Applied Mathematics, 2012, 3, 1023-1028
http://dx.doi.org/10.4236/am.2012.39151 Published Online September 2012 (http://www.SciRP.org/journal/am)

Fritz John Duality in the Presence of Equality and Inequality Constraints

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Received June 10, 2012; revised August 3, 2012; accepted August 10, 2012

ABSTRACT

A dual for a nonlinear programming problem in the presence of equality and inequality constraints which represent many realistic situation, is formulated which uses Fritz John optimality conditions instead of the Karush-Kuhn-Tucker optimality conditions and does not require a constraint qualification. Various duality results, namely, weak, strong, strict-converse and converse duality theorems are established under suitable generalized convexity. A generalized Fritz John type dual to the problem is also formulated and usual duality results are proved. In essence, the duality results do not require any regularity condition if the formulations of dual problems uses Fritz John optimality conditions.

Keywords: Second-Order Invexity; Second-Order Pseudoinvexity; Second-Order Quasi-Invexity; Nonlinear Programming; Fritz John Type Dual

1. Introduction

Consider the following mathematical programming problems.

(NP): Minimize $f(x)$

Subject to

$$g(x) \leq 0$$

(NEP): Minimize $f(x)$

Subject to

$$g(x) \leq 0 \tag{1}$$

$$h(x) = 0 \tag{2}$$

where $f: R^n \to R$, $g: R^n \to R^m$ and $h: R^n \to R^k$ are differentiable functions. The best-known necessary optimality conditions for the mathematical programming problem (NP) and (NEP) are Fritz John necessary optimality conditions and Karush-Kuhn-Tucker type optimality conditions. The Fritz John type [1] optimality condition which predates the Karush-Kuhn-Tucker type optimality conditions by a few years are more general in a sense. In order for Karush-Kuhn-Tucker type optimality conditions to hold, a constraint qualification or regularity condition on the constraint is required. On the other hand, no such constraint qualification is needed for Fritz John optimality conditions to hold.

Fritz John [2] established the following optimality conditions for (NP):

Proposition 1. (Fritz John type necessary conditions). If $\bar{x}$ is an optimal solution of (NP), then there exist $\bar{r} \in R$ and a vector $\bar{y} \in R^m$ such that

$$\bar{r}\nabla f(\bar{x}) + \bar{y}^T \nabla g(\bar{x}) = 0$$

$$\bar{y}^T g(\bar{x}) = 0$$

$$(\bar{r}, \bar{y}) \geq 0$$

$$(\bar{r}, \bar{y}) \neq 0$$

Using these optimality conditions, Weir and Mond [3] formulated the for Fritz John type dual $(F_r D)$ to (NP) and established usual duality theorems, this eliminating the requirement of a constraint qualification:

$(F_r D)$: Maximize $f(u)$

Subject to

$$r\nabla f(u) + y^T \nabla g(u) = 0$$

$$y^T g(u) \geq 0$$

$$(r, y) \geq 0$$

$$(r, y) \neq 0.$$

Originally, Fritz John derived his optimality condition for the case of inequality constraint alone. If equality constraint are present in a mathematical programming problem and they are converted into two inequality constraints, then the Fritz John optimality conditions become useless because every feasible point satisfying them. Later Mangasarian and Fromovitz [4] derived necessary optimality condition for (NEP) without replacing an equality constraint by two inequalities and hence making it possible to handle equalities and inequalities together as many realistic problems contain both equality and inequality constraint. Mangasarian and Fromovitz [4] established the following Fritz John type optimality condition given in the following propositions:

Proposition 2. (Generalized Fritz John necessary optimality Conditions [4]):

If $\bar{x}$ is an optimal solution of (NEP), then there exist $\bar{r} \in R$, $\bar{y} \in R^m$ and $\bar{z} \in R^k$ such that

$$\bar{r}\,\nabla f(\bar{x}) + \bar{y}^T \nabla g(\bar{x}) + \bar{z}^T \nabla h(\bar{x}) = 0 \quad (3)$$

$$\bar{y}^T g(\bar{x}) = 0 \quad (4)$$

$$(\bar{r}, \bar{y}) \geq 0 \quad (5)$$

$$(\bar{r}, \bar{y}, \bar{z}) \neq 0 \quad (6)$$

2. Sufficiency of Fritz John Optimality Conditions

Before proceeding to the main results of our analysis we give the following definitions which are required for their validation.

1) The function $\phi : R^n \to R$ is strictly pseudoconvex on R^n for all $x \neq u$,

$$(x-u)^T \nabla\phi(u) \geq 0 \Rightarrow \phi(x) > \phi(u)$$

Equivalently

$$\phi(x) \leq \phi(u) \Rightarrow (x-u)^T \nabla\phi(u) < 0$$

2) For $y \in R^m$ and $\psi : R^n \to R^m, y^T\psi$ is said to be semi-strictly pseudoconvex if $y^T\psi$ is strictly pseudoconvex for all $y \geq 0, y \neq 0$.

Theorem 1. (Sufficient Optimality Conditions): Assume that

1) $f(.)$ is pseudoconvex,
2) $\bar{y}^T g(.)$ is semi strictly pseudoconvex and
3) $z^T h(.)$ is quasiconvex,

If there exist $\bar{x}$, $r \in R$, $y \in R^m$ and $z \in R^k$ such that (3)-(8) are satisfied, then $\bar{x}$ is an optimal solution of (NEP).

Proof: Suppose $\bar{x}$ is not optimal, *i.e.*, and then there exists $x \neq \bar{x}$ Such that

$$f(x) < f(\bar{x})$$

Since $f(.)$ is pseudoconvex, this implies

$$(x-\bar{x})^T \nabla f(\bar{x}) < 0$$

and

$$(x-\bar{x})^T \bar{r}\,\nabla f(\bar{x}) \leq 0 \quad (7)$$

with strict-inequality in the above if $\bar{r} > 0$

Since $\bar{x}$ is feasible for (*NEP*) we have

$$y^T g(x) \leq \bar{y}^T g(\bar{x})$$

Because of semi strict pseudoconvexity of $\bar{y}^T g(.)$, This implies

$$(x-\bar{x})^T \nabla\bar{y}^T g(\bar{x}) \leq 0 \quad (8)$$

With strict inequality with $y^i(.) > 0$, $i \in \{1,2,3,\cdots,m\}$.

Also $\bar{z}h(x) = \bar{z}^T h(\bar{x})$

Because of quasi-convexity of $z^T h(.)$ at $\bar{x}$,

$$(x-\bar{x})^T \nabla z^T h(\bar{x}) \leq 0 \quad (9)$$

Combining (7), (8) and (9), we have

$$(x-\bar{x})^T \left(\bar{r}\,\nabla f(\bar{x}) + \nabla y^T g(\bar{x}) + \nabla z^T h(\bar{x})\right) < 0,$$

Contradicting (3). Hence $\bar{x}$ is an optimal solution of (NEP).

3. Fritz John Type Duality

We propose the following dual (F_rED) to (*NEP*), using Fritz John optimality conditions stated in the preceding section instead of Karush-Kuhn-Tucker conditions [5,6] and established duality results, thus the requirement of a constraint qualification [4] is eliminated:

Dual Problem:

(F_rED): Maximize $f(u)$

Subject to

$$\nabla\left(rf(u) + y^T g(x) + z^T h(x)\right) = 0 \quad (10)$$

$$y^T g(u) \geq 0 \quad (11)$$

$$z^T h(u) \geq 0 \quad (12)$$

$$(r, y) \geq 0 \quad (13)$$

$$(r, y, z) \neq 0 \quad (14)$$

Theorem 2. (Weak Duality): Assume that

(A_1): x is feasible for (NEP) and (u,r,y,z) is feasible for (F_rED).

(A_2): For all feasible (x,u,r,y,z), $f(.)$ is pseudoconvex, $y^T g(.)$ is semi-strictly pseudoconvex and $z^T h(.)$ is quasiconvex.

Then

$$\inf(NEP) \geq \sup(F_rED)$$

Proof: Suppose $f(x) < f(u)$ this, because of pseudoconvexity of $f(.)$ yields $(x-u)^T \nabla f(u) < 0$, Multiplying this, by $r \geq 0$, We have

$$(x-u)^T \nabla r\, f(u) \leq 0 \quad (15)$$

With strict inequality in (15) if $r > 0$

From the Constraints of (NEP) and (F_rED), we have

$$y^T g(x) \le y^T g(u)$$

which by semi-strictly pseudoconvexity of $y^T g(.)$ implies

$$(x-u)^T \nabla y^T g(u) \le 0 \quad (16)$$

with strict inequality in (16) if $y_i > 0, i \in (1,2,3,\cdots,m)$

As earlier $y^T h(x) \le y^T h(u)$

This along with quasiconvexity of $y^T h(.)$ implies

$$(x-u)^T \nabla y^T h(u) \le 0 \quad (17)$$

Combining (15), (16), (17), we have

$$(x-u)^T \nabla\left(rf(u) + y^T g(u) + z^T h(u)\right) < 0$$

Contradicting

$$(x-u)^T \nabla\left(rf(u) + y^T g(u) + z^T h(u)\right) = 0$$

Hence $f(x) \ge f(u)$

This implies $\inf(NEP) \ge \sup(F_r ED)$.

Theorem 3. (Strong Duality):

If $\bar{x}$ is an optimal solution of (NED) then there exist $r \in R$, $y \in R^m$ and $z \in R^k$ such that $(\bar{x}, r, y, z)$ is feasible for (NED) and the corresponding values of (NEP) and $(F_r ED)$ are equal. If, also *f* is pseudoconvex, $y^T g(.)$ is semi-strictly pseudoconvex and $z^T h(.)$ is quasi-convex, then $(\bar{x}, r, y, z)$ is an optimal solution of (NED).

Proof: Since $\bar{x}$ is an optimal solution of (NEP), by Proposition 2. There exist $r \in R$, $y \in R^m$ and $z \in R^k$ such that

$$\nabla\left(rf(\bar{x}) + y^T g(\bar{x}) + z^T h(\bar{x})\right) = 0,$$

$$y^T g(\bar{x}) = 0,$$

$$g(\bar{x}) \le 0$$

$$h(\bar{x}) = 0$$

$$(r, y) \ge 0, \ (r, y, z) \ne 0$$

This implies $(\bar{x}, r, y, z)$ is feasible for $(F_r ED)$. Equality of objective function of (NEP) and $(F_r ED)$ is abovious optimality follows, in view of the hypothesis of the theorem1.

Theorem 4. (Strict Converse Duality): Assume that

1) $f(.)$ is strictly pseudoconvex, $y^T g(.)$ is semi-strictly pseudoconvex and as $z^T h(.)$ is quasiconvex and

2) The problem (NEP) has an optimal solution $\bar{x}$.

If $(\bar{u}, r, y, z)$ is an optimal solution of $(F_r ED)$, Then $\bar{u} = \bar{x}$ *i.e.* $\bar{u}$ is an optimal solution of (NEP).

Proof: We assume that $\bar{x} \ne \bar{u}$ and exhibit a contradiction, it follows from Proposition 2 that there exist $r \in R$, $y \in R^m$ and $z \in R^k$ such that $(\bar{x}, r, y, z)$ is optimal solution of $(F_r ED)$, since $(\bar{u}, r, y, z)$ is also an optimal solution for $(F_r ED)$, It follows that

$$f(\bar{x}) = f(\bar{u})$$

by strict pseudoconvexity of $f(.)$ we have

$$(\bar{x} - \bar{u})^T \nabla f(\bar{u}) < 0 \quad (18)$$

Also from the constraints of (NED) and $(F_r ED)$ we have $y^T g(\bar{x}) \le y^T g(\bar{u})$.

By the semi strictly convexity of $y^T g(.)$, this implies

$$(\bar{x} - \bar{u}) \nabla y^T g(\bar{x}) \le 0 \quad (19)$$

with strict inequality in the above, if $y_i > 0$

Also $z^T h(\bar{x}) \le z^T h(\bar{u})$ which by quasi-convexity of $zh(.)$ at $\bar{u}$, implies

$$(\bar{x} - \bar{u})^T \nabla z^T h(\bar{u}) \le 0 \quad (20)$$

Combining (18), (19), and (20), we have

$$(\bar{x} - \bar{u}) \nabla\left(rf(\bar{u}) + y^T g(\bar{u}) + z^T h(\bar{u})\right) < 0$$

which contradicts

$$(\bar{x} - \bar{u}) \nabla\left(rf(\bar{u}) + y^T g(\bar{u}) + z^T h(\bar{u})\right) = 0$$

Hence $\bar{u} = \bar{x}$ *i.e.* $\bar{u}$ is an optimal solution.

Theorem 5. (Converse Duality): Let $(\bar{x}, r, y, z)$ be an optimal solution of $(F_r ED)$. Assume that

(A_1): $f(.)$ is pseudoconvex, $y^T g(.)$ is semi strictly-pseudoconvex and $z^T h(.)$ is quasiconvex.

(A_2): Hessian matrix $\nabla^2\left(\bar{r}f(\bar{x}) + \bar{y}^T g(\bar{x}) + \bar{z}^T h(\bar{x})\right)$ is positive or negative definite, and

(A_3): the set $\left\{\nabla \bar{y}^T g(\bar{x}), \nabla \bar{z}^T h(x)\right\}$ is linearly independent, and

Then $\bar{x}$ is an optimal solution of (NEP).

Proof: By Preposition 2, there exist $\tau \in R$, $\theta \in R^n$ $\phi \in R$, $\psi \in R$, $\xi \in R$ and $\eta \in R^m$ such that

$$\begin{aligned} &\tau \nabla f(\bar{x}) + \theta \nabla^2\left(rf(\bar{x}) + y^T g(\bar{x}) + z^T h(\bar{x})\right) \\ &+ \phi \nabla\left(\bar{y}^T g(\bar{x})\right) + \psi \nabla z^T h(\bar{x}) = 0 \end{aligned} \quad (21)$$

$$\theta^T \nabla f(\bar{x}) + \xi = 0 \quad (22)$$

$$\theta \nabla g(\bar{x}) + \phi g(\bar{x}) + \eta = 0 \quad (23)$$

$$\theta \nabla h(\bar{x}) + \psi h(\bar{x}) = 0 \quad (24)$$

$$\phi\left(y^T g(\bar{x})\right) = 0 \quad (25)$$

$$\psi\left(z^T h(\bar{x})\right) = 0 \quad (26)$$

$$\xi r = 0 \quad (27)$$

$$\eta^T y = 0 \quad (28)$$

$$(\tau, \phi, \psi, \eta, \xi) \ge 0 \quad (29)$$

$$(\tau, \theta, \phi, \psi, \eta, \xi) \ne 0. \quad (30)$$

Multiplying (23) by $y \geq 0$ and using (25) and (28), we obtain

$$\theta^T \nabla y^T g(\bar{x}) = 0 \quad (31)$$

Multiplying (24) by z and $h(\bar{x}) = 0$, we have

$$\theta \nabla z^T h(\bar{x}) = 0 \quad (32)$$

Multiplying equality constraint of $(F_r ED)$ by θ^T and using (31) and (32) We have $\theta^T \nabla r f(\bar{x}) = 0$

Multiplying (21) by θ and using (31) and (32), we have

$$\tau \theta \nabla f(\bar{x}) + \theta \nabla^2 \left(r f(\bar{x}) + y^T g(\bar{x}) + z^T h(\bar{x}) \right) \theta = 0$$

Multiplying the above equation by r and using (33), we have

$$(r\theta)^T \nabla^2 \left(r f(\bar{x}) + y^T g(\bar{x}) + z^T h(\bar{x}) \right)(r\theta) = 0$$

This because of hypothesis (A_2) implies $r\theta = 0$. In view of (A_3) the equality constraint of $(F_r ED)$ implies $r \neq 0$, *i.e.*, $r > 0$.consequently $\theta = 0$.

Multiplying (21) by r and using $\theta = 0$, we have

$$\tau r \nabla f(\bar{x}) + r\phi\left(\nabla y^T g(\bar{x})\right) + r\psi\left(\nabla z^T h(\bar{x})\right) = 0$$

Using the equality constraint (10) in the above, we have

$$-\tau\left(\nabla y^T g(\bar{x}) + \nabla z^T h(\bar{x})\right) + r\phi \nabla y^T g(\bar{x}) + r\psi \nabla z^T h(\bar{x}) = 0$$

This reduces to

$$(\phi - \tau/r)\nabla y^T g(\bar{x}) + (\psi - \tau/r)\nabla z^T h(\bar{x}) = 0$$

By the linear independence hypothesis (A_3). this implies

$$(\phi - \tau/r) = 0 \quad \text{and} \quad (\psi - \tau/r) = 0$$

Now if $\tau = 0$, then from above, we have $\phi = 0$, $\psi = 0$ and from (22) and (23), We have $\xi = 0$, $\eta = 0$, consequently we have $(t, \phi, \psi, \theta, \xi, \eta) = 0$ contradicting to (30).

Hence $\tau > 0$, $\phi > 0$, and $\psi > 0$.

Using $\theta = 0$ in (23) and (24), we have

$$\phi g(\bar{x}) + \eta = 0, \quad \phi g(\bar{x}) + \eta = 0$$

This implies $g(\bar{x}) \leq 0$ and $h(\bar{x}) = 0$.

Thus $\bar{x}$ is feasible for $(F_r ED)$ and the objective functions of (NEP) and $(F_r ED)$ are equal in their formulations. Under, the state generalized Convexity, Theorem 1 implies that, $\bar{x}$ is an optimal solution of (NEP).

4. Generalized Fritz John Duality

Let $M = \{1, 2, \cdots, m\}$ and $L = \{1, \cdots, l\}$, $I_\alpha \subseteq M$, $\alpha = 0, 1, \cdots, t$. with $I_\alpha \cap I_\beta = \phi$, $\alpha \neq \beta$ and $\bigcup_{\alpha=0}^{t} I_\alpha = M$. and $J_\alpha \subseteq L$ with $J_\alpha \cap J_\beta \neq \phi$, $\alpha \neq \beta$ and $L = \bigcup_{\alpha=0}^{t} J_\alpha$. Let $K = \{0, 1, 2, \cdots, t\}$ and $N \subset K$. The following is the generalized Fritz John type dual to (NEP).

$(GF_r ED)$: Maximize $f(u)$

Subject to

$$\bar{r} \nabla f(u) + \nabla y^T g(u) + \nabla z^T h(u) = 0$$

$$\sum_{i \in I_\alpha} y_i\, g_i(u) \geq 0, \ \alpha = 0, 1, 2, \cdots, t$$

$$\sum_{J \in I_\alpha} z_j\, h_j(u) \geq 0, \ \alpha = 0, 1, 2, \cdots, t$$

$$(\bar{r}, y) \geq 0$$

$$(\bar{r}, y, z) \neq 0.$$

Theorem 6: If $f(.)$ is pseudoconvex, $\sum_{i \in I_\alpha} y_i g(.)$, $\alpha \in N$ is semi-strictly pseudoconvex, $\sum_{i \in I_\alpha} y_i g(.)$, $\alpha \in K \setminus N$ and $\sum_{j \in I_\alpha} z_j h_j(.), \alpha = 0, 1, 2, \cdots, t$ is quasiconvex,

Then $\inf(NEP) \geq \sup(GF_r ED)$

Proof: Let x be feasible for (NEP) and (u, r, y, z) feasible for $(GF_r ED)$ Suppose $f(x) > f(u)$ This by pseudoconvexity of $f(.)$ yields

$$\begin{aligned} (x-u)^T \nabla f(u) &< 0 \\ (x-u)^T \nabla r f(u) &\leq 0 \end{aligned} \quad (34)$$

with strict inequality in (34) if $r > 0$.

From the constraint of (NEP) and $(GF_r ED)$, we have

$$\sum_{i \in I_\alpha} y_i g_i(x) \leq \sum_{i \in I_\alpha} y_i g_i(u), \alpha \in N \quad (35)$$

Which because of semistrictly pseudoconvexity of $\sum_{i \in I_\alpha} y_i g_i(x)$ $\alpha = 0, 1, 2, \cdots, t$ implies

$$(x-u)^T \nabla \sum_{i \in I_\alpha} y_i g_i(u) \leq 0, \ \alpha \in N \quad (36)$$

with strict inequality in (36) if some $y_i > 0, i \in I_\alpha, \alpha \in N$.

Also

$$\sum_{i \in I_\alpha} y_i g_i(x) - \sum_{i \in I_\alpha} y_i g_i(u) \leq 0, \alpha \in K \setminus N$$

And

$$\sum_{j \in I_\alpha} z_j h_j(x) - \sum_{j \in I_\alpha} z_j g_j(u) \leq 0, \alpha = 0, 1, 2, \cdots, t$$

Which by quasiconvex of $\sum_{i \in I_\alpha} y_i g_i(.), \alpha \in K \setminus N$ and

$$\sum_{j \in I_\alpha} z_j h_j(.), \alpha = 0, 1, 2, \cdots, t$$

respectively imply

$$\left(x-u\right)^T \nabla\left(\sum_{i\in I_\alpha} y_i g_i\left(u\right)\right)\le 0, \alpha\in K\setminus N$$

and

$$\left(x-u\right)^T \nabla\left(\sum_{j\in I_\alpha} z_j h_j\left(u\right)\right)\le 0, \alpha\in N$$

combining (34), (35), (36) and above equation we have

$$\left(x-u\right)^T\left(r\nabla f\left(u\right)+\nabla y^T g\left(u\right)+\nabla z^T h\left(u\right)\right)<0$$

contradicting the equality constraint of (GF_rED). Hence $f(x)\ge f(u)$

Implying $\inf(NEP)\ge\sup(GF_rED)$.

Theorem 7. (Strong Duality):

If $\bar{x}$ is an optimal solution of (NEP) and there exist $\bar{r}\in R$, $\bar{y}\in R^m$ and $\bar{z}\in R^k$, such that $(\bar{r},\bar{x},\bar{y},\bar{z})$ is feasible for (GF_rED) and the corresponding value of (NEP) and (GF_rED) are equal. If, the hypotheses of Theorem 1 hold, then $(\bar{x},\bar{r},\bar{y},\bar{z})$ is an optimal solution of (GF_rED).

Proof: By Proposition 2, there exist $\bar{r}\in R$, $\bar{y}\in R^m$ and $\bar{z}\in R^k$ such that

$$\nabla\left(\bar{r}f\left(x\right)+\bar{y}^T g\left(x\right)+\bar{z}^T h\left(x\right)\right)=0,$$

$$\bar{y}^T g\left(\bar{x}\right)=0,$$

$$\left(\bar{r},\bar{y}\right)\ge 0,$$

$$\left(\bar{r},\bar{y},\bar{z}\right)\ne 0$$

Since $\bar{y}_i\, g_i(\bar{x})=0$, $i=1,2,\cdots,m$ and $\bar{z}_j h_j(\bar{x})=0$, feasibility of $(\bar{r},\bar{x},\bar{y},\bar{z})$ for (GF_rED) is obvious. Optimality follows, give the pseudoconvexity of $f(.)$ and semi-strict pseudoconvexity of $\sum_{i\in I_\alpha}\bar{y}_i g_i(.), \alpha\in N$, quasiconvexity of $\sum_{i\in I_\alpha}\bar{y}_i g_i(.), \alpha\in N$, and quasiconvexity of $\sum_{j\in I_\alpha}\bar{z}_j h_j(.), \alpha\in N, \alpha=0,1,2,\cdots,t$ from Theorem 1.

Theorem 8: (Mangasarian [4] Type Strict Converse Duality): Assume that

(A_1): $f(.)$ is strictly pseudoconvex,

(A_2): $\sum_{i\in I_\alpha} y_i g_i(.), \alpha\in N$ is semi-strictly pseudoconvex and

(A_3): $\sum_{i\in I_\alpha} y_i g_i(.), \alpha\in K\setminus N$ and $\sum_{j\in I_\alpha} z_j h_j(.)$, $\alpha=0,1,2,\cdots,t$ are quasiconvex.

(A_4): $\bar{x}$ is an optimal solution of (NEP).

If $(\bar{r},\bar{u},\bar{y},\bar{z})$ is an optimal solution of (GF_rED) then $\bar{x}=\bar{u}$ *i.e.* $\bar{u}$ is an optimal solution of (NEP).

Proof: Assume that $\bar{x}=\bar{u}$ and exhibit a contradiction. Since $\bar{x}$ is an optimal solution of (NEP), by Proposition 2, it implies that there exist $r\in R$, $y\in R^m$ and $z\in R^k$ such that $(\bar{x},r,y,z)$ is an optimal solution of (GF_rED).

Since $(\bar{u},r,y,z)$ is an optimal solution for (GF_rED), it follows that $f(\bar{x})=f(\bar{u})$

This, in view of strict pseudoconvexity of $f(.)$ implies

$$\left(\bar{x}-u\right)^T\nabla f\left(u\right)<0 \tag{37}$$

From the constraint of (NEP) and (GF_rED), we have

$$\sum_{i\in I_\alpha} y_i g_i\left(\bar{x}\right)\le\sum_{i\in I_\alpha} z_i g_i\left(\bar{u}\right), \alpha=0,1,2,\cdots,t \tag{38}$$

and

$$\sum_{j\in I_\alpha} z_j h_j\left(\bar{x}\right)\le\sum_{j\in I_\alpha} z_j h_j\left(\bar{u}\right), \alpha=0,1,2,\cdots,t \tag{39}$$

The inequality (38), in view of semi-strict pseudo convexity of $\sum_{i\in I_\alpha} y_i g_i(.), \alpha\in N$ implies

$$\left(\bar{x}-\bar{u}\right)^T\nabla\left(\sum_{i\in I_\alpha} y_i g_i\left(\bar{u}\right)\right)\le 0, \alpha\in N \tag{40}$$

with strict inequality in (40) if $y_i>0, i\in I_\alpha, \alpha\in N$.

By quasiconvexity of $\sum_{i\in I_\alpha} y_i g_i(.), \alpha\in K\setminus N$, (38) implies

$$\left(\bar{x}-\bar{u}\right)^T\nabla\left(\sum_{i\in I_\alpha} y_i g_i\left(\bar{u}\right)\right)\le 0 \tag{41}$$

The inequality (39), because of quasiconvexity of $\sum_{j\in I_\alpha} z_j h_j(.)$, $\alpha=0,1,2,3,\cdots,t$ yields,

$$\left(\bar{x}-\bar{u}\right)^T\nabla\left(\sum_{j\in I_\alpha} z_j h_j\left(\bar{u}\right)\right)\le 0, \alpha=0,1,\cdots,t \tag{42}$$

Combining (37), (40), (41) and (42), we have

$$\left(\bar{x}-\bar{u}\right)^T\left(r\nabla f\left(\bar{u}\right)+\nabla y^T g\left(\bar{u}\right)+\nabla z^T h\left(\bar{u}\right)\right)<0$$

which contradicts the feasibility of $(r,\bar{u},y,z)$ for (GF_rED). Hence $\bar{x}=\bar{u}$.

Theorem 9 (Converse Duality): Let

(C_1): $(\bar{r},\bar{x},\bar{y},\bar{z})$ be an optimal solution of (GF_rED).

(C_2): $f(.)$ be pseudoconvex, $\sum_{i\in I_\alpha}\bar{y}_i g_i(.), \alpha\in N$ semi-strictly pseudoconvex, $\sum_{i\in I_\alpha}\bar{y}_i g_i(.), \alpha\in K\setminus N$ quasiconvex.

(C_3): The Hessian matrix

$$\nabla^2\left(\bar{r}f\left(\bar{x}\right)+\bar{y}^T g\left(\bar{x}\right)+\bar{z}^T h\left(\bar{x}\right)\right)$$

is positive or negative definite, and

(C_4): The set

$$\left\{\nabla\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right),\nabla\left(\sum_{i\in I_\alpha}\bar{z}_i h_i(\bar{x})\right):\alpha=0,1,2,\cdots,t\right\}$$

is linearly independent. Then $\bar{x}$ is feasible for (NEP).

Proof: By Proposition 2, there exist $\tau\in R,\upsilon\in R^n$, $\theta_\alpha\in R$, $\phi_\alpha\in R$, $\alpha=0,1,2,\cdots,t$ $\xi\in R$, and $\eta\in R^m$ such that

$$\tau\nabla f(\bar{x})+\upsilon\nabla^2\left(rf(\bar{x})+y^Tg(\bar{x})+z^Th(\bar{x})\right)$$
$$+\sum_{\alpha=0}^{k}\left\{\theta_\alpha\nabla\left(\sum_{i\in I_\alpha}y_ig_i(\bar{x})\right)+\sum_{\alpha=0}^{k}\left\{\phi_\alpha\nabla\left(\sum_{j\in j_\alpha}z_jh_j(\bar{x})\right)\right\}\right\}=0 \tag{43}$$

$$\upsilon^T\nabla f(\bar{x})+\xi=0 \tag{44}$$

$$\upsilon^T\nabla g_i(\bar{x})+\theta_\alpha g_i+\eta_i=0, i\in I_\alpha, \alpha=0,1,2,\cdots,t \tag{45}$$

$$\upsilon\nabla h_j(\bar{x})+\phi_\alpha h_j=0,\ j\in I_\alpha, \alpha=0,1,2,\cdots,t \tag{46}$$

$$\theta_\alpha\left(\sum_{i\in I_\alpha}y_ig_i(\bar{x})\right)=0, \alpha=0,1,2,\cdots,t \tag{47}$$

$$\phi_\alpha\left(\sum_{j\in I_\alpha}z_jh_j(\bar{x})\right)=0, \alpha=0,1,2,\cdots,t \tag{48}$$

$$\xi^Tr=0 \tag{49}$$

$$\eta^Ty=0 \tag{50}$$

$$(\tau,\upsilon,\eta,\xi,\theta_0,\theta_1,\cdots,\theta_t,\phi_0,\phi_1,\cdots,\phi_t)\neq 0, \tag{51}$$

$$(\tau,\upsilon,\eta,\xi,\theta_0,\theta_1,\cdots,\theta_t)\geq 0, \tag{52}$$

Multiplying (45) and (46) by y_i and z_j respectively and using (47) and (48), we have

$$\upsilon^T\nabla\left(\sum_{i\in I_\alpha}y_ig_i(\bar{x})\right)=0 \tag{53}$$

$$\upsilon^T\nabla\left(\sum_{j\in I_\alpha}z_jh_j(\bar{x})\right)=0 \tag{54}$$

Multiplying (44) by *r*, we have

$$r\upsilon\nabla f(\bar{x})=0 \tag{55}$$

Multiplying (43) by υ and using (53), (54) and (53), we have

$$\left(r\upsilon^T\right)\left(r\nabla^2f(\bar{x})+\nabla^2y^Tg(\bar{x})+\nabla^2z^Th(\bar{x})\right)=0$$

By positive or negative definite and by hypothesis (C_3), we have $r\upsilon=0$.

In view of (C_4), equality constraint of (GF_rED) implies that $r>0$. Hence $\upsilon=0$. using $\upsilon=0$ we have

$$\sum_{\alpha=0}^{r}(r\theta_\alpha-\tau)\nabla\left(\sum_{i\in I_\alpha}y_ig_i(\bar{x})\right)$$
$$+\sum_{\alpha=0}^{r}(r\phi_\alpha-\tau)\nabla\left(\sum_{j\in I_\alpha}z_jh_j(\bar{x})\right)=0$$

which in view of the hypothesis (C_4) gives $r\theta_\alpha-\tau=0$, $r\phi_\alpha-\tau=0$, $\alpha=0,1,2,\cdots,t$. From (44) and (45), we have $\xi=0$ and $\eta=0$. consequently we have

$$(\tau,\upsilon,\eta,\xi,\theta_0,\theta_1,\cdots,\theta_t,\phi_0,\phi_1,\cdots,\phi_t)=0,$$

Contradicting Fritz John Condition (51). Hence $\tau>0$. since $\tau>0,\upsilon=0,\eta\geq 0,r>0$, The Equations (45) and (46), implies $g(\bar{x})=0, h(\bar{x})=0$.

Thus $\bar{x}$ is feasible for (NEP) and optimality follows as earlier.

5. Conclusion

In this exposition, we have formulated a dual and generalized dual by Fritz John optimality conditions instead of the Karush-Kuhn-Tucker optimality conditions. Consequently no constraint qualification is required and hence such formulations enjoy computational advantage over those formulated by using Karush-Kuhn-Tucker. The problems of these results can be revisited in multiobjective and dynamic setting.

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