular basis and shower after work and third control variable represents clean up both work place and home. We first show the existence of the optimal control system. Then, we derive the conditions under which it is optimal to eradicate the leptospirosis infection and examine the impact of a possible educational/vaccinaction campaign using Pontryagin's Maximum Principle. We also solve the optimality system numerically, which consists of the original state system, the adjoint system and their boundary conditions by using the data presented for leptospirosis epidemic in Thailand. We conclude by discussing results of the numerical simulations in detail.

The structure of the paper is organized as follows. Section 2 is devoted to the formulation of the basic mathematic model. In Section 3, we present the control problem and develop reproductive number. In Section 4, we present the endemic equilibria for both systems with and without control and bifurcation analysis. In Section 5, we present the existence of the control problem and derive the necessary conditions for an optimal control and the corresponding state system by using Pontryagin's Maximum Principle. Section 6 is devoted to numerical solution of the optimality system and finally, we conclude our work.

2. Basic Mathematical Model

Basic epidemic models allow for variations in the different stages(classes) of the infection. Several researcher developed different mathematical models to identifying the stages which depends on the dynamics of the disease and the composition of the population. In these mathematical models an individual can be in any one of the stages of infection. Susceptible (S), the individual is able to contract the infection; exposed (E), the individual has

contracted the disease but is not yet infectious or symptomatic; infectious (I), the individual is contagious and may or may not be showing symptoms; and removed (R), an individual can be removed from the population by recovering with immunity, being quarantined or by death. In this work, we present the basic model proposed by [20], consisting a non-linear system of seven differential equations. We consider a given human population which we divide into four categories: susceptible, exposed, infected and recovered classes.

For each category, we assume the population changes over time. Thus, we write the number of humans in each category $S_h(t)$ susceptible, $E_h(t)$ exposed, $I_h(t)$ infected and $R_h(t)$ recovered human as functions of time t. The total human population is denoted by $N_h(t)$ with $N_h(t) = S_h(t) + E_h(t) + I_h(t) + R_h(t)$. Similarly, we write the number of vector in each category: $S_v(t)$ susceptible, $E_v(t)$ exposed, and $I_v(t)$ infected vector, respectively as functions of time t. The total vector class is denoted by $N_v(t)$ with $N_v(t) = S_v(t) + E_v(t) + I_v(t)$. The complete system of non-linear differential equation is given by:

$$\frac{dS_h}{dt} = b_1 - \mu S_h - \beta_1 S_h I_v - \alpha_1 S_h + \lambda_h R_h,$$

$$\frac{dE_h}{dt} = \beta_1 S_h I_v + \alpha_1 S_h - \mu E_h - \alpha_h E_h,$$

$$\frac{dI_h}{dt} = \alpha_h E_h - \mu I_h - \mu_h I_h - \delta_h I_h,$$

$$\frac{dR_h}{dt} = \delta_h I_h - \mu R_h - \lambda_h R_h, \quad (1)$$

$$\frac{dS_v}{dt} = b_2 - \delta_o S_v - \beta_2 S_v I_h,$$

$$\frac{dE_v}{dt} = \beta_2 S_v I_h - \delta_o E_v - \alpha_v E_v,$$

$$\frac{dI_v}{dt} = \alpha_v E_v - \delta_o I_v - \delta_v I_v.$$

With initials conditions

$$S_h \geq 0, \quad E_h \geq 0, \quad I_h \geq 0, \quad R_h \geq 0, \quad S_v \geq 0,$$
$$E_v \geq 0, I_v \geq 0 \quad (2)$$

The parameters involved in the basic model are as under:

b_1 is the recruitment rate of human population,
β_1 is the transmission coefficient,
β_2 is the transmission coefficient,
α_1 is the Transmission coefficient,
μ is the natural mortality rate of human,
μ_h is the death rate of infected human,
b_2 is the recruitment rate of vector,
δ_o is the natural mortality rate of vector,
δ_v is the death rate of infected vector,

α_v is the rate at which exposed vector move to exposed class,

α_h is the rate at which exposed human move to exposed class.

3. The Control Problem

Optimal control is one of the techniques to minimize (maximize) the infection in the human class of individuals. Several articles have been published on different population models by applying the optimal control techniques to reduce the infection at the human population using different control variables [17,19]. In this section, we present an optimal control technique by using multiple control variables to reduce the spread of leptospirosis infection in a community. Our educational/vaccination campaign consisting of the following control variables:

$u_1(t)$: represents (cover all cuts, water dry, full-cover boots, shoes and long sleeve shirts when handling animals),

$u_2(t)$: represents (wash hands thoroughly on a regular basis and shower after work),

$u_3(t)$: represents (clean up both work place and home).

Our control strategies by using the above three control variables can be easily implemented to eradicate the spread of this disease in the community.

The control set for the control variables is defined as,

$$U = \{u = (u_1, u_2, u_3) \text{ is Lebesgue measureable}$$
$$0 \le u_i(t) \le 1, \text{for } i = 1, 2, 3\} \tag{3}$$

The above control variables in the system (1) are adjusted in the following form

$$\frac{dS_h}{dt} = b_1 - \mu S_h - \beta_1 S_h (1 - u_1(t)) I_v$$
$$- \alpha_1 S_h (1 - u_1(t)) + \lambda_h R_h,$$
$$\frac{dE_h}{dt} = \beta_1 S_h (1 - u_1(t)) I_v + \alpha_1 S_h (1 - u_1(t))$$
$$- (\mu + \alpha_h) E_h (1 - u_2(t)),$$
$$\frac{dI_h}{dt} = \alpha_h E_h (1 - u_2(t)) - \mu I_h - \mu_h I_h - \delta_h I_h, \tag{4}$$
$$\frac{dR_h}{dt} = \delta_h I_h - \mu_0 R_h - \lambda_h R_h,$$
$$\frac{dS_v}{dt} = b_2 - \delta_0 S_v - \beta_2 S_v I_h - c_1 u_3(t) S_v,$$
$$\frac{dE_v}{dt} = \beta_2 S_v I_h - \delta_0 E_v - \alpha_v E_v - c_1 u_3(t) E_v,$$
$$\frac{dI_v}{dt} = \alpha_v E_v - \delta_0 I_v - \delta_v I_v - c_1 u_3(t) I_v,$$

with the initials conditions given in (2).

Here $c_1 > 0$ represents the constant at which the rate of vector decreases at time t. The factor $(1 - u_1(t))$ and $(1 - u_2(t))$, are used to reduce the force of infections.

Our aim is to decrease the number of susceptible, exposed human and total vector population and increase the recovered human population. In order to do this, we define the objective functional is given by

$$F(u_1, u_2, u_3) = \int_0^T \left(A_1 S_h + A_2 E_h + A_3 (S_v + E_v + I_v) \right.$$
$$\left. + \frac{1}{2} (D_1 u_1^2 + D_2 u_2^2 + D_1 u_3^2) \right) dt \tag{5}$$

The objective functional includes the susceptible individuals, exposed individuals, and the class of vector population. The constants A_i and D_i for $i = 1, 2, 3$ are weight/balance factors to keep the balanced of individuals in the objective functional. The Lagrange for the control problem (4) is given by

$$L(N_h, N_v, u) = A_1 S_h + A_2 E_h + A_3 (S_v + E_v + I_v)$$
$$+ \frac{1}{2} (D_1 u_1^2 + D_2 u_2^2 + D_1 u_3^2) \tag{6}$$

To do this, we define the Hamiltonian H for the control problem as follows:

$$H(t, (N_h, N_v, \lambda, u)$$
$$= L(N_h, N_v, u)$$
$$+ \lambda_1(t) \left(b_1 - \mu S_h - (\beta_1 S_h I_v - \alpha_1 S_h)(1 - u_1(t)) + \lambda_h R_h \right)$$
$$+ \lambda_2(t) \left(\beta_1 S_h (1 - u_1(t)) I_v + \alpha_1 S_h (1 - u_1(t)) \right.$$
$$\left. - (\mu + \alpha_h) E_h (1 - u_2(t)) \right)$$
$$+ \lambda_3(t) \left(\alpha_h E_h (1 - u_2(t)) - \mu I_h - \mu_h I_h - \delta_h I_h \right)$$
$$+ \lambda_4(t) \left(\delta_h I_h - \mu_0 R_h - \lambda_h R_h \right)$$
$$+ \lambda_5(t) \left(b_2 - \delta_0 S_v - \beta_2 S_v I_h - c_1 u_3(t) S_v \right)$$
$$+ \lambda_6(t) \left(\beta_2 S_v I_h - \delta_0 E_v - \alpha_v E_v - c_1 u_3(t) E_v \right)$$
$$+ \lambda_7(t) \left(\alpha_v E_v - \delta_0 I_v - \delta_v I_v - c_1 u_3(t) I_v \right). \tag{7}$$

4. Reproductive Number R_o and R_u

In order to understand the dynamical behavior, we find the threshold quantity, also known as the basic reproductive number. This number is obtained by setting the right hand side of all equations equal to zero of the system (1) without control and the system (4) with control and do some rearrange to get the following two basic reproductive numbers. We obtain two reproductive numbers R_o and R_u form the above two systems without and with optimal control, respectively. The threshold quantity denoted by R_o for the system (1) without op-

timal control variable is given by,

$$R_o = \frac{a_1 \alpha_h \alpha_1 T_4 T_5 \delta_0 + \alpha_v \alpha_h \beta_1 \beta_2 a_2 a_1}{T_1 T_2 T_4 T_5 (\mu_0 + \alpha_1) \delta_0},$$

where,

$$T_1 = (\mu_0 + \alpha_h), T_2 = (\mu_0 + \mu_h + \delta_h),?$$
$$T_3 = (\mu_0 + \lambda_h), T_4 = (\delta_0 + \alpha_v), T_5 = (\delta_0 + \delta_v)$$

The threshold quantity R_u for control problem in the control system (4) is given by

$$R_u = \frac{\alpha_h}{(1 - u_1) T_1 T_2} \left(\frac{P_1 P_2 P_3}{\beta_1 \alpha_v \beta_2 b_2} + \frac{1}{\alpha_1} \right),$$

where,

$$P_1 = (\delta_0 + \delta_v + c_1 u_3), \quad P_2 = (\delta_0 + \alpha_v + c_1 u_3),$$
$$P_3 = (\delta_0 + \beta_2 I_h^* + c_1 u_3), \quad u_1, u_2, \quad u_3, \quad T_1 \text{ and } T_2 \text{ are}$$
defined above for both threshold quantity R_o and R_u.

5. Endemic Equilibria and Backward Bifurcation

In this section, we find the endemic equilibria of the control system (4) and check that the backward bifurcation of the optimal control problem exists or not. For the endemic equilibria we set left hand side of the control system (4) equal to zero, to obtain

$$S_h^* = \frac{T_1 T_2 I_h^* P_1 P_2 P_3}{\alpha_h \left[\beta_1 (1 - u_1) \alpha_v \beta_2 b_1 I_h^* + \alpha_1 (1 - u_1) P_1 P_2 P_3 \right]},$$

$$E_h^* = \frac{T_2 I_h^*}{\alpha_h (1 - u_2)}, \quad R_h^* = \frac{\delta_h I_h^*}{(T_3)},$$

$$S_v^* = \frac{b_2}{(\delta_0 + \beta_2 I_h^* + c_1 u_3)},$$

$$E_v^* = \frac{\beta_2 b_2 I_h^*}{(\delta_0 + \delta_v + c_1 u_3)(\delta_0 + \beta_2 I_h^* + c_1 u_3)},$$

$$I_v^* = \frac{\alpha_v \beta_2 b_2 I_h^*}{P_1 P_2 P_3}.$$

Here $P_2 = (\delta_0 + \alpha_v + c_1 u_3), \quad P_1 = (\delta_0 + \delta_v + c_1 u_3),$

$$T_1 = (\mu + \alpha_h), \quad T_2 = (\mu + \mu_h + \delta_h), \quad T_3 = \mu + \lambda_h.$$

In order to find the backward bifurcation, we put the above endemic equilibria in the first equation of the system (4), with setting left hand side equal to zero to get

$$f(I_h^*) = a I_h^{*2} + b I_h^* + c = 0,$$

where,

$$a = T_3 T_1 T_2 \beta_1 (1 - u_1) \alpha_v \beta_2 b_2 - \lambda_h \delta_h \alpha_h \beta_1 (1 - u_1) \alpha_v \beta_2 b_2,$$

$$b = T_3 T_1 T_2 \mu P_1 P_2 P_3 + \alpha_1 (1 - u_1) P_1 P_2 P_3 T_3 T_1 T_2$$
$$- \delta_h \lambda_h \alpha_1 \alpha_h (1 - u_1) P_1 P_2 P_3 - T_3 b_1 \alpha_h \beta_1 (1 - u_1) \alpha_v \beta_2 b_2,$$

$$c = T_3 b_1 P_3 \alpha_1 \alpha_h P_1 P_2 (1 - u_1)(1 - R_u).$$

Here the coefficient a is positive always and c depends upon the value of R_u, if the value of $R_u < 1$, then c is positive, otherwise negative. The positive solution of the above equation depends upon the value of b and c. For the value of $K_u > 1$, the above equation leads to two different roots one positive and negative. If we substitute $K_u = 1$, then the equation has no positive solution. This is possible if and only if $b < 0$. For $b < 0$ and $K_u = 1$, the equilibria depends upon K_u then there exists an open interval having two positive roots that is

$$I_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad I_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

For $c > 0$ either $b^2 < 4ac$ or $b \geq 0$, then the above have no positive solution. For backward bifurcation, we set $b^2 - 4ac = 0$, $R_u = R_c$ and solving for the critical value of K_c, which is given by

$$R_c = 1 - \frac{b^2}{4a T_3 b_1 P_3 \alpha_1 \alpha_h P_1 P_2 (1 - u_1)(1 - R_u)}$$

The numerical simulation of the backward bifurcation is obtained by using MATLAB. First we find the numerical results represented in **Figures 1**-**3** for control variable u_1, u_2, u_3 respectively. **Figure 4** shows the numerical result without control system and **Figure 5** shows the numerical result of the system with control for all the three control variables.

6. Existence of Control Problem

In this section, we show the existence of the control system (4). Let $S_h(t)$, $E_h(t)$, $I_h(t)$, $R_h(t)$, $S_v(t)$, $E_v(t)$ and $I_v(t)$ be the state variables with control variables $u_1(t)$, $u_2(t)$ and $u_3(t)$. We can write the system (4) in the following form:

$$X' = AX + F(X), \tag{8}$$

where

$$X = \begin{bmatrix} S_h(t) \\ E_h(t) \\ I_h(t) \\ R_h(t) \\ S_v(t) \\ E_v(t) \\ I_v(t) \end{bmatrix},$$

$$
A = \begin{bmatrix}
-\mu & 0 & 0 & \lambda_h & 0 & 0 & 0 \\
0 & -(\mu + \alpha_h)(1 - u_2) & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_h & -(\mu + \mu_h + \delta_h) & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_h & -(\mu + \delta_h) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\delta_0 & 0 \\
0 & 0 & 0 & 0 & 0 & -(\delta_0 + \alpha_v) & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_v & -(\delta_0 + \delta_v)
\end{bmatrix},
$$

$$
F = \begin{bmatrix}
-\beta_1 S_h (1 - u_1) I_v - \alpha_1 S_h (1 - u_1) + b_1 \\
\beta_1 S_h (1 - u_1) I_v + \alpha_1 S_h \\
(1 - u_2) \alpha_h E_h \\
0 \\
-\beta_2 S_v I_h + b_2 - c_1 u_3 S_v \\
\beta_2 S_v I_h - c_1 u_3 E_v \\
-c_1 u_3
\end{bmatrix},
$$
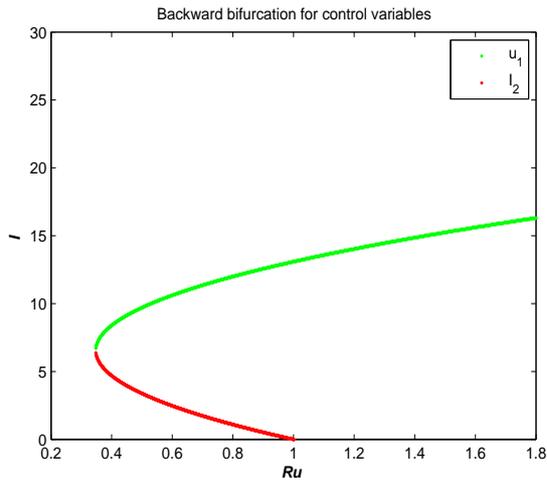
Figure 1. The plot represents the backward bifurcation for control variable u_1.

Figure 2. The plot represents the backward bifurcation for the control variable u_2.

Figure 3. The plot represents the backward bifurcation for control u_3.
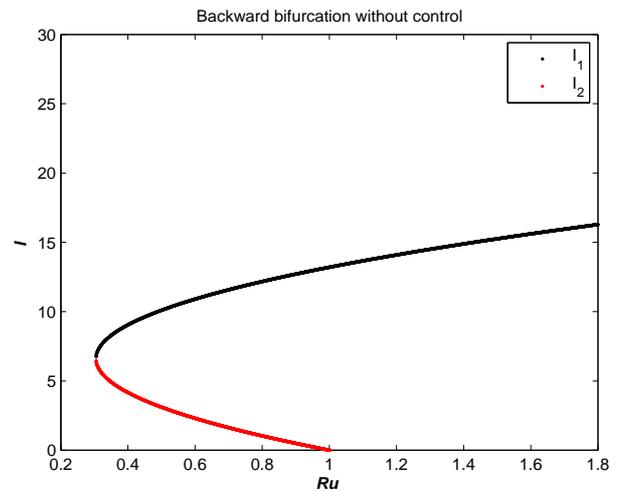
Figure 4. The plot represents the backward bifurcation without control variables.
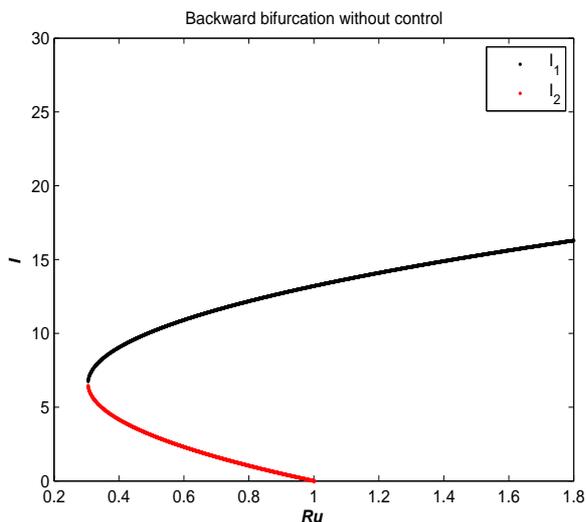
Figure 5. The plot represents the backward bifurcation with control variables u_1, u_2, u_3.

where X' denotes the derivative with respect to time t. The system (8) is a non-linear system with bounded co-efficients. We set

$$G(X) = A(X) + F(X), \qquad (9)$$

The second term on the right hand side of (9) satisfies

$$|F(X_1) - F(X_2)|$$

$$\leq C_1 \left(\left| \left(S_{1h}(t) - S_{2h}(t) \right) \right| + C_2 \left| E_{1h}(t) - E_{2h}(t) \right| \right.$$

$$+ C_3 \left| \left(I_{1h}(t) - I_{2h}(t) \right) \right| + C_4 \left| \left(R_{1h}(t) - R_{2h}(t) \right) \right|$$

$$+ C_5 \left| \left(S_{1v}(t) - S_{2v}(t) \right) \right| + C_6 \left(\left| E_{1v}(t) - E_{2v}(t) \right| \right)$$

$$+ C_7 \left| \left(I_{1v}(t) - I_{2v}(t) \right) \right| \right)$$

$$\leq C \left(\left| \left(S_{1h}(t) - S_{2h}(t) \right) \right| + \left| E_{1h}(t) - E_{2h}(t) \right| \right.$$

$$+ \left| \left(I_{1h}(t) - I_{2h}(t) \right) \right|$$

$$+ \left| \left(R_{1h}(t) - R_{2h}(t) \right) \right| + \left| \left(S_{1v}(t) - S_{2v}(t) \right) \right|$$

$$+ \left| E_{1v}(t) - E_{2v}(t) \right| + \left| \left(I_{1v}(t) - I_{2v}(t) \right) \right| \right)$$

where the positive constant $C = \max(C_1, C_2, C_3, C_4, C_5, C_6, C_7)$ is independent of the state variables. Also we have

$$|G(X_1) - G(X_2)| \leq C|X_1 - X_2|,$$

where $C = C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + \|M\| < \infty$.
So, it follows that the function G is uniformly Lipschitz

continuous. From the definition of control variables and non-negative initial conditions we can see that a solution of the system (8) exists see [21]. For the existence of our control problem, we revisit the optimal control problem presented in (4) with initial conditions (2) to state and prove the following theorem.

Theorem 5.1: There exists an optimal control $u^* = (u_1^*, u_2^*, u_3^*) \in U$ such that

$J(u_1^*, u_2^*, u_3^*) = \min_{(u_1, u_2, u_3) \in U} J(u_1, u_2, u_3)$, subject to the control system (4) with the initial conditions (2).

Proof: For the proof of this result, we use the same result presented in [22]. Since the control and the state variable are nonnegative. Our goal is to minimize the objective functional in the optimal control problem, the necessary convexity of the objective functional in u_1, u_2, u_3 are satisfied. The set of control variables $(u_1, u_2, u_3) \in U$ is also convex and closed by the definition. The optimal system is bounded which determines the compactness needed for the existence of optimal control. The integrand in the objective functional (5) is given by

$$\left(A_1 S_h + A_2 E_h + A_3 (S_v + E_v + I_v) \right)$$

$$+ \frac{1}{2} \left(D_1 u_1^2 + D_2 u_2^2 + D_1 u_3^2 \right)$$

is convex in the control set *U*. Also we can easily see that, there exists a constant $\sigma > 1$ and positive numbers ω_1 and ω_2 such that

$$J(u_1, u_2, u_3) \geq \omega_1 \left(|u_1|^2 + |u_2|^2 + |u_3|^2 \right)^{\frac{\sigma}{2}} - \omega_2,$$

which shows the existence of an optimal control problem. $\square$

To find the optimal solution to the control problem (4), we using the necessary conditions presented in [23,24] are given by

$$\frac{dx}{dt} = \frac{\partial H(t, x, u, \lambda)}{\partial \lambda},$$

$$0 = \frac{\partial H(t, x, u, \lambda)}{\partial u}, \qquad (10)$$

$$\frac{d\lambda}{dt} = -\frac{\partial H(t, x, u, \lambda)}{\partial x}.$$

Now we apply the necessary conditions to Hamiltonian (7), for our optimal solution.

Theorem 5.2: Suppose $S_h^*, E_h^*, I_h^*, R_h^*, S_v^*, E_v^*$ and I_v^* be the optimal state solutions with associated optimal control variables (u_1^*, u_2^*, u_3^*) for the optimal control problem (4), with the initial conditions (2). Then there exists adjoint variables λ_i, for $i = 1, 2, \cdots, 7$ satisfying

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$$\frac{d\lambda_2}{dt} = (\mu + \alpha_h)(1 - u_2)\lambda_2 - \alpha_h(1 - u_2)\lambda_3 - A_2,$$

$$\frac{d\lambda_3}{dt} = \lambda_3(\mu + \mu_h + \delta_h) - \lambda_4\delta_h + \beta_2 S_v(\lambda_5 - \lambda_6),$$

$$\frac{d\lambda_4}{dt} = (\lambda_4 - \lambda_1)\lambda_h + \lambda_4\mu,$$

$$\frac{d\lambda_5}{dt} = \lambda_5\delta_0 + \beta_2 I_h(\lambda_5 - \lambda_6) + \lambda_5 c_1 u_3 - A_3, \qquad (11)$$

$$\frac{d\lambda_6}{dt} = \lambda_6(\delta_0 + \alpha_v + c_1 u_3) - \alpha_v\lambda_7 - A_3,$$

$$\frac{d\lambda_7}{dt} = (\lambda_1 - \lambda_2)\beta_1 S_h(1 - u_1)$$
$$+ \lambda_7(\delta_0 + \delta_v + c_1 u_3) - A_3,$$

with transversality conditions (or boundary conditions)

$$\lambda_i(T_{end}) = 0, \qquad i = 1, 2, \cdots, 7. \qquad (12)$$

Furthermore, optimal controls u_1^*, u_2^* and u_3^* are given by

$$u_1^* = \max\left\{\min\left\{\frac{(\lambda_2 - \lambda_1)\beta_1 S_h^* I_v^* + \alpha_1 S_h^*(\lambda_2 - \lambda_1)}{D_1}, 1\right\}, 0\right\},$$
$$(13)$$

$$u_2^* = \max\left\{\min\left\{\frac{\alpha_h E_h^* \lambda_3 - \lambda_2(\mu + \alpha_h)E_h^*}{D_2}, 1\right\}, 0\right\}, \quad (14)$$

$$u_3^* = \max\left\{\min\left\{\frac{c_1\lambda_5 S_v^* + c_1\lambda_6 E_v^* + \lambda_7 c_1 I_v^*}{D_3}, 1\right\}, 0\right\}. \quad (15)$$

Proof: To prove the above result, *i.e.* the adjoint equation and the transversallity conditions, we use the Hamiltonian (7). The adjoint system was obtained by Pontryagin's Maximum Principle [23,24].

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S_h}, \cdots, \frac{d\lambda_7}{dt} = -\frac{\partial H}{\partial I_v}, \qquad (16)$$

with $\lambda_i(T) = 0$. To obtained the required characterization of the optimal control given by (13) to (15), solving equations,

$$\frac{\partial H}{\partial u_1} = 0, \frac{\partial H}{\partial u_2} = 0, \text{ and } \frac{\partial H}{\partial u_3} = 0 \qquad (17)$$

in the interior of the control set and by the control space U, we derive Equations (13) to (15).

6. Numerical Results

In this section, we present numerical simulations of the system (1) and the control system (4). We use forward Runge-Kutta order four schemes to solve both the system (1) and the control system (4). For the numerical solution of the adjoint system (11), we use backward Runge-Kutta

order four schemes because of the transversality conditions or boundary conditions (12). For numerical simulation we consider parameters value presented in **Table 1** using the MATLAB. Throughout this simulation we use the bold line for the system without control and the dashes line represents the control system.

Figure 6 shows the population of both the system of control and without control. The number of susceptible individuals increases in the control system than that of the system without control.

In **Figure 7** the plot shows the population of exposed human in both systems with and without control. The bold line shows the population of exposed individuals in the system of without control and the dashes line shows the population of exposed individuals in the system of with control.

Figure 8 shows the population of infected individuals in both the system with and without control.

Figure 9 represents the population of susceptible vector in both the system of with and without control.

Table 1. Parameters values in the numerical simulation.

Notation	Description	Value
δ_v	Disease death rate of vector	0.094
b_1	Human recruitment rate	2
b_2	Vector recruitment rate	3
β_1	Transmission rate between S_h and I_v	0.0074
β_2	Transmission between S_h and I_v	0.002
α_1	Transmission between S_h and I_h	0.001
λ_h	Proportionality constant	2.85×10^{-3}
α_h	The rate at which E_h moves to I_h	0.092
δ_h	Recovery rate from infection	0.0004
μ_h	Disease death rate of human	5×10^{-5}
μ	Natural death rate of human	9×10^{-4}
α_v	The rate at which E_v moves to I_v	0.005
δ_o	Natural death rate of vector	9×10^{-4}
δ_v	Disease death rate of vector	0.0094
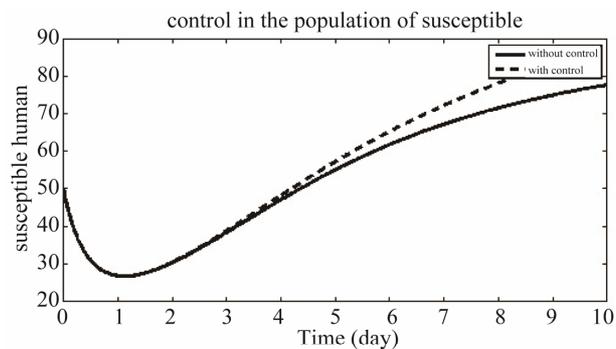
Figure 6. The plot represents the population of susceptible individuals.
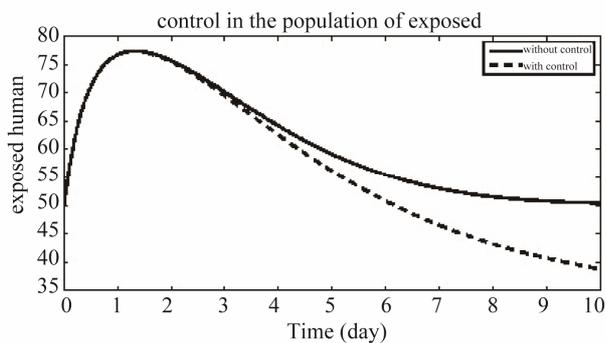
Figure 7. The plot represents the population of exposed individuals.
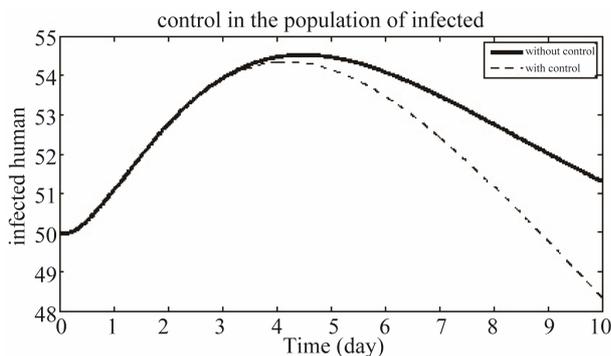
Figure 8. The plot represents the population of infected individuals.
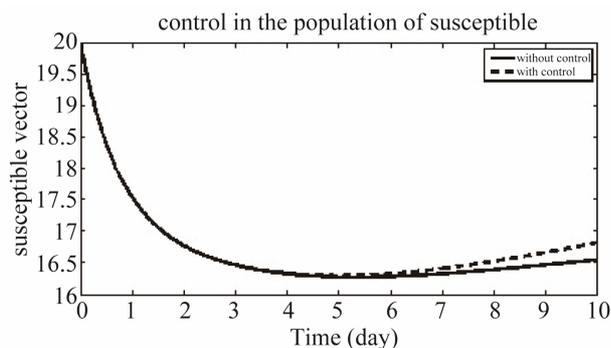
Figure 9. The plot represents the population of susceptible vector.

Figure 10 shows the population of exposed vector in both the system of with and without control system.

Figure 11 shows the population of infected vector in both the system of with and without control. Our numerical results show that the number of susceptible, exposed human and total vector population decrease and the number recovered human population increase which is the goal of this paper.

7. Conclusion

In this paper, we introduced optimal control campaign to
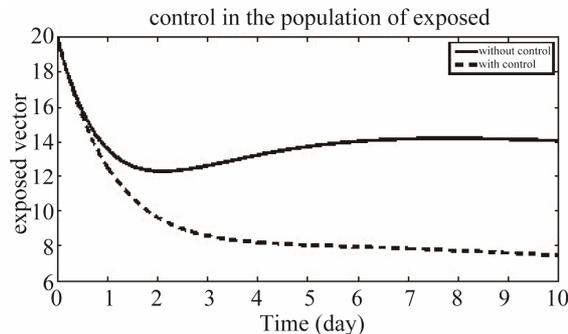
Figure 10. The plot represents the population of exposed vector.
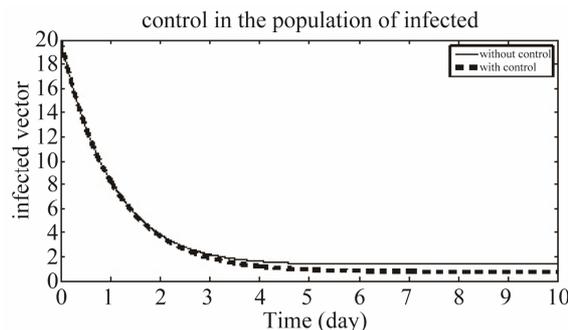
Figure 11. The plot represents the population of infected vector.

the model to maximize the population of recovered human and minimize the population of susceptible and infected human and total vector population using three control variables. First, we developed a mathematical model and then formulated the optimal control problem. We also investigated the endemic equilibria for both systems with and without control and presented their existence. We also found two basic reproductive numbers including the control variables. For certain values of these basic reproductive numbers some one can find that the disease spread in a community or not. We solved both system numerically and shown our work objective graphically.

8. Acknowledgements

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