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Explicit Inversion for Two Brownian-Type Matrices

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ABSTRACT

We present explicit inverses of two Brownian-type matrices, which are defined as Hadamard products of certain already known matrices. The matrices under consideration are defined by $3n - 1$ parameters and their lower Hessenberg form inverses are expressed analytically in terms of these parameters. Such matrices are useful in the theory of digital signal processing and in testing matrix inversion algorithms.

Keywords: Brownian Matrix; Hadamard Product; Hessenberg Matrix; Numerical Complexity; Test Matrix

1. Introduction

Brownian matrices are frequently involved in problems concerning "digital signal processing". In particular, Brownian motion is one of the most common linear models used for representing nonstationary signals. The covariance matrix of a discrete-time Brownian motion has, in turn, a very characteristic structure, the so-called "Brownian matrix".

In [1] (Equation (2)) the explicit inverse of a class of matrices $G_n = \left[\beta_{ij}\right]$ with elements

$$\beta_{ij} = \begin{cases} b_j, & i \le j, \\ a_j, & i > j. \end{cases} \tag{1}$$

is given. On the other hand, the analytic expressions of the inverses of two symmetric matrices $K = \left[\kappa_{ij}\right]$ and $N = \left[\nu_{ij}\right]$, where

$$\kappa_{ij} = k_i \text{ and } \nu_{ij} = k_j, i \le j, \tag{2}$$

respectively, are presented in [2] (first equation in p. 113, and Equation (1), respectively). The matrix K is a special case of Brownian matrix and G_n is a lower Brownian matrix, as they have been defined in [3] (Equation (2.1)). Earlier, in [4] (paragraph following Equation (3.3)) the term "pure Brownian matrix" for the type of the matrix K has introduced. Furthermore, in [5] (discussion concerning Equations (28)-(30)) the so-called "diagonal innovation matrices" (DIM) have been treated, special cases of which are the matrices K and N.

In the present paper, we consider two matrices A_1 and A_2 defined by

$$A_1 = K \circ G_n \text{ and } A_2 = N \circ G_n, \tag{3}$$

where the symbol $\circ$ denotes the Hadamard product. Hence, the matrices have the forms

$$A_1 = \begin{bmatrix} k_1 b_1 & k_1 b_2 & k_1 b_3 & \cdots & k_1 b_{n-1} & k_1 b_n \\ k_1 a_1 & k_2 b_2 & k_2 b_3 & \cdots & k_2 b_{n-1} & k_2 b_n \\ k_1 a_1 & k_2 a_2 & k_3 b_3 & \cdots & k_3 b_{n-1} & k_3 b_n \\ \cdots & & & & & \\ k_1 a_1 & k_2 a_2 & k_3 a_3 & \cdots & k_{n-1} b_{n-1} & k_{n-1} b_n \\ k_1 a_1 & k_2 a_2 & k_3 a_3 & \cdots & k_{n-1} a_{n-1} & k_n b_n \end{bmatrix} \tag{4}$$

and

$$A_2 = \begin{bmatrix} k_1 b_1 & k_2 b_2 & k_3 b_3 & \cdots & k_{n-1} b_{n-1} & k_n b_n \\ k_2 a_1 & k_2 b_2 & k_3 b_3 & \cdots & k_{n-1} b_{n-1} & k_n b_n \\ k_3 a_1 & k_3 a_2 & k_3 b_3 & \cdots & k_{n-1} b_{n-1} & k_n b_n \\ \cdots & & & & & \\ k_{n-1} a_1 & k_{n-1} a_2 & k_{n-1} a_3 & \cdots & k_{n-1} b_{n-1} & k_n b_n \\ k_n a_1 & k_n a_2 & k_n a_3 & \cdots & k_n a_{n-1} & k_n b_n \end{bmatrix}. \tag{5}$$

Let us now define for a matrix $B = \left[b_{ij}\right]$ the terms "pure upper Brownian matrix" and "pure lower Brownian matrix", for the elements of which the following relations are respectively valid

$$b_{i,j+1} = b_{ij}, i \le j, \text{ and } b_{i+1,j} = b_{ij}, i \ge j. \tag{6}$$

The matrix A_1 (Equation (4)) is a lower Brownian matrix. Furthermore, the matrix PNP, where $P = \left[p_{ij}\right]$ is the permutation matrix with elements

$$p_{ij}=\begin{cases}1, & i+j=n+1,\\ 0, & \text{otherwise},\end{cases} \quad (7)$$

is a pure Brownian matrix and PG_nP a pure lower Brownian matrix. Hence, their Hadamard product $(PNP)\circ(PG_nP)$ gives a pure lower Brownian matrix, that is, the matrix PA_2P.

In the following sections, we deduce in analytic form the inverses and determinants of the matrices A_1 and A_2; and we study the numerical complexity on evaluating A_1^{-1} and A_2^{-1}.

2. The Inverse and Determinant of A_1

The inverse of A_1 is a lower Hessenberg matrix expressed analytically by the $3n-1$ parameters defining A_1. In particular, the inverse $A_1^{-1}=\left[\alpha_{ij}\right]$ has elements given by the relations

$$\alpha_{ij}=\begin{cases}\dfrac{k_{i+1}b_{i-1}-k_{i-1}a_{i-1}}{c_{i-1}c_i}, & i=j\neq 1,n,\\[2mm] \dfrac{k_2}{k_1c_1}, & i=j=1,\\[2mm] \dfrac{b_{n-1}}{c_{n-1}c_n}, & i=j=n,\\[2mm] (-1)^{i+j}\dfrac{d_{j-1}g_i\prod\limits_{\nu=j+1}^{i-1}k_\nu f_\nu}{\prod\limits_{\nu=j-1}^{i}c_\nu}, & i-j\geq 1,\\[2mm] -\dfrac{1}{c_i}, & j-i=1,\\[2mm] 0, & j-i>1,\end{cases} \quad (8)$$

where

$$\begin{cases}c_i=k_{i+1}b_i-k_ia_i, & i=1,2,\cdots,n-1, \quad c_0=1, \quad c_n=b_n,\\ d_i=k_{i+1}a_{i+1}b_i-k_ia_ib_{i+1}, & i=1,2,\cdots,n-2, \quad d_0=a_1,\\ f_i=a_i-b_i, & i=2,3,\cdots,n-1,\\ g_i=k_{i+1}-k_i, & i=2,3,\cdots,n-1, \quad g_n=1,\end{cases} \quad (9)$$

with

$$\prod_{\nu=j+1}^{i-1}k_\nu f_\nu=1 \text{ if } i=j+1, \quad (10)$$

and with the obvious assumptions

$$k_1\neq 0 \text{ and } c_i\neq 0, i=1,2,\cdots,n. \quad (11)$$

To prove that the relations (8)-(10) give the inverse matrix A_1^{-1}, we reduce A_1 to the identity matrix I by applying a number of elementary row transformations. Then the product of the corresponding elementary matrices gives the inverse matrix of A_1. These transformations are defined by the following sequence of row operations.

Operation 1 (applied on A_1 and on the identity matrix I):

$$\text{row } i-\frac{k_i}{k_{i+1}}\times\text{row }(i+1), \; i=1,2,\cdots,n-1,$$

which transforms A_1 into the lower triangular matrix C_1 given by

$$\begin{bmatrix}\frac{k_1(k_2b_1-k_1a_1)}{k_2} & 0 & 0 & \cdots & 0 & 0\\ \frac{k_1a_1(k_3-k_2)}{k_3} & \frac{k_2(k_3b_2-k_2a_2)}{k_3} & 0 & \cdots & 0 & 0\\ \frac{k_1a_1(k_4-k_3)}{k_4} & \frac{k_2a_2(k_4-k_3)}{k_4} & \frac{k_3(k_4b_3-k_3a_3)}{k_4} & \cdots & 0 & 0\\ \cdots & & & & & \\ \frac{k_1a_1(k_n-k_{n-1})}{k_n} & \frac{k_2a_2(k_n-k_{n-1})}{k_n} & \frac{k_3a_3(k_n-k_{n-1})}{k_n} & \cdots & \frac{k_{n-1}(k_nb_{n-1}-k_{n-1}a_{n-1})}{k_n} & 0\\ k_1a_1 & k_2a_2 & k_3a_3 & \cdots & k_{n-1}a_{n-1} & k_nb_n\end{bmatrix},$$

and the identity matrix I into the upper bidiagonal matrix F_1 with main diagonal

$$(1,1,\cdots,1)$$

and upper first diagonal

$$\left(-\frac{k_1}{k_2},-\frac{k_2}{k_3},\cdots,-\frac{k_{n-1}}{k_n}\right).$$

Operation 2 (applied on C_1 and F_1):

$$\text{row } i - \frac{k_i g_i}{k_{i+1} g_{i-1}} \times \text{row } (i-1),\ i = n, n-1, \cdots, 3,\ k_{n+1} = 1,$$

which derives a lower bidiagonal matrix C_2 with main diagonal

$$\left(\frac{k_1 c_1}{k_2}, \frac{k_2 c_2}{k_3}, \cdots, \frac{k_{n-1} c_{n-1}}{k_n}, k_n c_n\right)$$

and lower first diagonal

$$\left(\frac{k_1 a_1 g_2}{k_3}, \frac{k_2 k_3 g_3 f_2}{k_4 g_2}, \cdots, \frac{k_{n-2} k_{n-1} g_{n-1} f_{n-2}}{k_n g_{n-2}}, \frac{k_{n-1} k_n f_{n-1}}{g_{n-1}}\right);$$

while the matrix F_1 is transformed into the tridiagonal matrix F_2 given by

$$\begin{bmatrix} 1 & -\frac{k_1}{k_2} & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\frac{k_2}{k_3} & \cdots & 0 & 0 \\ 0 & -\frac{k_3 g_3}{k_4 g_2} & 1+\frac{k_2 g_3}{k_4 g_2} & \cdots & 0 & 0 \\ \cdots \\ 0 & 0 & 0 & \cdots & 1+\frac{k_{n-2} g_{n-1}}{k_n g_{n-2}} & -\frac{k_{n-1}}{k_n} \\ 0 & 0 & 0 & \cdots & -\frac{k_n}{g_{n-1}} & 1+\frac{k_{n-1}}{g_{n-1}} \end{bmatrix}.$$

Operation 3 (applied on C_2 and F_2):

$$\text{row } 2 - \frac{k_2 a_1 g_2}{k_3 c_1} \times \text{row } 1 \text{ and}$$

$$\text{row } i - k_i \frac{k_i g_i f_{i-1}}{k_{i+1} g_{i-1} c_{i-1}} \times \text{row } (i-1),\ i = 3, 4, \cdots, n,$$

which derives the diagonal matrix

$$C_3 = \left\lceil \frac{k_1 c_1}{k_2} \quad \frac{k_2 c_2}{k_3} \quad \cdots \quad \frac{k_{n-1} c_{n-1}}{k_n} \quad k_n c_n \right\rfloor,$$

and, respectively, the lower Hessenberg matrix F_3 given by

$$\begin{bmatrix} 1 & -\frac{k_1}{k_2} & \cdots & 0 & 0 \\ -\frac{k_2 a_1 g_2}{k_3 c_0 c_1} & \frac{k_2 (k_3 b_1 - k_1 a_1)}{k_3 c_1} & \cdots & 0 & 0 \\ \frac{k_3 a_1 g_3 k_2 f_2}{k_4 c_0 c_1 c_2} & -\frac{k_3 d_1 g_3}{k_4 c_1 c_2} & \cdots & 0 & 0 \\ \cdots \\ \frac{s k_{n-1} a_1 g_{n-1} k_2 f_2 \cdots k_{n-2} f_{n-2}}{k_n c_0 c_1 \cdots c_{n-2}} & \frac{s k_{n-1} d_1 g_{n-1} k_3 f_3 \cdots k_{n-2} f_{n-2}}{k_n c_1 c_2 \cdots c_{n-2}} & \cdots & \frac{k_{n-1} (k_n b_{n-2} - k_{n-2} a_{n-2})}{k_n c_{n-2}} & -\frac{k_{n-1}}{k_n} \\ \frac{s k_n a_1 g_n k_2 f_2 \cdots k_{n-1} f_{n-1}}{c_0 c_1 \cdots c_{n-1}} & \frac{s k_n d_1 g_n k_3 f_3 \cdots k_{n-1} f_{n-1}}{c_1 c_2 \cdots c_{n-1}} & \cdots & -\frac{k_n d_{n-2}}{c_{n-2} c_{n-1}} & \frac{k_n b_{n-1}}{c_{n-1}} \end{bmatrix},$$

with the symbol s standing for the quantity $(-1)^{i+j}$.

Operation 4 (applied on C_3 and F_3):

$$\frac{k_{i+1}}{k_i c_i} \times \text{row } i,\ i = 1, 2, \cdots, n,$$

which transforms C_3 into the identity matrix I and the matrix F_3 into the inverse A_1^{-1}.

The determinant of A_1 takes the form

$$\det(A_1) = k_1 b_n (k_2 b_1 - k_1 a_1)(k_3 b_2 - k_2 a_2) \cdots (k_n b_{n-1} - k_{n-1} a_{n-1}). \quad (12)$$

Evidently, A_1 is singular if $k_1 = 0$ or, considering the relation (9), if $c_i = 0$ for some $i \in \{1, 2, \cdots, n\}$.

3. The Inverse and Determinant of A_2

In the case of A_2, its inverse $A_2^{-1} = [\alpha_{ij}]$ is a lower

Hessenberg matrix with elements given by the relations

$$\alpha_{ij}=\begin{cases}\dfrac{k_{i-1}b_{i-1}-k_{i+1}a_{i-1}}{c_{i-1}c_i}, & i=j\neq 1,n,\\ \dfrac{1}{c_1}, & i=j=1,\\ \dfrac{k_{n-1}b_{n-1}}{k_n c_{n-1}c_n}, & i=j=n,\\ (-1)^{i+j}\dfrac{d_{j-1}g_i\prod_{\nu=j+1}^{i-1}k_\nu f_\nu}{\prod_{\nu=j-1}^{i}c_\nu}, & i-j\geq 1,\\ -\dfrac{1}{c_i}, & j-i=1,\\ 0, & j-i>1,\end{cases} \tag{13}$$

where

$$\begin{cases}c_i=k_ib_i-k_{i+1}a_i, & i=1,2,\cdots,n-1,\ c_0=1,\quad c_n=b_n,\\ d_i=k_ia_{i+1}b_i-k_{i+1}a_ib_{i+1}, & i=1,2,\cdots,n-2,\ d_0=a_1,\\ f_i=a_i-b_i, & i=2,3,\cdots,n-1,\\ g_i=k_i-k_{i+1}, & i=2,3,\cdots,n-1,\ g_n=1,\end{cases} \tag{14}$$

with

$$\prod_{\nu=j+1}^{i-1}k_\nu f_\nu=1 \text{ if } i=j+1, \tag{15}$$

and with the obvious assumptions

$$k_n\neq 0 \text{ and } c_i\neq 0, i=1,2,\cdots,n. \tag{16}$$

In order to prove that the relations (13)-(15) give the inverse matrix A_2^{-1}, we follow a similar manner to that of Section 2.

Operation 1 (applied on A_2 and on the identity matrix I):

$$\text{row } i-\text{row }(i+1), i=1,2,\cdots,n-1,$$

which transforms A_2 into the lower triangular matrix D_1 equal to

$$\begin{bmatrix} k_1b_1-k_2a_1 & 0 & \cdots & 0 & 0\\ a_1(k_2-k_3) & k_2b_2-k_3a_2 & \cdots & 0 & 0\\ \cdots & & & & \\ a_1(k_{n-1}-k_n) & a_2(k_{n-1}-k_n) & \cdots & k_{n-1}b_{n-1}-k_na_{n-1} & 0\\ k_na_1 & k_na_2 & \cdots & k_na_{n-1} & k_nb_n\end{bmatrix},$$

and the identity matrix I into the bidiagonal matrix L_1 with main diagonal

$$(1,1,\cdots,1,1)$$

and upper first diagonal

$$(-1,-1,\cdots,-1,-1).$$

Operation 2 (applied on D_1 and L_1):

$$\text{row } n-\frac{k_n}{g_{n-1}}\times\text{row }(n-1) \text{ and}$$

$$\text{row } i-\frac{g_i}{g_{i-1}}\times\text{row }(i-1),\ i=n-1,n-2,\cdots,3,$$

which derives the lower bidiagonal matrix D_2 with main diagonal

$$(c_1,c_2,\cdots,c_{n-1},k_nc_n)$$

and lower first diagonal

$$\left(a_1g_2,\frac{g_3k_2f_2}{g_2},\cdots,\frac{g_{n-1}k_{n-2}f_{n-2}}{g_{n-2}},\frac{k_nk_{n-1}f_{n-1}}{g_{n-1}}\right);$$

while the matrix L_1 is transformed into the tridiagonal matrix L_2 with main diagonal

$$\left(1,1,1+\frac{g_3}{g_2},\cdots,1+\frac{g_{n-1}}{g_{n-2}},1+\frac{k_n}{g_{n-1}}\right),$$

upper first diagonal

$$(-1,-1,\cdots,-1,-1)$$

and lower first diagonal

$$\left(0,-\frac{g_3}{g_2},\cdots,-\frac{g_{n-1}}{g_{n-2}},-\frac{k_n}{g_{n-1}}\right).$$

Operation 3 (applied on D_2 and L_2):

$$\text{row } 2-\frac{a_1g_2}{c_1}\times\text{row } 1,$$

$$\text{row } i-\frac{g_ik_{i-1}f_{i-1}}{g_{i-1}c_{i-1}}\times\text{row }(i-1),\cdots,$$

$$\text{row } n-\frac{k_nk_{n-1}f_{n-1}}{g_{n-1}c_{n-1}}\times\text{row }(n-1),$$

with $i=3,4,\cdots,n-1$, which yields the diagonal matrix D_3,

$$D_3=\lceil c_1\ c_2\cdots c_{n-1}\ k_nc_n\rfloor,$$

and the lower Hessenberg matrix L_3 equal to

$$\begin{bmatrix} 1 & -1 & \cdots & 0 & 0 \\ -\frac{a_1 g_2}{c_0 c_1} & \frac{k_1 b_1 - k_3 a_1}{c_1} & \cdots & 0 & 0 \\ \frac{a_1 g_3 k_2 f_2}{c_0 c_1 c_2} & -\frac{d_1 g_3}{c_1 c_2} & \cdots & 0 & 0 \\ \cdots & & & & \\ \frac{s a_1 g_{n-1} k_2 f_2 \cdots k_{n-2} f_{n-2}}{c_0 c_1 \cdots c_{n-2}} & \frac{s d_1 g_{n-1} k_3 f_3 \cdots k_{n-2} f_{n-2}}{c_1 \cdots c_{n-2}} & \cdots & \frac{k_{n-2} b_{n-2} - k_n a_{n-2}}{c_{n-2}} & -1 \\ \frac{s k_n a_1 g_n k_2 f_2 \cdots k_{n-1} f_{n-1}}{c_0 c_1 c_2 \cdots c_{n-1}} & \frac{s k_n d_1 g_n k_3 f_3 \cdots k_{n-1} f_{n-1}}{c_1 c_2 \cdots c_{n-1}} & \cdots & -\frac{k_n d_{n-2} g_n}{c_{n-2} c_{n-1}} & \frac{k_{n-1} b_{n-1}}{c_{n-1}} \end{bmatrix},$$

where the symbol s stands for $(-1)^{i+j}$.

Operation 4 (applied on D_3 and L_3):

$$\frac{1}{c_i} \times \text{row } i,\ i = 1, 2, \cdots, n-1, \text{ and } \frac{1}{k_n c_n} \times \text{row } n,$$

which transforms D_3 into the identity matrix *I* and L_3 into the inverse A_2^{-1}.

The determinant of A_2 has the form

$$\det(A_2) = k_n b_n (k_1 b_1 - k_2 a_1)(k_2 b_2 - k_3 a_2) \cdots (k_{n-1} b_{n-1} - k_n a_{n-1}), \quad (17)$$

which shows in turn that the matrix A_2 is singular if $k_n = 0$, or, adopting the conventions (14), if $c_i = 0$ for some $i \in \{1, 2, \cdots, n\}$.

4. Numerical Complexity

The relations (8) and (13) lead to recurrence formulae, by which the inverses A_1^{-1} and A_2^{-1}, respectively, are computed in $O(n^2)$ multiplications/divisions and $O(n)$ additions/substractions. In fact, the recursive algorithm

$$\alpha_{i,i+1} = -1/c_i,\ i = 1, 2, \cdots, n-1, \quad (18)$$

$$\alpha_{ii} = -\alpha_{i,i+1} + \frac{b_{i-1} g_i}{c_{i-1} c_i},\quad i = 2, 3, \cdots, n-1,$$
$$\alpha_{11} = \frac{k_2}{k_1 c_1},\quad \alpha_{nn} = \frac{b_{n-1}}{c_{n-1} c_n}, \quad (19)$$

$$\alpha_{i,i-1} = -\frac{d_{i-2} g_i}{c_{i-2} c_{i-1} c_i},\ i = 2, 3, \cdots, n, \quad (20)$$

$$\alpha_{i,i-s-1} = -\frac{d_{i-s-2} k_{i-s} f_{i-s}}{d_{i-s-1} c_{i-s-2}} \alpha_{i,i-s},$$
$$i = 3, 4, \cdots, n,\quad s = 1, 2, \cdots, i-2, \quad (21)$$

where c_i, d_i, f_i, and g_i are given by the relation (9), computes A_1^{-1} in $5n^2/2 + 5n/2 - 6$ mult/div (since the coefficients of $\alpha_{i,i-s}$ depends only on the second subscript) and $5n - 9$ add/sub.

In terms of j, the above algorithm takes the form

$$\alpha_{j-1,j} = -1/c_{j-1},\ j = 2, 3, \cdots, n,$$

$$\alpha_{jj} = -\alpha_{j,j+1} + \frac{b_{j-1} g_j}{c_{j-1} c_j},\ j = 2, 3, \cdots, n-1,$$

$$\alpha_{11} = \frac{k_2}{k_1 c_1},\ \alpha_{nn} = \frac{b_{n-1}}{c_{n-1} c_n},$$

$$\alpha_{j+1,j} = -\frac{d_{j-1} g_{j+1}}{c_{j-1} c_j c_{j+1}},\ j = 1, 2, \cdots, n-1,$$

$$\alpha_{j+s+1,j} = -\frac{g_{j+s+1} k_{j+s} f_{j+s}}{g_{j+s} c_{j+s+1}} \alpha_{j+s,j},$$
$$j = 1, 2, \cdots, n-2,\quad s = 1, 2, \cdots, n-j-1.$$

For the computation of A_2^{-1} the algorithms (18)-(21) changes only in the estimation of the diagonal elements, for which we have

$$\alpha_{ii} = -\alpha_{i,i+1} + \frac{a_{i-1} g_i}{c_{i-1} c_i},\ i = 2, 3, \cdots, n-1,$$

$$\alpha_{11} = -\alpha_{12},\ \alpha_{nn} = \frac{k_{n-1} b_{n-1}}{k_n c_{n-1} c_n},$$

where c_i, d_i, f_i, and g_i are given by the relation (14). Therefore, considering the relations (9) and (14), it is clear that the number of mult/div and add/sub in computing A_2^{-1} is the same with that of A_1^{-1}.

5. Concluding Remarks

The matrices A_1 and A_2 represent generalizations of known classes of test matrices. For instance, the test matrices given in [6] (Equations (2.1) and (2.2)) and in [1] (Eq. (2)) belong to the categories presented. Furthermore, by restricting the *a*'s and *b*'s to unity, A_1 and A_2 reduce to the matrices given in [2]. Also, the matrices in [7] (pp. 41, 42, 49) are special cases of A_1 and A_2. On the other hand, concerning the recursive algorithms given in Section 4, we have performed numerical experiments by assigning random values to the parameters of A_1, and with a variety of the order n from 256 to 1024. We have found that computing A_1^{-1} by the recursive algorithms (18)-(21) is ~100 times faster than using the LU decomposition when n = 256 and increases gradually to ~1000 times faster when n = 1024.

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