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On the k-Lucas Numbers of Arithmetic Indexes

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ABSTRACT

In this paper, we study the k-Lucas numbers of arithmetic indexes of the form $an + r$, where n is a natural number and r is less than r. We prove a formula for the sum of these numbers and particularly the sums of the first k-Lucas numbers, and then for the even and the odd k-Lucas numbers. Later, we find the generating function of these numbers. Below we prove these same formulas for the alternated k-Lucas numbers. Then, we prove a relation between the k-Fibonacci numbers of indexes of the form $2^r n$ and the k-Lucas numbers of indexes multiple of 4. Finally, we find a formula for the sum of the square of the k-Fibonacci even numbers by mean of the k-Lucas numbers.

Keywords: k-Fibonacci Numbers; k-Lucas Numbers; Generating Function

1. Introduction

Let us remember the k-Lucas numbers $L_{k,n}$ are defined [1] by the recurrence relation $L_{k,n+1} = k\,L_{k,n} + L_{k,n-1}$ with the initial conditions $L_{k,0} = 2,\ L_{k,1} = k$

Among other properties, the Binnet Identity establishes $L_{k,} = \sigma_1^n + \sigma_2^n$ being $\sigma_1 = \dfrac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \dfrac{k-\sqrt{k^2+4}}{2}$ the characteristic roots of the recurrence equation $r^2 - k\,r - 1 = 0$.

Evidently, $\sigma_1\sigma_2 = -1,\ \sigma_1+\sigma_2 = k,\ \sigma_1-\sigma_2 = \sqrt{k^2+4}$.

Moreover, it is verified [1, Theorem 2.4] that $L_{k,n} = F_{k,n-1} + F_{k,n+1}$.

If we apply iteratively the equation $L_{k,n+1} = k\,L_{k,n} + L_{k,n-1}$ then we will find a formula that relates the k–Lucas numbers to the k–Fibonacci numbers:

$$L_{k,n} = L_{k,n-(p-1)}F_{k,p} + L_{k,n-p}F_{k,p-1} \qquad (1.1)$$

This formula is similar to the Convolution formula for the k–Fibonacci numbers $F_{k,n+m} = F_{k,n+1}F_{k,m} + F_{k,n}F_{k,m-1}$ [2,3].

Moreover, we define $L_{k,-n} = (-1)^{n+1}L_{k,n}$. Then, if we do p = −n in Formula (1.1) obtain $L_{k,n} = (-1)^{n+1}\left(L_{k,2n+1}F_{k,n} - L_{k,2n}F_{k,n+1}\right)$.

2. On the k-Lucas Numbers of Arithmetic Index

We begin this section with a formula that relates each other some k-Lucas numbers.

2.1. Theorem 1 (The k-Lucas Numbers of Arithmetic Index)

If a is a nonnull natural number and $r = 0, 1, 2, \cdots a-1$, then

$$L_{k,a(n+1)+r} = L_{k,a}L_{k,an+r} - (-1)^a L_{k,a(n-1)+r} \qquad (2.1)$$

Proof. In [4] it is proved $F_{k,a(n+1)+r} = \left(F_{k,a-1}+F_{k,a+1}\right)F_{k,an+r} - (-1)^a F_{k,a(n-1)+r}$. Then

$$\begin{aligned} L_{k,a(n+1)+r} &= F_{k,a(n+1)+r-1} + F_{k,a(n+1)+r+1} \\ &= \left(F_{k,a-1}+F_{k,a+1}\right)F_{k,an+r-1} \\ &\quad -(-1)^a F_{k,a(n-1)+r-1} + \left(F_{k,a-1}+F_{k,a+1}\right)F_{k,an+r+1} \\ &\quad -(-1)^a F_{k,a(n-1)+r+1} \\ &= L_{k,a}L_{k,an+r} - (-1)^a L_{k,a(n-1)+r} \end{aligned}$$

If $r = 0$, then $L_{k,a(n+1)} = L_{k,a}L_{k,an} - (-1)^a L_{k,a(n-1)}$

In this case, if $a = 2p + 1$, then an odd k-Lucas number can be expressed in the form

$$L_{k,(2p+1)(n+1)} = L_{k,2p+1}L_{k,(2p+1)n} + L_{k,(2p+1)(n-1)}$$

Applying iteratively Formula (2.1), the general term, for $0 \le m \le n$, can be written like a non-linear combination of the form $L_{k,an+r} = \sum_{j=0}^{m}(-1)^{j(a+1)}\binom{m}{j}L_{k,a}^{m-j}L_{k,a(n-m-j)+r}$

In particular, if $m = n$, then

$$L_{k,an+r} = \sum_{j=0}^{n}(-1)^{j(a+1)}\binom{n}{j}L_{k,a}^{n-j}L_{k,r-aj}$$

2.2. Generating Function of the Sequence $\{L_{k,an+r}\}$

Let $l(k,x;a,r)$ be the generating function of the sequence $\{L_{k,an+r}\}$. That is,

$$l(k,x;a,r) = L_{k,r} + L_{k,a+r}x + L_{k,a+2r}x^2 + L_{k,a+3r}x^3 + \cdots$$

Then,

$$L_{k,a}x \cdot l(k,x;a,r) = L_{k,a}L_{k,r}x + L_{k,a}L_{k,a+r}x^2 + L_{k,a}L_{k,a+2r}x^3 + \cdots$$

and

$$(-1)^a x^2 l(k\ x\ a\ r) = (-1)^a L_{k,r}x^2 + (-1)^a L_{k,a+r}x^3 + \cdots$$

from where

$$l(k,x;a,r)\cdot\left(1 - L_{k,a}x + (-1)^a x^2\right) = L_{k,r} + \left(L_{k,a+r} - L_{k,a}\right)x$$

no more to take into account Formula (2.1). So, the generating function of the sequence $\{L_{k,an+r}\}$ is

$$l(k,x;a,r) = \frac{\left(L_{k,r} + L_{k,a+r} - L_{k,a}L_{k,r}\right)x}{1 - L_{k,a}x + (-1)^a x^2}.$$

As particular case, if $a = 1$, then $r = 0$ and the generating function of the k-Lucas sequence $\{L_{k,n}\}$ is $l(k,x) = \frac{2 - kx}{1 - kx - x^2}$, that, for the classical Lucas sequence is $l(x) = \frac{2 - x}{1 - x - x^2}$

If we want to take out the two bisection sequences of the classical Lucas sequence (k = 1), the respective generating functions are $a = 2$ and $r = 0$:

$l(x;2,0) = \frac{2 - 3x}{1 - 3x + x^2}$ that generates the sequence $\{L_{2n}\} = \{2,3,7,18,47,123,\cdots\}$

$a = 2$ and $r = 1$: $l(x;2,1) = \frac{1 + x}{1 - 3x + x^2}$ that generates the sequence $\{L_{2n+1}\} = \{1,4,11,29,76,199,\cdots\}$.

2.3. Theorem 2 (Sum of the k-Lucas Numbers of Arithmetic Index)

If a is a nonnull natural number and $r = 0, 1, 2, \cdots a - 1$, then

$$\sum_{j=0}^{n} L_{k,aj+r} = \frac{L_{k,a(n+1)+r} - (-1)^a L_{k,an+r} + (-1)^r L_{k,a-r} - L_{k,r}}{L_{k,a} - (-1)^a - 1} \tag{2.2}$$

Proof.

$$\sum_{j=0}^{n} L_{k,aj+r} = \sum_{j=0}^{n}\left(\sigma_1^{aj+r} + \sigma_2^{aj+r}\right) = \frac{\sigma_1^{an+r} - \sigma_1^r}{\sigma_1^a - 1} + \frac{\sigma_2^{an+r} - \sigma_2^r}{\sigma_2^a - 1}$$

$$= \frac{(-1)^a \sigma_1^{an+r} - \sigma_1^{a(n+1)+r} - \sigma_1^r\sigma_2^a + \sigma_1^r + (-1)^a \sigma_2^{an+r} - \sigma_2^{a(n+1)+r} - \sigma_2^r\sigma_1^a + \sigma_1^r}{(-1)^a - \sigma_1^a - \sigma_2^a + 1}$$

$$= \frac{(-1)^a L_{k,an+r} - L_{k,a(n+1)+r} - (-1)^r L_{k,a-r} + L_{k,r}}{-L_{k,a} + (-1)^a + 1}$$

because

$$\sigma_1^r\sigma_2^a + \sigma_1^a\sigma_2^r = \sigma_1^r\sigma_2^{a-r}\sigma_2^r + \sigma_1^{a-r}\sigma_1^r\sigma_2^r = (-1)^r\left(\sigma_2^{a-r} + \sigma_1^{a-r}\right)$$

and after applying the formula for the sum of a geometric progression.

2.4. Corollary 1 (Sum of Consecutive Odd k-Lucas Numbers)

If $r = 0$ and $a = 2p + 1$, Equation (2.2) is

$$\sum_{j=0}^{n} L_{k,(2p+1)j} = \frac{L_{k,(2p+1)(n+1)} + L_{k,(2p+1)n} - 2}{L_{k,2p+1}} + 1$$

In this case, the sum of the first k-Lucas numbers is (for p = 0),

$$\sum_{j=0}^{n} L_{k,j} = \frac{1}{k}\left(L_{k,n+1} + L_{k,n} - 2\right) + 1 \tag{2.3}$$

that for the classical Lucas numbers is $\sum_{j=0}^{n} L_j = L_{n+2} - 1$

2.5. Corollary 2 (Sum of Consecutive Even k-Lucas Numbers)

If $r = 0$ and $a = 2p$, then Equation (2.2) is

$$\sum_{j=0}^{n} L_{k,2pj} = \frac{L_{k,2p(n+1)} - L_{k,2pn}}{L_{2p} - 2} + 1 \tag{2.4}$$

In this case, if $p = 1$ we obtain the formula for the sum of the first even k-Lucas numbers $\sum_{j=0}^{n} L_{k,2j} = \frac{L_{k,2n+1}}{k} + 1$, and for the classical Lucas numbers is $\sum_{j=0}^{n} L_{2j} = L_{2n+1} + 1$

2.6. Theorem 3 (Sum of Alternated k-Lucas Numbers of Arithmetic Index)

For $a > 0$ and $r = 0, 1, 2, \ldots a-1$, the sum of alternated k-Lucas numbers is

$$\sum_{j=0}^{n} (-1)^j L_{k,aj+r} = \frac{(-1)^{a+n} L_{k,an+r} + (-1)^n L_{k,a(n+1)+r} + (-1)^n L_{k,a-r} + L_{k,r}}{L_{k,a} + (-1)^a + 1}$$

Proof. As in the previous theorem,

$$\sum_{j=0}^{n} (-1)^j L_{k,aj+r} = \sum_{j=0}^{n} (-1)^j \left(\sigma_1^{aj+r} + \sigma_2^{aj+r}\right)$$

$$= \frac{(-1)^n \sigma_1^{an+r}\left(-\sigma_1^a\right) - \sigma_1^r}{-\sigma_1^a - 1} + \frac{(-1)^n \sigma_2^{an+r}\left(-\sigma_2^a\right) - \sigma_2^r}{-\sigma_2^a - 1}$$

$$= \frac{(-1)^n (-1)^a \sigma_1^{an+r} + (-1)^n \sigma_1^{a(n+1)+r} + (-1)^r \sigma_2^{a-r} + \sigma_1^r + (-1)^n (-1)^a \sigma_2^{an+r} + (-1)^n \sigma_2^{a(n+1)+r} + (-1)\sigma_1^{a-r} + \sigma_2^r}{(-1)^a + \sigma_1^a + \sigma_2^a + 1}$$

$$= \frac{(-1)^a \left(\sigma_1^{an+r} + \sigma_2^{an+r}\right) + (-1)^n \left(\sigma_1^{a(n+1)+r} + \sigma_2^{a(n+1)+r}\right) + (-1)^r \left(\sigma_2^{a-r} + \sigma_1^{a-r}\right) + \sigma_1^r + \sigma_2^r}{L_{k,a} + (-1)^a + 1}$$

$$= \frac{(-1)^{a+n} L_{k,an+r} + (-1)^n L_{k,a(n+1)+r} + (-1)^n L_{k,a-r} + L_{k,r}}{L_{k,a} + (-1)^a + 1}$$

2.7. Corollary 3 (Sum of Consecutive Alternated Odd k-Lucas Numbers)

As particular case, if $a = 2p + 1$ and $r = 0$,

$$\sum_{j=0}^{n} (-1)^j L_{k,(2p+1)j} = (-1)^n \frac{L_{k,(2p+1)(n+1)} - L_{k,(2p+1)n} + 2(-1)^n}{L_{k,2p+1}} + 1$$

Then, for $p = 0$ we obtain the sum of the first alternated k-Lucas numbers

$\sum_{j=0}^{n} (-1)^j L_{k,2j+1} = (-1)^n \frac{L_{k,n+1} - L_{k,n} + 2(-1)^n}{k} + 1$, that for the classical Lucas numbers is

$$\sum_{j=0}^{n} (-1)^j L_{2j+1} = (-1)^n L_{k,n-1} + 3 .$$

2.8. Corollary 4 (Sum of Consecutive Alternated Even k-Lucas Numbers)

If $r = 0$ and $a = 2p + 1$, then

$$\sum_{j=0}^{n} (-1)^j L_{k,2pj} = (-1)^n \frac{L_{k,2p(n+1)} + L_{k,2pn} + L_{k,2p} + 2(-1)^n}{L_{k,2p} + 2}$$

And for the first consecutive alternated even k-Lucas numbers

$$\sum_{j=0}^{n} (-1)^j L_{k,2j} = (-1)^n \frac{L_{k,2(n+1)} + L_{k,2n} + 2(-1)^n + k^2 + 2}{k^2 + 4},$$

that for the classical Lucas numbers is

$$\sum_{j=0}^{n} (-1)^j L_{2j} = (-1)^n \frac{1}{5}\left(5F_{2n+1} + 3 + 2(-1)^n\right).$$

3. On the k-Fibonacci Numbers of Indexes n and the k-Lucas Numbers

In this section we will study a relation between the numbers $F_{k,2^r n}$ and $L_{k,n}$.

3.1. Theorem 4 (A Relation between Some k-Fibonacci and the k-Lucas Numbers)

For $r \geq 1$, it is

$$\frac{F_{k,2^r n}}{F_{k,n}} = \prod_{j=0}^{r-1} L_{k,2^j n} \qquad (3.1)$$

Proof.

$$\frac{F_{k,2^r n}}{F_{k,n}} = \frac{\sigma_1^{2^r n} - \sigma_2^{2^r n}}{\sigma_1^n - \sigma_2^n} = \left(\sigma_1^{2^{r-1} n} + \sigma_2^{2^{r-1} n}\right) \frac{\sigma_1^{2^{r-1} n} - \sigma_2^{2^{r-1} n}}{\sigma_1^n - \sigma_2^n}$$

$$= L_{k,2^{r-1} n} \left(\sigma_1^{2^{r-2} n} + \sigma_2^{2^{r-2} n}\right) \frac{\sigma_1^{2^{r-2} n} - \sigma_2^{2^{r-2} n}}{\sigma_1^n - \sigma_2^n}$$

$$= L_{k,2^{r-1} n} L_{k,2^{r-2} n} \cdots \frac{\sigma_1^{2n} - \sigma_2^{2n}}{\sigma_1^n - \sigma_2^n} = \prod_{j=0}^{r-1} L_{k,2^j n}$$

In particular, if $r = 1$, it is $\frac{F_{k,2n}}{F_{k,n}} = L_{k,n}$

Taking into account $L_{k,m} = \sigma_1^m + \sigma_2^m$, if we expand Formula (3.1), we find that this formula can be expressed as $\frac{F_{k,2^r n}}{F_{k,n}} = \prod_{j=0}^{2^{r-1}-1} (-1)^{(j+1)n} L_{k,2(2j+1)n}$ or, that is the same,

$$\frac{F_{k,2^r n}}{F_{k,n}} = \prod_{j=0}^{2^{r-2}-1} L_{k,2(4j+3)n} + (-1)^n \sum_{j=0}^{2^{r-1}-1} L_{k,(4j+1)n}$$

Then, applying Formula (2.2) to the second hand right of this equation with $n = 2^{r-2} - 1$, $a = 4n$, and $r = 3n$ for the first term and $r = n$ for the second,

$$\frac{F_{k,2^r n}}{F_{k,n}} = \frac{L_{k,(2^r+3)n} - L_{k,(2^r-1)n} + (-1)^n L_{k,n} - L_{k,3n}}{L_{k,4n} - 2}$$
$$+ (-1)^n \frac{L_{k,(2^r+1)n} - L_{k,(2^r-3)n} + (-1)^n L_{k,3n} - L_{k,n}}{L_{k,4n} - 2} \quad (3.2)$$
$$= \frac{L_{k,(2^r+3)n} - L_{k,(2^r-1)n} + (-1)^n \left(L_{k,(2^r+1)n} - L_{k,(2^r-3)n} \right)}{L_{k,4n} - 2}$$

We tray to simplify the second hand right of this equation. For that, we will prove the following Lemma.

3.2. Lemma 1

$$L_{k,(a+4)n} - L_{k,an} = \left(k^2+4\right) F_{k,(a+2)n} F_{k,2n} \quad (3)$$

Proof. We will apply the following formulas:

$$L_{k,p} = F_{k,p-1} + F_{k,p+1} \quad \text{(relation)}$$

$$F_{k,-p} = (-1)^{p+1} F_{k,p} \quad \text{(negative)}$$

$$F_{k,a+b} = F_{k,a} F_{k,b-1} + F_{k,a+1} F_{k,b} \quad \text{(convolution)}$$

$$F_{k,p+1} = k F_{k,p} + F_{k,p-1} \quad \text{(definition)}$$

Then:

$$L_{k,(a+4)n} - L_{k,an}$$
$$= F_{k,(a+4)n-1} + F_{k,(a+4)n+1} - F_{k,an-1} - F_{k,an+1} \quad \text{(by relation)}$$
$$= F_{k,(a+2)n+(2n-1)} + F_{k,(a+2)n+(2n+1)}$$
$$- F_{k,(a+2)n+(-2n-1)} - F_{k,(a+2)n+(-2n+1)}$$
$$= F_{k,(a+2)n} F_{k,2n-2} + F_{k,(a+2)n+1} F_{k,2n-1}$$
$$+ F_{k,(a+2)n} F_{k,2n} + F_{k,(a+2)n+1} F_{k,2n+1} \quad \text{(by convolution)}$$
$$- F_{k,(a+2)n} F_{k,-2n-2} - F_{k,(a+2)n+1} F_{k,-2n-1}$$
$$- F_{k,(a+2)n} F_{k,-2n} - F_{k,(a+2)n+1} F_{k,-2n+1}$$
$$= F_{k,(a+2)n} F_{k,2n-2} + F_{k,(a+2)n+1} F_{k,2n-1}$$
$$+ F_{k,(a+2)n} F_{k,2n} + F_{k,(a+2)n+1} F_{k,2n+1} \quad \text{(by negative)}$$
$$+ F_{k,(a+2)n} F_{k,2n+2} - F_{k,(a+2)n+1} F_{k,2n+1}$$
$$+ F_{k,(a+2)n} F_{k,2n} - F_{k,(a+2)n+1} F_{k,2n-1}$$
$$= F_{k,(a+2)n} \left(F_{k,2n} - k F_{k,2n-1} + F_{k,2n} + k F_{k,2n+1} + F_{k,2n} \right)$$
$$= F_{k,(a+2)n} \left(k^2+4\right) F_{k,2n} \quad \text{(by definition)}$$

And applying this Lemma to Equation (3.2), we will have:

$$\frac{F_{k,2^r n}}{F_{k,n}} = \frac{\left(k^2+4\right) F_{k,(2^r+1)n} F_{k,2n} + (-1)^n \left(k^2+4\right) F_{k,(2^r-1)n} F_{k,2n}}{L_{k,4n} - 2},$$

that is

$$\frac{F_{k,2^r n}}{F_{k,n}} = \frac{\left(k^2+4\right) F_{k,2n}}{L_{k,4n} - 2} \left(F_{k,(2^r+1)n} + (-1)^n F_{k,(2^r-1)n} \right)$$

from where

$$\frac{F_{k,(2^r+1)n} + (-1)^n F_{k,(2^r-1)n}}{F_{k,2^r n}} = \frac{L_{k,4n} - 2}{\left(k^2+4\right) F_{k,n} F_{k,2n}}$$
$$= \frac{L_{k,4n} - 2}{\left(\sigma_1^n - \sigma_2^n\right)\left(\sigma_1^{2n} - \sigma_2^{2n}\right)}$$
$$= \frac{L_{k,4n} - 2}{\left(\sigma_1^{3n} + \sigma_2^{3n}\right) - (-1)^n \left(\sigma_1^n + \sigma_2^n\right)} = \frac{L_{k,4n} - 2}{L_{k,3n} - (-1)^n L_{k,n}}$$

If in Equation (3.3) it is a = 0, then it is $F_{k,2n}^2 = \frac{L_{k,4n} - 2}{k^2+4}$, and applying the Formulas (2.5) and (2.4),

$$\sum_{j=0}^{n} F_{k,2j}^2 = \frac{1}{k^2+4} \sum_{j=0}^{n} \left(L_{k,4j} - 2 \right)$$
$$= \frac{1}{k^2+4} \left(\frac{L_{k,4(n+1)} - L_{k,4n}}{L_{k,4} - 2} + 1 - 2(n+1) \right)$$

That is

$$\sum_{j=0}^{n} F_{k,2j}^2 = \frac{1}{k^2+4} \left(\frac{k L_{k,4n+3} + L_{k,4n+1}}{k\left(k^2+4\right)} - 2n - 1 \right)$$

In particular, for the classical Lucas numbers (k = 1), it is $\sum_{j=0}^{n} F_{2j}^2 = \frac{1}{5} \left(\frac{L_{4n+3} + L_{4n+1}}{5} - 2n - 1 \right)$.

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