

Green's Function Solution for the Dual-Phase-Lag Heat Equation

Reem Alkhairy

Mathematical Department, Dammam University, Dammam, KSA
Email: ralkhairy@ud.edu.sa, r_alkhairi@hotmail.com

Received July 26, 2012; revised August 28, 2012; accepted September 4, 2012

ABSTRACT

The present work is devoted to define a generalized Green's function solution for the dual-phase-lag model in homogeneous materials in a unified manner. The high-order mixed derivative with respect to space and time which reflect the lagging behavior is treated in special manner in the dual-phase-lag heat equation in order to construct a general solution applicable to wide range of dual-phase-lag heat transfer problems of general initial-boundary conditions using Green's function solution method. Also, the Green's function for a finite medium subjected to arbitrary heat source and arbitrary initial and boundary conditions is constructed. Finally, four examples of different physical situations are analyzed in order to illustrate the accuracy and potentialities of the proposed unified method. The obtained results show good agreement with works of [1-4].

Keywords: Dual-Phase-Lag Heat Conduction; Green's Function; Integral Transforms

1. Introduction

Recently the dual-phase-lag (DPL) heat conduction model has stimulated considerable interest in the heat transfer community, by offering alternative interpretations and new perspectives to a large body of non-Fourier thermal behaviors in energy transportation process under special considerations, such as heat conduction in biological materials, heat transport in amorphous media, layered-film heating in superconductors, fins and reactor walls. and many commonly used devices, such as personal computers or cellular phones. Needless to say, numerous efforts have been invested to the development of an explicit mathematical solution to the heat conduction equation under the DPL model. Most of these analytical solutions to the DPL heat conduction problems in the literature were formulated ad hoc, only applicable to specific formulations of initial-boundary conditions. Other than the notoriously annoying fictitious numerical oscillations frequently encountered in solving hyperbolic partial differential equations (HPDE), the intrinsic complexity of the DPL heat conduction equation alone (high-order mixed derivative with respect to space and time which dramatically alter the fundamental characteristics of the solution) poses a tremendous hindering obstacle against a general solution [5]. In the present work high-order mixed derivative with respect to space and time is treated in special manner in the dual-phase-lag heat equation in order to construct a general solution applicable to

wide range of dual-phase-lag heat transfer problems of general initial-boundary conditions using Green's function solution method.

The definition of Green's functions for a wave-type conduction equation and a general form of the Green's function solution method for finite bodies is introduced by Haji-Sheikh and Beck [6]. Loureiro *et al.* [7] studied the hyperbolic bioheat conduction equation using the explicit Green's approach method. The dual-phase-lag heat equation was used to generalize macroscopic model in treating the transient heat conduction in finite slabs irradiated by short pulse laser using Green's function method by [8,9]. For powerful reviewing of construction of several Green's functions for different boundary and initial condition of various physical equations, the reader is referred to [10].

The present work is devoted to define a generalized Green's function solution for the dual-phase-lag model in homogeneous materials. Also, the Green's function for a finite medium subjected to arbitrary heat source and arbitrary initial and boundary conditions is constructed. To examine the applicability of the present method, calculations are performed on four different previously solved researches [1-4]. The obtained results show good agreement with these researches.

2. The Dual-Phase-Lag Heat Equation

Let $\Omega \subset R^d$ be an open bounded domain with smooth

boundary $\Gamma = \partial\Omega$, where d is the number of space dimensions and let $I = [0, t_f]$ be the time domain with $t_f > 0$, the dual-phase lag model (DPL), given by Tzou [11], which allows either the temperature gradient (*cause*) to precede the heat flux vector (*effect*) or the heat flux vector (*cause*) to precede the temperature gradient (*effect*) in the transient process, can be represented, mathematically, by

$$\underline{q}(\underline{r}, t) + \tau_q \frac{\partial \underline{q}}{\partial t} = -K \left[\underline{\nabla} T(\underline{r}, t) + \tau_T \frac{\partial}{\partial t} (\underline{\nabla} T(\underline{r}, t)) \right] \quad (1)$$

in $\Omega \times I$

where $T(\underline{r}, t)$ and $\underline{q}(\underline{r}, t)$ are the temperature and heat flux distributions at position \underline{r} at time t , respectively. τ_q is the phase lag (relaxation time) of the heat flux vector, τ_T is the phase lag (relaxation time) of the temperature gradient, K is the thermal conductivity. Combining Equation (1) with the energy conservation law,

$$\rho c_p \frac{\partial T}{\partial t} = \underline{\nabla} \cdot \underline{q} + Q \quad \text{in } \Omega \times I \quad (2)$$

leads to the energy transport equation (the dual-phase-lag heat equation) in the form

$$\begin{aligned} \nabla^2 T + \tau_T \frac{\partial}{\partial t} [\nabla^2 T] + \frac{1}{K} \left[Q(\underline{r}, t) + \tau_q \frac{\partial Q(\underline{r}, t)}{\partial t} \right] \\ = \frac{1}{\alpha} \frac{\partial T}{\partial t} + \frac{\tau_q}{\alpha} \frac{\partial^2 T}{\partial t^2} \quad \text{in } \Omega \times I. \end{aligned} \quad (3)$$

where c_p is specific heat at constant pressure, ρ is the density, $Q(\underline{r}, t)$ is the heat generation per unit volume and α is the thermal diffusivity. The high-order mixed derivative with respect to space and time is dramatically alter the fundamental characteristics of the solution and completely destroys the wave structure resulting from the wavy term, the second-order derivative term with respect to time, and the energy equation is parabolic in nature. It predicts a higher temperature level in the heat-penetration zone than diffusion but does not have a sharp wavefront in heat propagation.

The smooth boundary Γ can be imposed on either prescribed temperature or prescribed heat flux. In addition to the prescribed boundary values, the initial condition on temperature may be also specified as below

$$T(\underline{r}, t)|_{t=0} = T(\underline{r}, 0) \quad \text{in } \Omega \quad (4)$$

while according to the conservation law (2), with the consideration that the initial value of the heat flux $\underline{q}(\underline{r}, 0) = 0$, the initial value of the time derivative of the temperature distribution may takes the form

$$\left. \frac{\partial T}{\partial t} \right|_{t=0} = \frac{1}{\rho c_p} Q(\underline{r}, 0) \quad \text{in } \Omega \quad (5)$$

3. Solution with Green's Function

The Green's functions are an important tool in solving partial differential equations since the solution of the problem subjected to any kind of initial conditions, boundary conditions and internal heat generation can be obtained through integral equations once the Green's function is known. The Green's function $G(\underline{r}, t | \underline{r}', t')$ for finite or semi-infinite medium of constant physical properties with arbitrary initial and boundary conditions which correspond to the dual-phase-lag heat conduction Equation (3) is defined as the solution of

$$\begin{aligned} \nabla^2 G + \tau_T \frac{\partial}{\partial t} [\nabla^2 G] \\ + \frac{1}{K} \left[\delta(\underline{r} - \underline{r}') \delta(t - t') + \tau_q \frac{\partial [\delta(\underline{r} - \underline{r}') \delta(t - t')]}{\partial t} \right] \\ = \frac{1}{\alpha} \frac{\partial G}{\partial t} + \frac{\tau_q}{\alpha} \frac{\partial^2 G}{\partial t^2}. \quad \text{in } \Omega \times I \end{aligned} \quad (6)$$

For convenience of subsequent analysis, the following dimensionless variables are defined

$$\underline{R} = \frac{\underline{r}}{2\sqrt{\alpha\tau_q}}, \quad \underline{R}' = \frac{\underline{r}'}{2\sqrt{\alpha\tau_q}}, \quad (7a)$$

$$\eta = \frac{t}{2\tau_q}, \quad \tau = \frac{t'}{2\tau_q}$$

$$B = \frac{\tau_T}{2\tau_q} \quad (7b)$$

$$\theta(\underline{R}, \eta) = \frac{T}{T_m}, \quad \psi(\underline{R}, \eta) = \frac{Q\tau_q}{\rho c_p T_m} \quad (7c)$$

Using the above dimensionless variables, Equations (3) and (6) are expressed as

$$\begin{aligned} \nabla_R^2 \left[\theta + B \frac{\partial \theta}{\partial \eta} \right] + 4\psi + 2 \frac{\partial \psi}{\partial \eta} \\ = 2 \frac{\partial \theta}{\partial \eta} + \frac{\partial^2 \theta}{\partial \eta^2}, \quad \text{in } \Omega \times I. \end{aligned} \quad (8)$$

$$\begin{aligned} \nabla_R^2 \left[G + B \frac{\partial G}{\partial \eta} \right] + 4\delta(\underline{R} - \underline{R}') \delta(\eta - \tau) \\ + 2 \frac{\partial}{\partial \eta} \left[\delta(\underline{R} - \underline{R}') \delta(\eta - \tau) \right] \\ = 2 \frac{\partial G}{\partial \eta} + \frac{\partial^2 G}{\partial \eta^2}, \quad \text{in } \Omega \times I. \end{aligned} \quad (9)$$

where δ is Dirac delta function. For convenience of algebra, Equation (9) can be reduced to a simpler form. To accomplish this task, one can define a Green's function $G = G_1 + G_2$ so that

$$\begin{aligned} \nabla_R^2 \left[G_1 + B \frac{\partial G_1}{\partial \eta} \right] + 4\delta(\underline{R} - \underline{R}') \delta(\eta - \tau) \\ = 2 \frac{\partial G_1}{\partial \eta} + \frac{\partial^2 G_1}{\partial \eta^2}, \quad \text{in } \Omega \times I. \end{aligned} \tag{10a}$$

$$\begin{aligned} \nabla_R^2 \left[G_2 + B \frac{\partial G_2}{\partial \eta} \right] + 2 \frac{\partial}{\partial \eta} \left[\delta(\underline{R} - \underline{R}') \delta(\eta - \tau) \right] \\ = 2 \frac{\partial G_2}{\partial \eta} + \frac{\partial^2 G_2}{\partial \eta^2}, \quad \text{in } \Omega \times I. \end{aligned} \tag{10b}$$

Examining the above two equations, one can hypothesize that $G_2 = \frac{1}{2} \frac{\partial G_1}{\partial \eta}$; this acceptable since both G_1 and G_2 have homogeneous boundary conditions and their initial conditions, including all time derivatives, are zero. To show this relation between G_1 and G_2 , simply substitute for G_2 in Equation (10b) and get

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial \eta} \left\{ \nabla_R^2 \left[G_1 + B \frac{\partial G_1}{\partial \eta} \right] \right\} \\ + 2 \frac{\partial}{\partial \eta} \left[\delta(\underline{R} - \underline{R}') \delta(\eta - \tau) \right] \\ = \frac{\partial^2 G_1}{\partial \eta^2} + \frac{1}{2} \frac{\partial^3 G_1}{\partial \eta^3}, \end{aligned} \tag{11}$$

that reduces to the equation,

$$\begin{aligned} \frac{\partial}{\partial \eta} \left\{ \nabla_R^2 \left[G_1 + B \frac{\partial G_1}{\partial \eta} \right] + 4\delta(\underline{R} - \underline{R}') \delta(\eta - \tau) \right\} \\ = \frac{\partial}{\partial \eta} \left\{ 2 \frac{\partial G_1}{\partial \eta} + \frac{\partial^2 G_1}{\partial \eta^2} \right\}, \quad \text{in } \Omega \times I. \end{aligned} \tag{12}$$

Notice that any function G_1 that satisfies Equation (10a) also satisfy Equation (10b). Therefore, instead of solving for G from Equation (9), it is sufficient to solve a simpler Equation (10a), and then utilize the relation

$$G = G_1 + G_2 = G_1 + \frac{1}{2} \frac{\partial G_1}{\partial \eta} \quad \text{in } \Omega \times I. \tag{13}$$

Changing the spatial variables in Equation (10a) to “prime” space and time from η to τ yields

$$\begin{aligned} \nabla_{R'}^2 \left[G_1 - B \frac{\partial G_1}{\partial \tau} \right] + 4\delta(\underline{R} - \underline{R}') \delta(\eta - \tau) \\ = -2 \frac{\partial G_1}{\partial \tau} + \frac{\partial^2 G_1}{\partial \tau^2}, \quad \text{in } \Omega' \times I. \end{aligned} \tag{14}$$

Moreover, the dual-phase-lag heat equation in (\underline{R}', τ) space is

$$\begin{aligned} \nabla_{R'}^2 \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] + 4\psi + 2 \frac{\partial \psi}{\partial \tau} \\ = 2 \frac{\partial \theta}{\partial \tau} + \frac{\partial^2 \theta}{\partial \tau^2}, \quad \text{in } \Omega' \times I. \end{aligned} \tag{15}$$

Multiplying Equation (15) by $\left[G_1 - B \frac{\partial G_1}{\partial \tau} \right]$ and Equation (14) by $\left[\theta + B \frac{\partial \theta}{\partial \tau} \right]$, then subtracting the results to produce equation

$$\begin{aligned} \left[G_1 - B \frac{\partial G_1}{\partial \tau} \right] \nabla_{R'}^2 \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \\ - \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \nabla_{R'}^2 \left[G_1 - B \frac{\partial G_1}{\partial \tau} \right] \\ + \left[G_1 - B \frac{\partial G_1}{\partial \tau} \right] \left[4\psi + 2 \frac{\partial \psi}{\partial \tau} \right] \\ - 4 \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \delta(\underline{R} - \underline{R}') \delta(\eta - \tau) \\ = \left[G_1 - B \frac{\partial G_1}{\partial \tau} \right] \left[2 \frac{\partial \theta}{\partial \tau} + \frac{\partial^2 \theta}{\partial \tau^2} \right] \\ - \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \left[-2 \frac{\partial G_1}{\partial \tau} + \frac{\partial^2 G_1}{\partial \tau^2} \right] \end{aligned} \tag{16}$$

Both sides of Equation (16) are integrated, \underline{R}' over volume Ω , and τ from 0 to $\eta + \varepsilon$, where ε is a small positive number. Then, following the application of the Green’s theorem and after letting ε to go to zero, one gets

$$\begin{aligned} 4 \int_0^{\eta+\varepsilon} \int_{\Omega} \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \delta(\underline{R} - \underline{R}') \delta(\eta - \tau) d\Omega' d\tau \\ = 4 \left[\theta(\underline{R}, \eta) + B \frac{\partial \theta(\underline{R}, \eta)}{\partial \eta} \right] \\ = \theta_{\text{source}} + \theta_{B,C} + \theta_{I,C} \end{aligned} \tag{17}$$

where the source contribution to the temperature distribution is given by

$$\begin{aligned} \theta_{\text{source}}(\underline{R}, \eta) \\ = \int_0^{\eta} \int_{\Omega} \left[4\psi + 2 \frac{\partial \psi}{\partial \tau} \right] \left[G_1 - B \frac{\partial G_1}{\partial \tau} \right] d\Omega' d\tau \\ = \int_0^{\eta} \int_{\Omega} \left[4\psi + 2 \frac{\partial \psi}{\partial \tau} \right] \left[G_1 + 2BG_2 \right] d\Omega' d\tau \end{aligned} \tag{18}$$

while the boundary conditions contribution to temperature distribution is

$$\begin{aligned}
 &\theta_{B,C}(\underline{R}, \eta) \\
 &= \int_0^\eta \int_\Omega \left\{ \left[G_1 - B \frac{\partial G_1}{\partial \tau} \right] \nabla_{R'}^2 \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \right. \\
 &\quad \left. - \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \nabla_{R'}^2 \left[G_1 - B \frac{\partial G_1}{\partial \tau} \right] \right\} d\Omega' d\tau \\
 &= \int_0^\eta \int_\Gamma \left\{ \left[G_1 + 2BG_2 \right] \frac{\partial}{\partial n'} \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \right. \\
 &\quad \left. - \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \frac{\partial}{\partial n'} \left[G_1 + 2BG_2 \right] \right\} d\Gamma' d\tau \tag{19} \\
 &= \int_0^\eta d\tau \int_\Gamma \left\{ \left[G_1 \frac{\partial \theta}{\partial n'} - \theta \frac{\partial G_1}{\partial n'} \right] + 2B \left[G_2 \frac{\partial \theta}{\partial n'} - \theta \frac{\partial G_2}{\partial n'} \right] \right. \\
 &\quad + B \left[G_1 \frac{\partial}{\partial n'} \left(\frac{\partial \theta}{\partial \tau} \right) - \frac{\partial \theta}{\partial \tau} \frac{\partial G_1}{\partial n'} \right] \\
 &\quad \left. + 2B^2 \left[G_2 \frac{\partial}{\partial n'} \left(\frac{\partial \theta}{\partial \tau} \right) - \frac{\partial \theta}{\partial \tau} \frac{\partial G_2}{\partial n'} \right] \right\} d\Gamma'
 \end{aligned}$$

where Γ is the boundary of the volume Ω and \underline{n} is the unit vector outward normal to the boundary Γ . Notice that due to the causality principle one has

$$G_1(\underline{R}, \eta | \underline{R}', \tau) = 0 \quad \text{and} \quad \frac{\partial G_1(\underline{R}, \eta | \underline{R}', \tau)}{\partial \eta} = 0 \quad \text{for} \quad \eta < \tau$$

and consequently, the initial conditions contribution to the temperature distribution is

$$\begin{aligned}
 &\theta_{I,C}(\underline{R}, \eta) \\
 &= \int_0^{\eta+\epsilon} \int_\Omega \left\{ \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \left[-2 \frac{\partial G_1}{\partial \tau} + \frac{\partial^2 G_1}{\partial \tau^2} \right] \right. \\
 &\quad \left. - \left[G_1 - B \frac{\partial G_1}{\partial \tau} \right] \left[2 \frac{\partial \theta}{\partial \tau} + \frac{\partial^2 \theta}{\partial \tau^2} \right] \right\} d\Omega' d\tau \\
 &= \int_0^{\eta+\epsilon} \int_\Omega \left\{ \left[\theta + B \frac{\partial \theta}{\partial \tau} \right] \left[-2 \frac{\partial G_1}{\partial \tau} + \frac{\partial^2 G_1}{\partial \tau^2} \right] \right. \tag{20} \\
 &\quad \left. - \left[G_1 + 2BG_2 \right] \left[2 \frac{\partial \theta}{\partial \tau} + \frac{\partial^2 \theta}{\partial \tau^2} \right] \right\} d\Omega' d\tau \\
 &= \int_\Omega \left\{ 2G(\underline{R}, \eta | \underline{R}', 0) \theta(\underline{R}', 0) \right. \\
 &\quad \left. + \left[G_1(\underline{R}, \eta | \underline{R}', 0) + 2BG_2(\underline{R}, \eta | \underline{R}', 0) \right] \frac{\partial \theta}{\partial \tau} \Big|_{\tau=0} \right\} d\Omega'
 \end{aligned}$$

Thus, the temperature distribution can be expressed as

$$\theta(\underline{R}, \eta) = \frac{\exp\left(\frac{-\eta}{B}\right)}{4B} \int_0^\eta \exp\left(\frac{\nu}{B}\right) \tag{21}$$

$$\left[\theta_{\text{source}}(\underline{R}, \nu) + \theta_{B,C}(\underline{R}, \nu) + \theta_{I,C}(\underline{R}, \nu) \right] d\nu, \quad B \neq 0$$

For the hyperbolic model, *i.e.* $B = 0$, the temperature

distribution can be expressed as

$$\begin{aligned}
 \theta(\underline{R}, \eta) &= \int_0^\eta \int_\Omega \left[4\psi + 2 \frac{\partial \psi}{\partial \tau} \right] G_1(\underline{R}, \eta | \underline{R}', \tau) d\Omega' d\tau \\
 &+ \int_0^\eta d\tau \int_\Gamma \left[G_1 \frac{\partial \theta}{\partial n'} - \theta \frac{\partial G_1}{\partial n'} \right] d\Gamma' \\
 &+ \int_\Omega \left\{ 2G(\underline{R}, \eta | \underline{R}', 0) \theta(\underline{R}', 0) + G_1(\underline{R}, \eta | \underline{R}', 0) \frac{\partial \theta}{\partial \tau} \Big|_{\tau=0} \right\} d\Omega' \tag{22}
 \end{aligned}$$

which coincide with that given by [6], with the consideration that the present formulae is dimensionless one.

4. Construction of Green's Function for Finite Medium

Green's function for finite medium can be derived by solving Equation (10a) for G_1 with homogeneous initial and boundary conditions. Applying a suitable finite transform to Equation (10a) using homogeneous boundary conditions either of Neuman or Dirrichlet kind or even radiation boundary conditions, yields to

$$\frac{d^2 \overline{G_1}}{d\eta^2} + A \frac{d\overline{G_1}}{d\eta} + \lambda_n^2 \overline{G_1} = 4f_n(\underline{R}') \delta(\eta - \tau) \tag{23}$$

where λ_n are the eigen values corresponding to the eigen functions f_n and

$$\overline{G_1}(n, \eta | \underline{R}', \tau) = \int_\Omega G_1(\underline{R}, \eta | \underline{R}', \tau) f_n(\underline{R}) d\Omega \tag{24a}$$

$$G_1(\underline{R}, \eta | \underline{R}', \tau) = \sum_{n=1}^\infty \frac{\overline{G_1}(n, \eta | \underline{R}', \tau)}{N_n} f_n(\underline{R}) \tag{24b}$$

$$\int_\Omega f_n(\underline{R}) f_m(\underline{R}) d\Omega = \begin{cases} 0 & \text{when } n \neq m \\ N_n & \text{when } n = m \end{cases} \tag{24c}$$

$$A = 2 + B\lambda_n^2 \tag{24d}$$

where N_n is the orthogonality constant.

Now, solving Equation (23) with homogeneous initial conditions and then using the inversion Formula (24b), the first component of the Green's function can be expressed as

$$\begin{aligned}
 &G_1(\underline{R}, \eta | \underline{R}', \tau) \\
 &= 8 \sum_{n=1}^\infty \frac{1}{N_n} K_1(n, \eta - \tau, B) f_n(\underline{R}) f_n(\underline{R}'). \tag{25a}
 \end{aligned}$$

where

$$K_1(n, \eta, B) = \exp\left(\frac{-A\eta}{2}\right) \frac{\sinh\left[\frac{\sqrt{A^2 - 4\lambda_n^2} \eta}{2}\right]}{\sqrt{A^2 - 4\lambda_n^2}} \tag{25b}$$

Then using equation $\left(G_2 = \frac{1}{2} \frac{\partial G_1}{\partial \eta}\right)$, the second component $G_2(\underline{R}, \eta | \underline{R}', \tau)$ of the Green's function can be expressed as

$$G_2(\underline{R}, \eta | \underline{R}', \tau) = 2 \sum_{n=1}^{\infty} \frac{1}{N_n} K_2(n, \eta - \tau, B) f_n(\underline{R}) f_n(\underline{R}') \tag{26a}$$

where

$$K_2(n, \eta, B) = \exp\left(\frac{-A\eta}{2}\right) \left[\cosh\left[\frac{\sqrt{A^2 - 4\lambda_n^2}}{2} \eta\right] - A \frac{\sinh\left[\frac{\sqrt{A^2 - 4\lambda_n^2}}{2} \eta\right]}{\sqrt{A^2 - 4\lambda_n^2}} \right] \tag{26b}$$

Thus the Green's function $G(\underline{R}, \eta | \underline{R}', \tau)$ can be written in the form

$$G(\underline{R}, \eta | \underline{R}', \tau) = G_1 + G_2 = 2 \sum_{n=1}^{\infty} \frac{1}{N_n} Ker(n, \eta - \tau, B) f_n(\underline{R}) f_n(\underline{R}') \tag{27a}$$

where

$$Ker(n, \eta, B) = \exp\left(\frac{-A\eta}{2}\right) \left[\cosh\left[\frac{\sqrt{A^2 - 4\lambda_n^2}}{2} \eta\right] + (4 - A) \frac{\sinh\left[\frac{\sqrt{A^2 - 4\lambda_n^2}}{2} \eta\right]}{\sqrt{A^2 - 4\lambda_n^2}} \right] \tag{27b}$$

Note that

$$Ker(0, \eta, B) = 1. \tag{27c}$$

Note that the above postulated Green's function can be modified to a semi-infinite medium by extending Ω to a semi-open domain and consequently the integral transform (24a) and its inversion (24b) should be modified.

5. Discussion

Since the lagging behavior is a special response to time, the consideration of one-dimensional problems in space is sufficient to illustrate its fundamental characteristics. In addition, from a mathematical point of view, the lagging behavior introduces the highest order differentials in the energy equation, reflected by the mixed-derivative and the wave term. These terms characterize the funda-

mental solutions of the energy equation employing the dual-phase-lag model. Consideration of multidimensional problem will not alter the qualitative behavior depicted by the one-dimensional problem.

With the objective of showing the applicability and generality of the given Green's function method to deal with any heat generation and any kind of initial and boundary conditions, three one-dimensional examples and one two dimensional example are discussed.

5.1. Example 1

In this example the overshooting phenomenon was investigated by M. Xu *et al.* [1]. The overshooting phenomenon is studied based on the one-dimensional dual-phase-lagg heat conduction model. The thermal wave interference is found to trigger the overshooting of temperature field. A condition for the occurrence of overshooting phenomenon is established for the one-dimensional dual-phase-lagging heat conduction in a finite medium. According to this condition, the overshooting phenomenon may occur in heat conduction across gold films with the thickness ranging from 4.8555 nm to 19.581 mm.

The purpose of this example is to show the method of determining the temperature distribution $\theta(X, \eta)$ in a slab using the one dimensional dual-phase-lag Green's function when its faces are imposed to constant boundary conditions by solving the system

$$\frac{\partial^2 \theta}{\partial X^2} + B \frac{\partial^3 \theta}{\partial X^2 \partial \eta} = 2 \frac{\partial \theta}{\partial \eta} + \frac{\partial^2 \theta}{\partial \eta^2}, \tag{28a}$$

$$0 \leq X \leq L,$$

$$\theta(0, \eta) = \theta(L, \eta) = 1 \tag{28b}$$

$$\theta(X, 0) = \frac{\partial \theta(X, 0)}{\partial \eta} = 0 \tag{28c}$$

Accordingly, the temperature distribution in terms of Green's function is given in the form

$$\theta(X, \eta) = \frac{\exp\left(\frac{-\eta}{B}\right)}{4B} \int_0^\eta \exp\left(\frac{\nu}{B}\right) \theta_{B-C}(X, \nu) d\nu \tag{29a}$$

where

$$\theta_{B-C}(X, \eta) = - \left[\frac{\partial G_1}{\partial X'} + 2B \frac{\partial G_2}{\partial X'} \right]_{X'=L} + \left[\frac{\partial G_1}{\partial X'} + 2B \frac{\partial G_2}{\partial X'} \right]_{X'=0} \tag{29b}$$

Using Green's functions from Equations (25a) and (26a) with recognizing that the eigen functions of this example are $f_n(X) = \sin(\lambda_n X)$ with eigen values

$\lambda_n = \frac{n\pi}{L}$, and the normalization constant $N_n = \frac{L}{2}$, the temperature distribution (29a) can be written as

$$\theta(X, \eta) = 1 - \frac{2}{L} \sum_{n=1}^{\infty} \Theta_n(\eta) (1 - (-1)^n) \sin\left(\frac{n\pi X}{L}\right) \quad (30a)$$

where

$$\Theta_n(\eta) = \frac{\exp\left(\frac{-A\eta}{2}\right)}{\lambda_n} \left[\cosh\left[\frac{\sqrt{A^2 - 4\lambda_n^2}}{2}\eta\right] + \frac{A \sinh\left[\frac{\sqrt{A^2 - 4\lambda_n^2}}{2}\eta\right]}{\sqrt{A^2 - 4\lambda_n^2}} \right] \quad (30b)$$

Equation (30b) is plotted using Mathematica program ver. 5. **Figure 1** shows the temperature distribution for a thin gold film where the averaging values of the two lag times over nominal range of temperature are $\tau_q = 10^{-11}s$ $\tau_r = 10^{-13}s$. thus $B = 4.7222E-2$, with dimensionless thickness $L = 1$ at dimensionless times $\eta = 0.2, 0.605, 0.8$.

The obtained results show good agreement with those depicted by [1] who solved this problem using separation of variables method.

5.2. Example 2

The objective of this example is to test the proposed Green's method using heat source and prescribed initial conditions with insulated boundaries. In this example the one dimensional dual-phase-lag heat equation in a thin film subjected to symmetrical time dependent laser heating is investigated by Alkhaury [2] by solving the system

$$\frac{\partial^2 \theta}{\partial X^2} + B \frac{\partial^3 \theta}{\partial \eta \partial X^2} + 4\psi + 2 \frac{\partial \psi}{\partial \eta} \quad (31a)$$

$$= 2 \frac{\partial \theta}{\partial \eta} + \frac{\partial^2 \theta}{\partial \eta^2},$$

$$\theta(X, 0) = 0, \quad (31b)$$

$$\frac{\partial \theta}{\partial \eta}(X, 0) = 2\psi(X, 0), \quad (31c)$$

$$\frac{\partial \theta}{\partial X}(0, \eta) = \frac{\partial \theta}{\partial X}(L, \eta) = 0. \quad (31d)$$

where

$$\psi(X, \eta) = \psi_l(X, \eta) + \psi_r(X, \eta), \quad (32a)$$

$$\psi_l(X, \eta) = \psi_0 R(\eta) \exp(-\beta X), \quad (32b)$$

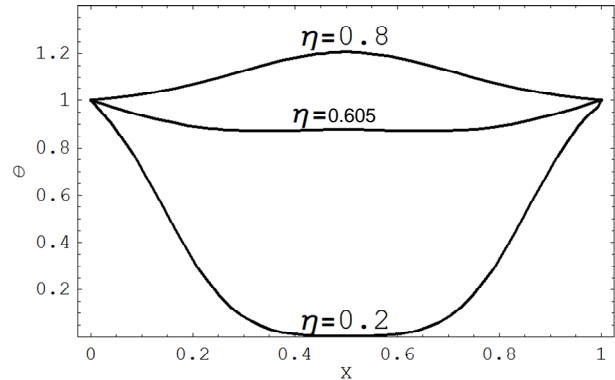


Figure 1. Variation of dimensionless temperature θ for a thin gold film of $B = 4:7222E-2$; with dimensionless thickness $L = 1$ at dimensionless times $\eta = 0.2, 0.605, 0.8$.

$$\psi_r(X, \eta) = \psi_0 R(\eta) \exp[-\beta(L - X)]. \quad (32c)$$

where $R(\eta)$ is the characteristic of the laser beam intensity, ψ_0 is the dimensionless capacity of internal heat source, β is the dimensionless absorption coefficient and the subscripts l, r refer to the left and right edges of the film, respectively. In our example, a light heat pulse is adopted, *i.e.*,

$$R(\eta) = \exp(-\sigma_1 \eta). \quad (33)$$

where $\sigma_1 = \frac{1}{\sigma}$, and σ is the laser heating duration.

Accordingly, the temperature distribution in terms of Green's function is given in the form

$$\theta(X, \eta) = \frac{\exp\left(\frac{-\eta}{B}\right)}{4B} \int_0^\eta \exp\left(\frac{\nu}{B}\right) \quad (34a)$$

$$[\theta_{\text{source}}(X, \nu) + \theta_{l,c}(X, \nu)] d\nu,$$

where

$$\theta_{\text{source}}(X, \eta) = \int_0^L \int_0^\eta \left[4\psi(X', \tau) + 2 \frac{\partial \psi(X', \tau)}{\partial \tau} \right] \quad (34b)$$

$$[G_1(X, \eta|X', \tau) + 2BG_2(X, \eta|X', \tau)] dX' d\tau$$

$$\theta_{l,c}(X, \eta)$$

$$= 2 \int_0^L [G_1(X, \eta|X', 0) + 2BG_2(X, \eta|X', 0)] \quad (34c)$$

$$\psi(X', 0) dX'$$

Using Green's functions from Equations (25a) and (26a) with recognizing that the eigen functions of this example are $f_n(X) = \cos(\lambda_n X) + \cos(\lambda_n(L - X))$ with eigen values $\lambda_n = \frac{n\pi}{L}$, and the normalization constant

$N_n = \frac{L}{2}$, the temperature distribution (34a) can be written as

$$\theta(X, \eta) = \frac{2\psi_0\beta}{L} \sum_{n=0}^{\infty} \frac{[2 - \delta_{0n}][1 - (-1)^n \exp(-\beta L)]}{(\beta^2 + \lambda_n^2)} \Theta_n(\eta) \quad (35a)$$

$$\times \{ \cos(\lambda_n X) + \cos(\lambda_n(L - X)) \}$$

where

$$\Theta_n(\eta) = C_1 \left\{ -\exp(-\sigma_1 \eta) + \exp\left(\frac{-A\eta}{2}\right) \right.$$

$$\left. \cdot \left[\cosh\left[\frac{\sqrt{A^2 - 4\lambda_n^2}}{2}\eta\right] + C_2 \frac{A \sinh\left[\frac{\sqrt{A^2 - 4\lambda_n^2}}{2}\eta\right]}{\sqrt{A^2 - 4\lambda_n^2}} \right] \right\} \quad (35b)$$

where

$$C_1 = \frac{\sigma_1 - 2}{2(\sigma_1^2 - A\sigma_1 + \lambda_n^2)} \quad (35c)$$

$$C_2 = \frac{2(2\sigma_1 + \lambda_n^2) - A(\sigma_1 + 2)}{\sigma_1 - 2} \quad (35d)$$

Figure 2 depict the results of calculations of Equation (35a) in a film of thickness $L = 1$ for light heat source of dimensionless laser heating duration $\sigma = 0.01$, and dimensionless absorption coefficient $\beta = 1$ at dimensionless time $\eta = 0.4$, for various dimensionless controlling coefficients $B = 0.0, 0.005, 0.05$.

With increasing B from zero, it is clear that the sharp wave fronts are smoothed and the portions of the disturbance are dissipated. The behavior of temperature response for $0 < B < 0.5$ is called wavelike behavior. **Figure 2** manifests that the wavelike behavior has smaller amplitude of temperature rise than the wavy one ($B = 0$) and the increase of B results attenuation of the amplitude but not any change in the wide of the portion of the thermal disturbance.

The obtained results show good agreement with those depicted by [2] who solved this problem using the integral transforms and the variation of parameters method.

5.3. Example 3

Example 2 was also investigated by [3], but for hyper-

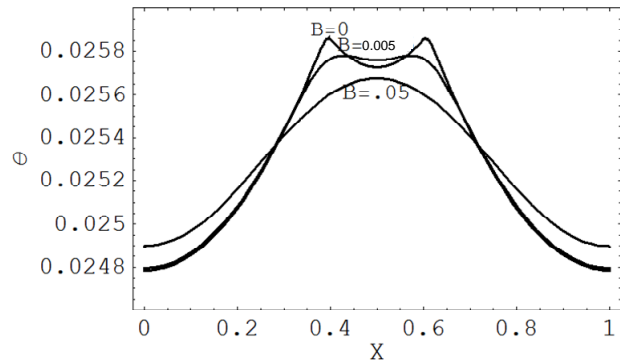


Figure 2. Variation of dimensionless temperature θ for a film of $L = 1$; $\eta = 0.4$; $\beta = 1$ for light heat source $R(\eta) = \exp\left(-\frac{\eta}{\sigma}\right)$ of dimensionless laser heating duration $\sigma = 0.01$.

bolic heat model *i.e.*, $B = 0$. For purpose of comparison, the present Green's method is applied to the corresponding hyperbolic system of Example 2 for instantaneous heat source whose time characteristic of the laser beam intensity $R(\eta)$ is given as

$$R(\eta) = \delta(\eta) \quad (36)$$

According, the temperature distribution in terms of Green's function is given in the form

$$\theta(X, \eta) = \int_0^{\eta} \int_0^L \left[4\psi + 2 \frac{\partial \psi}{\partial \tau} \right] G_1(X, \eta | X', \tau) dX' d\tau \quad (37)$$

$$+ \int_{\Omega} \left\{ G_1(X, \eta | X', 0) \frac{\partial \theta}{\partial \tau} \Big|_{\tau=0} \right\} dX'$$

Using integration by parts and the causality principle Equation (37) can be written as

$$\theta(X, \eta) = 4 \int_0^{\eta} \int_0^L \psi(X', \tau) G(X, \eta | X', \tau) dX' d\tau. \quad (38)$$

Using Green's functions from Equation (37) with recognizing that the eigen functions of this example are $f_n(X) = \cos(\lambda_n X) + \cos(\lambda_n(L - X))$ with eigen values $\lambda_n = \frac{n\pi}{L}$, and the normalization constant $N_n = \frac{L}{2}$, the temperature distribution (38) can be written as

$$\theta(X, \eta) = \frac{2\psi_0\beta}{L} \sum_{n=0}^{\infty} \frac{[2 - \delta_{0n}][1 - (-1)^n \exp(-\beta L)]}{(\beta^2 + \lambda_n^2)} \Theta_n(\eta) \quad (39)$$

$$\times \{ \cos(\lambda_n X) + \cos(\lambda_n(L - X)) \}$$

where

$$\Theta_n(\eta) = \text{Ker}(n, \eta, 0) \quad (40)$$

Figure 3 depict the results of calculations of Equation

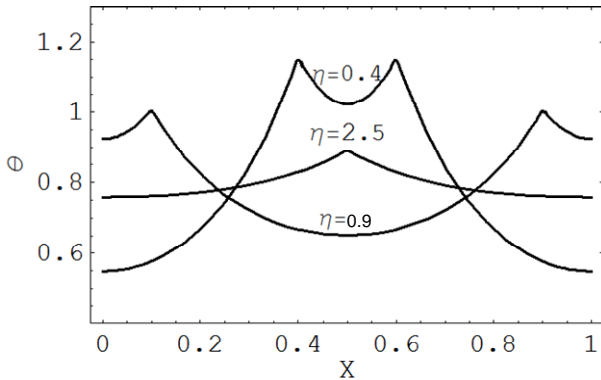


Figure 3. Variation of dimensionless temperature θ for a film of $L = 1$; $\beta = 5$ for instantaneous heat source $R(\eta) = \delta(\eta)$.

(39) for a film of thickness $L = 1$ with instantaneous heat source of dimensionless absorption coefficient $\beta = 5$, at various dimensionless times $\eta = 0.4, 0.9, 2.5$.

The obtained results show good agreement with those depicted by [3] who solved this problem using Laplace transforms method.

5.4. Example 4

In this example the two-dimensional dual-phase-lag (DPL) model of heat conduction was investigated numerically by [4] for treating the transient heat conduction problems in finite rigid medium under short-pulse-laser heating with Gaussian distributions in both temporal and spatial profiles by solving the system

$$\nabla^2 \theta + B \frac{\partial}{\partial \eta} (\nabla^2 \theta) = 2 \frac{\partial \theta}{\partial \eta} + \frac{\partial^2 \theta}{\partial \eta^2}, \quad (41)$$

$$\theta(X, Y, 0) = 0, \quad \frac{\partial \theta}{\partial \eta}(X, Y, 0) = 0, \quad (42a)$$

$$\frac{\partial \theta}{\partial X}(0, Y, \eta) = \frac{\partial \theta}{\partial X}(L_x, Y, \eta) = 0, \quad (42b)$$

$$\frac{\partial \theta}{\partial Y}\left(X, -\frac{L_y}{2}, \eta\right) = \frac{\partial \theta}{\partial Y}\left(X, \frac{L_y}{2}, \eta\right) = 0$$

$$Q_x(0, Y, \eta) = \frac{1}{\Delta y \eta p} \exp\left[-\left(\frac{Y^2}{\Delta y^2} + \frac{\eta^2}{\eta p^2}\right)\right] \quad (42c)$$

where $\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$ and $Q_x(0, Y, \eta)$ is the heat flux at the boundary $X = 0$. Accordingly, the temperature distribution in terms of Green's function is given in the form

$$\theta(X, Y, \eta) = \frac{1}{L_x L_y} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [2 - \delta_{0n}] [2 - \delta_{0m}] \Theta_{nm}(\eta) \cdot \cos\left(\frac{m\pi X}{L_x}\right) \cos\left(\frac{2n\pi Y}{L_y}\right) \quad (43)$$

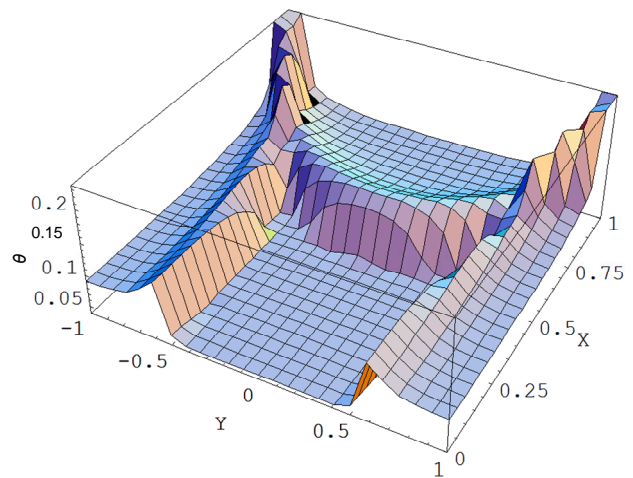


Figure 4. Variation of dimensionless temperature θ for a finite medium of dimensions $L_x = 1$; $L_y = 2$ irradiated by laser pulse with characteristic time $\eta p = 0.1$ and characteristic length $\Delta y = 0.1$ with controlling coefficient $B = 0.0$.

where

$$\Theta_{nm}(\eta) = \int_0^{\frac{\eta}{2}} \int_0^{\frac{\Delta y}{2}} Q_x(0, Y', \tau) [G_1 + 2BG_2] dY' d\tau. \quad (44)$$

Figure 4 depict the results of calculations of Equation (43) for a rectangular medium of dimensions $L_x = 1$, $L_y = 2$ irradiated by laser pulse with characteristic time $\eta p = 0.1$ and characteristic length $\Delta y = 0.1$ with controlling coefficient $B = 0.0$.

The obtained result show good agreement with that depicted by [4] who solved this problem numerically using finite-difference method method.

6. Conclusion

Hence, the purpose of the present paper is to describe the analytical solution of the dual-phase-lag heat equation in a unified manner by Green's function method. The high-order mixed derivative with respect to space and time which reflect the lagging behavior is treated in special manner in the dual-phase-lag heat equation in order to construct a general solution applicable to wide range of dual-phase-lag heat transfer problems of general initial-boundary conditions using Green's function solution method. Also, the Green's function for a finite medium subjected to arbitrary heat source and arbitrary initial and boundary conditions is constructed. Since the lagging behavior is a special response to time, the consideration of one-dimensional problems in space is sufficient to illustrate its fundamental characteristics. Therefore, three one dimensional examples and one two dimensional example of different physical situations are analyzed in order to illustrate the accuracy and potentialities of the

proposed unified method. The obtained results show good agreement with works of [1-4].

7. Acknowledgements

The author would like to thank the scientific deanship of University of Dammam for its generous support of this work through the project No. 2011082.

REFERENCES

- [1] M. Xu, J. Guo, L. Wang and L. Cheng, "Thermal Wave Interference as the Origin of the Overshooting Phenomenon in Dual-Phase-Lagging Heat Conduction," *International Journal of Thermal Sciences*, Vol. 50, No. 5, 2011, pp. 825-830. [doi:10.1016/j.ijthermalsci.2010.12.006](https://doi.org/10.1016/j.ijthermalsci.2010.12.006)
- [2] R. Al-Khairy, "Analytical Solution of Non-Fourier Temperature Response in a Finite Medium Symmetrically Heated on Both Sides," *Physics of Wave Phenomena*, Vol. 17, No. 4, 2009, pp. 277-285. [doi:10.3103/S1541308X09040049](https://doi.org/10.3103/S1541308X09040049)
- [3] M. Lewandowska and L. Malinowski, "An Analytical Solution of the Hyperbolic Heat Conduction Equation for the Case of a Finite Medium Symmetrically Heated on Both Sides," *International Communications in Heat and Mass Transfer*, Vol. 33, No. 1, 2006, pp. 61-69. [doi:10.1016/j.icheatmasstransfer.2005.08.004](https://doi.org/10.1016/j.icheatmasstransfer.2005.08.004)
- [4] P. Han, D. W. Tang and L. Zhou, "Numerical Analysis of Two Dimensional Lagging Thermal Behavior under Short-Pulse-Laser Heating on Surface," *International Journal of Engineering Science*, Vol. 44, No. 20, 2006, pp. 1510-1519. [doi:10.1016/j.jjengsci.2006.08.012](https://doi.org/10.1016/j.jjengsci.2006.08.012)
- [5] B. Shen and P. Zhang, "Notable Physical Anomalies Manifested in Non-Fourier Heat Conduction under the Dual-Phase-Lag Model," *International Journal of Heat and Mass Transfer*, Vol. 51, No. 7-8, 2008, pp. 1713-1727. [doi:10.1016/j.ijheatmasstransfer.2007.07.039](https://doi.org/10.1016/j.ijheatmasstransfer.2007.07.039)
- [6] A. Haji-Sheikh and J. V. Beck, "Green's Function Solution for Thermal Wave Equation in Finite Bodies," *International Journal of Heat and Mass Transfer*, Vol. 37, No. 17, 1994, pp. 2615-2626. [doi:10.1016/0017-9310\(94\)90379-4](https://doi.org/10.1016/0017-9310(94)90379-4)
- [7] F. Loureiroa, P. Oyarzúna, J. Santosa, W. Mansura and C. Vasconcellosb, "A Hybrid Time/Laplace Domain Method Base on Numerical Green's Functions Applied to Parabolic and Hyperbolic Bioheat Transfer Problems," *Mechanica Computational*, Vol. 29, 2010, pp. 5599-5611.
- [8] D. W. Tang and N. Araki, "Wavy, Wavelike, Diffusive Thermal Responses of Finite Rigid Slabs to High-Speed Heating of Laser-Pulses," *International Journal of Heat and Mass Transfer*, Vol. 42, No. 5, 1999, pp. 855-860. [doi:10.1016/S0017-9310\(98\)00244-0](https://doi.org/10.1016/S0017-9310(98)00244-0)
- [9] D. W. Tang and N. Araki, "Non-Fourier Heat Conduction Behavior in a Finite Mediums under Pulse Surface Heating," *Materials Science and Engineering*, Vol. A292, No. 2, 2000, pp. 173-178.
- [10] G. Duffy, "Green's Functions with Applications," Chapman & Hall, New York, 2001. [doi:10.1201/9781420034790](https://doi.org/10.1201/9781420034790)
- [11] D. Y. Tzou, "Macro-To Microscale Heat Transfer: The Lagging Behavior," Taylor and Francis, Bristol, 1990.