

# On Approximate Solutions of Second-Order Linear Partial Differential Equations

Yousry S. Hanna

National Research Institute of Astronomy and Geophysics, Helwan, Egypt  
 Email: yousry\_hanna@yahoo.com

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## ABSTRACT

In this paper, a Chebyshev polynomial approximation for the solution of second-order partial differential equations with two variables and variable coefficients is given. Also, Chebyshev matrix is introduced. This method is based on taking the truncated Chebyshev expansions of the functions in the partial differential equations. Hence, the result matrix equation can be solved and approximate value of the unknown Chebyshev coefficients can be found.

**Keywords:** Chebyshev Polynomial; Differential Equations; Polynomial Approximation

## 1. Introduction

Let the second-order partial differential equation be in the form [1,2]

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u = G(x, y)$$

We assume that it has a Chebyshev series solution in the form

$$\begin{aligned} u(x, y) &= \frac{1}{4}a_{0,0}T_{0,0}(x, y) + \frac{1}{2}a_{0,1}T_{0,1}(x, y) + \dots \\ &+ \frac{1}{2}a_{1,0}T_{1,0}(x, y) + a_{1,1}T_{1,1}(x, y) + \dots \\ &+ \frac{1}{2}a_{2,0}T_{2,0}(x, y) + a_{2,1}T_{2,1}(x, y) + \dots \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_{r,s} T_{r,s}(x, y) \end{aligned} \quad (1.2)$$

where  $\sum$  denotes a sum whose first term is halved. The unknown coefficients  $a_{r,s}$ ;  $r = 0, 1, 2, \dots$ ,  $s = 0, 1, 2, \dots$  can be determined by using so called Chebyshev matrix method.

## 2. Calculation of Chebyshev Coefficients

Let we have a function  $u(x, y)$ ,  $(x, y) \in [-1, 1]$  and its  $n$ th derivatives with respect to  $x$  can be expanded in Chebyshev series

$$u(x, y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_{r,s} T_{r,s}(x, y)$$

and

$$u^{(n,0)}(x, y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_{r,s}^{(n,0)} T_{r,s}(x, y)$$

Respectively, where  $a_{r,s}^{(n,0)}$  and  $a_{r,s}$  are Chebyshev coefficients; clearly,  $a_{r,s}^{(n,0)} = a_{r,s}$  and  $u^{(0,0)}(x, y) = u(x, y)$ . Then the recurrence relation between the coefficients of  $u^{(n,0)}(x, y)$  and  $u^{(n+1,0)}(x, y)$  is obtained as

$$2ra_{r,s}^{(n,0)} = a_{r-1,s}^{(n+1,0)} - a_{r+1,s}^{(n+1,0)}, \quad r \geq 1, s \geq 0 \quad (2.1)$$

From Equation (2.1), we can deduce the relations

$$\begin{aligned} 2(r+1)a_{r+1,s}^{(n,0)} &= a_{r,s}^{(n+1,0)} - a_{r+2,s}^{(n+1,0)}, \\ 2(r+3)a_{r+3,s}^{(n,0)} &= a_{r+2,s}^{(n+1,0)} - a_{r+4,s}^{(n+1,0)}, \\ 2(r+5)a_{r+5,s}^{(n,0)} &= a_{r+4,s}^{(n+1,0)} - a_{r+6,s}^{(n+1,0)}, \end{aligned}$$

And adding these side by side, we get

$$a_{r,s}^{(n+1,0)} = 2[(r+1)a_{r+1,s}^{(n,0)} + (r+3)a_{r+3,s}^{(n,0)} + (r+5)a_{r+5,s}^{(n,0)} + \dots]$$

or

$$a_{r,s}^{(n+1,0)} = 2 \sum_{i=0}^{\infty} (r+2i+1) a_{r+2i+1,s}^{(n,0)} \quad (2.2)$$

Specially, we can express the coefficients  $a_{r,s}^{(1,0)}$  and  $a_{r,s}^{(2,0)}$ , in terms of the  $a_{r,s}$  by means of Equation (2.2), in the forms

$$a_{r,s}^{(1,0)} = 2 \sum_{i=0}^{\infty} (r+2i+1) a_{r+2i+1,s}^{(0,0)} \quad (2.3)$$

and

$$a_{r,s}^{(2,0)} = 2 \sum_{i=0}^{\infty} (r+2i+1) a_{r+2i+1,s}^{(1,0)}$$

or

$$a_{r,s}^{(2,0)} = 4 \sum_{i=0}^{\infty} i(r+i)(r+2i)a_{r+2i,s}^{(0,0)} \quad (2.4)$$

Now, let us take  $r = 0, 1, 2, \dots, N$  and assume  $a_{r,s}^{(0,0)} = a_{r,s}^{(1,0)} = \dots = 0$  for  $r > N$ ; then the system (2.2) can be transformed into the matrix form,

$$A^{(n+1,0)} = 2MA^{(n,0)}, \quad n = 0, 1, 2, \dots, \quad (2.5)$$

where  $M$  is given in [3].

$$A^{(n,0)} = \begin{bmatrix} \frac{1}{4}a_{0,0}^{(n,0)} & \frac{1}{2}a_{0,1}^{(n,0)} & \dots & \frac{1}{2}a_{0,N}^{(n,0)} \\ \frac{1}{2}a_{1,0}^{(n,0)} & a_{1,1}^{(n,0)} & \dots & a_{1,N}^{(n,0)} \\ \vdots & \vdots & \dots & \vdots \\ \frac{1}{2}a_{N,0}^{(n,0)} & a_{N,1}^{(n,0)} & \dots & a_{N,N}^{(n,0)} \end{bmatrix}$$

For  $n = 0, 1, \dots$ , it follows from Equation (2.5) that

$$\begin{aligned} A^{(1,0)} &= 2MA^{(0,0)}, \\ A^{(2,0)} &= 2MA^{(1,0)} = 2^2 M^2 A^{(0,0)}, \\ A^{(3,0)} &= 2MA^{(2,0)} = 2^3 M^3 A^{(0,0)}, \\ &\vdots \\ A^{(n,0)} &= 2MA^{(n-1,0)} = 2^n M^n A^{(0,0)}, \\ &\vdots \end{aligned} \quad (2.6)$$

where clearly  $A^{(0,0)} = A$ .

Let us assume, in the range  $[-1, 1]$ , that the  $n$ th derivatives of  $u(x, y)$  with respect to  $y$  can be expanded in Chebyshev series

$$u^{(0,n)}(x, y) = \sum_{s=0}^{\infty} a_{r,s}^{(0,n)} T_{r,s}(x, y)$$

Respectively, where  $a_{r,s}^{(0,n)}$  and  $a_{r,s}$  are Chebyshev coefficients; clearly  $a_{r,s}^{(0,0)} = a_{r,s}$  and  $u^{(0,0)}(x, y) = u(x, y)$ . Then the recurrence relation between the coefficients of  $u^{(0,n)}(x, y)$  and  $u^{(0,n+1)}(x, y)$  is obtained as

$$2sa_{r,s}^{(0,n)} = a_{r,s-1}^{(0,n+1)} - a_{r,s+1}^{(0,n+1)}, \quad r \geq 0, s \geq 1 \quad (2.7)$$

From Equation (2.7), we can deduce the relations

$$\begin{aligned} 2(s+1)a_{r,s+1}^{(0,n)} &= a_{r,s}^{(0,n+1)} - a_{r,s+2}^{(0,n+1)}, \\ 2(s+3)a_{r,s+3}^{(0,n)} &= a_{r,s+2}^{(0,n+1)} - a_{r,s+4}^{(0,n+1)}, \\ 2(s+5)a_{r,s+5}^{(0,n)} &= a_{r,s+4}^{(0,n+1)} - a_{r,s+6}^{(0,n+1)}, \\ &\vdots \end{aligned}$$

and adding these side by side, we get

$$a_{r,s}^{(0,n+1)} = 2 \left[ (s+1)a_{r,s+1}^{(0,n)} + (s+3)a_{r,s+3}^{(0,n)} + (s+5)a_{r,s+5}^{(0,n)} + \dots \right]$$

or

$$a_{r,s}^{(0,n+1)} = 2 \sum_{i=0}^{\infty} (s+2i+1)a_{r,s+2i+1}^{(0,n)} \quad (2.8)$$

Specially, we can express the coefficients  $a_{r,s}^{(0,1)}$  and  $a_{r,s}^{(0,2)}$  in terms of the  $a_{r,s}$ , by means of Equation (2.8), in the forms

$$a_{r,s}^{(0,1)} = 2 \sum_{i=0}^{\infty} (s+2i+1)a_{r,s+2i+1}^{(0,0)} \quad (2.9)$$

and

$$a_{r,s}^{(0,2)} = 2 \sum_{i=0}^{\infty} (s+2i+1)a_{r,s+2i+1}^{(0,1)}$$

or

$$a_{r,s}^{(0,2)} = 4 \sum_{i=0}^{\infty} i(s+i)(s+2i)a_{r,s+2i}^{(0,0)} \quad (2.10)$$

Now, let us take  $s = 0, 1, \dots, N$  and assume  $a_{r,s}^{(0,0)} = a_{r,s}^{(0,1)} = \dots = 0$  for  $s > N$ ; then the system (2.8) can be transformed into the matrix form,

$$A_t^{(0,n+1)} = 2MA_t^{(0,n)}, \quad n = 0, 1, 2, \dots, \quad (2.11)$$

where

$$A_t^{(0,n)} = \begin{bmatrix} \frac{1}{4}a_{0,0}^{(0,n)} & \frac{1}{2}a_{1,0}^{(0,n)} & \dots & \frac{1}{2}a_{N,0}^{(0,n)} \\ \frac{1}{2}a_{0,1}^{(0,n)} & a_{1,1}^{(0,n)} & \dots & a_{N,1}^{(0,n)} \\ \vdots & \vdots & \dots & \vdots \\ \frac{1}{2}a_{0,N}^{(0,n)} & a_{1,N}^{(0,n)} & \dots & a_{N,N}^{(0,n)} \end{bmatrix}$$

For  $n = 0, 1, \dots$ , it follows from Equation (2.11) that

$$\begin{aligned} A_t^{(0,1)} &= 2MA_t^{(0,0)}, \\ A_t^{(0,2)} &= 2MA_t^{(0,1)} = 2^2 M^2 A_t^{(0,0)}, \\ A_t^{(0,3)} &= 2MA_t^{(0,2)} = 2^3 M^3 A_t^{(0,0)}, \\ &\vdots \\ A_t^{(0,n)} &= 2MA_t^{(0,n-1)} = 2^n M^n A_t^{(0,0)}, \\ &\vdots \end{aligned} \quad (2.12)$$

where clearly  $A_t^{(0,0)} = A_t = [A^{(0,0)}]^t$ . Furthermore,  $A_t^{(n,n)}$  can be expressed as follows:

$$\begin{aligned} A^{(0,n)} &= 2^n M^n A_t, \\ A^{(n,n)} &= 2^n M^n A (2^n M^n)^t \end{aligned} \quad (2.13)$$

### 3. Fundamental Relations

Now consider Equation (1.1), where  $A, B, C, D, E, F$  and  $G$  are functions of  $x$  and  $y$ , or constant, defined in the

range  $[-1,1]$ . Our purpose is to investigate the truncated Chebyshev series solution of Equation (1.1), under the given conditions, in the series form

$$u(x, y) = \sum_{r=0}^N \sum_{s=0}^N a_{r,s} T_{r,s}(x, y)$$

or in the matrix form

$$u(x, y) = T_x A T_y \tag{3.1}$$

where  $a_{r,s}$ ,  $r = 0, 1, 2, \dots, N$ ,  $s = 0, 1, 2, \dots, N$  are the Chebyshev coefficients to be determined  $T_{r,s}(x, y)$  are the bivariate Chebyshev polynomials defined in [4], and matrices  $T_x$ ,  $T_y$  and  $A$  are defined by

$$T_x = [T_0(x) \quad T_1(x) \quad \dots \quad T_N(x)],$$

$$T_y = [T_0(y) \quad T_1(y) \quad \dots \quad T_N(y)]^t,$$

$$A = \begin{bmatrix} \frac{1}{4}a_{0,0} & \frac{1}{2}a_{1,0} & \dots & \frac{1}{2}a_{N,0} \\ \frac{1}{2}a_{0,1} & a_{1,1} & \dots & a_{N,1} \\ \vdots & \vdots & \dots & \vdots \\ \frac{1}{2}a_{0,N} & a_{1,N} & \dots & a_{N,N} \end{bmatrix}^t$$

To obtain the solution of Equation (1.1) in the form of Equation (3.1), first we must reduce Equation (1.1) to a differential Equation whose coefficients are polynomials [5]. For this purpose, we assume that the functions  $A(x, y)$ ,  $B(x, y)$ ,  $C(x, y)$ ,  $D(x, y)$ ,  $E(x, y)$ ,  $F(x, y)$ , and  $G(x, y)$  can be expressed in the form

$$\begin{aligned} A(x, y) &= \sum_{n=0}^N \sum_{m=0}^N a_{n,m} x^n y^m, \\ B(x, y) &= \sum_{n=0}^N \sum_{m=0}^N b_{n,m} x^n y^m, \\ C(x, y) &= \sum_{n=0}^N \sum_{m=0}^N c_{n,m} x^n y^m, \\ D(x, y) &= \sum_{n=0}^N \sum_{m=0}^N d_{n,m} x^n y^m, \\ E(x, y) &= \sum_{n=0}^N \sum_{m=0}^N e_{n,m} x^n y^m, \\ F(x, y) &= \sum_{n=0}^N \sum_{m=0}^N f_{n,m} x^n y^m, \\ G(x, y) &= \sum_{n=0}^N \sum_{m=0}^N g_{n,m} x^n y^m \end{aligned} \tag{3.2}$$

Which are Taylor polynomials at  $(x, y) = (0, 0)$ . By using the expressions (3.2) in Equation (1.1), we get

$$\begin{aligned} &\sum_{n=0}^N \sum_{m=0}^N [a_{n,m} x^n y^m u_{xx} + b_{n,m} x^n y^m u_{xy} + c_{n,m} x^n y^m u_{yy} \\ &+ d_{n,m} x^n y^m u_x + e_{n,m} x^n y^m u_y + f_{n,m} x^n y^m u] \\ &= \sum_{n=0}^N \sum_{m=0}^N g_{n,m} x^n y^m \end{aligned} \tag{3.3}$$

The Chebyshev expansions of terms

$$x^n y^m u^{(p,t)} \quad (p = 0, 1, 2; n = 0, 1, \dots, N; m = 0, 1, \dots, N)$$

in Equation (3.3) are obtained by means of the formulae

$$\begin{aligned} x^n u^{(p,0)}(x, y) &= \sum_{r=0}^N \sum_{s=0}^N \sum_{i=0}^n 2^{-n} \binom{n}{i} a_{|r-n+2i|,s}^{(p,0)} T_{r,s}(x, y), \\ y^m u^{(p,0)}(x, y) &= \sum_{r=0}^N \sum_{s=0}^N \sum_{j=0}^m 2^{-m} \binom{m}{j} a_{r,|s-m+2j|}^{(p,0)} T_{r,s}(x, y), \\ x^n y^m u^{(p,0)}(x, y) &= \sum_{r=0}^N \sum_{s=0}^N \sum_{i=0}^n \sum_{j=0}^m 2^{-n-m} \binom{n}{i} \binom{m}{j} a_{|r-n+2i|,|s-m+2j|}^{(p,0)} T_{r,s}(x, y), \\ x^n y^m u^{(0,t)}(x, y) &= \sum_{r=0}^N \sum_{s=0}^N \sum_{i=0}^n \sum_{j=0}^m 2^{-n-m} \binom{n}{i} \binom{m}{j} a_{|r-n+2i|,|s-m+2j|}^{(0,t)} T_{r,s}(x, y), \\ x^n y^m u^{(p,t)}(x, y) &= \sum_{r=0}^N \sum_{s=0}^N \sum_{i=0}^n \sum_{j=0}^m 2^{-n-m} \binom{n}{i} \binom{m}{j} a_{|r-n+2i|,|s-m+2j|}^{(p,t)} T_{r,s}(x, y), \end{aligned} \tag{3.4}$$

### 4. Matrix Forms of Terms in the Equation

The matrix representation of Equation (3.4) can be given by

$$x^p y^v u^{(0,0)} = T_x M_p A (M_v)^t T_y \tag{4.1}$$

$$x^p y^v u^{(1,0)} = T_x M_p 2MA (M_v)^t T_y \tag{4.2}$$

$$x^p y^v u^{(0,1)} = T_x M_p A (M_v 2M)^t T_y \tag{4.3}$$

$$x^p y^v u^{(2,0)} = T_x M_p 2^2 M^2 A (M_v)^t T_y \tag{4.4}$$

$$x^p y^v u^{(0,2)} = T_x M_p A (M_v 2^2 M^2)^t T_y \tag{4.5}$$

$$x^p y^v u^{(1,1)} = T_x M_p 2MA (M_v 2M)^t T_y \tag{4.6}$$

$$\sum_{n=0}^N \sum_{m=0}^N g_{n,m} x^n y^m = G(x, y) = T_x S T_y \tag{4.7}$$

where  $S = D^t \bar{G} D = [s_{i,j}]$  and  $\bar{G} = [g_{i,j}]$ ,  $i, j = 0, 1, 2, \dots, N$ .

And for  $p = 0, 1, 2, \dots, N$ ;  $v = 0, 1, 2, \dots, N$ ,  $M_p = [m_{i,j}]$  ( $i = 0, 1, \dots, N+1$  and  $j = 0, 1, \dots, N+1$ ) is a matrix of size  $(N+1) \times (N+1)$ . The elements of  $M_p$  are given in [6].

Substituting the expressions (4.1)-(4.7) into Equation (3.3), and simplifying the result, we have the matrix equation

$$\sum_{n=0}^N \sum_{m=0}^N \left[ 4a_{n,m} M_n M^2 A(M_m)^t + 4b_{n,m} M_n M A(M_m M^2)^t + 4c_{n,m} M_n A(M_m M^2)^t + 2d_{n,m} M_n M A(M_m)^t + 2e_{n,m} M_n A(M_m M^2)^t + f_{n,m} M_n A(M_m)^t \right] = S \tag{4.8}$$

Which corresponds to a system  $(N+1) \times (N+1)$  algebraic equations for the unknown Chebyshev coefficients  $a_{n,m}; n, m = 0, 1, \dots, N$ . Briefly, we can assume that Equation (4.8) is given in the form

$$\sum_{\sigma=1}^6 W_{\sigma} A Y_{\sigma} = S \tag{4.9}$$

$$C = [a_{00} \ a_{01} \ \dots \ a_{0N} \ a_{10} \ a_{11} \ \dots \ a_{1N} \ \dots \ a_{N0} \ a_{N1} \ \dots \ a_{NN}]^t, \\ \bar{S} = [s_{00} \ s_{01} \ \dots \ s_{0N} \ s_{10} \ s_{11} \ \dots \ s_{1N} \ \dots \ s_{N0} \ s_{N1} \ \dots \ s_{NN}]^t$$

**5. Matrix Forms of Conditions**

Let the conditions of Equation (1.1) be given by

$$u(x, -1) + u(x, 0) + u(x, 1) + u^{(0,1)}(x, -1) + u^{(0,1)}(x, 0) + u^{(0,1)}(x, 1) = f(x) \tag{5.1}$$

$$u(-1, y) + u(0, y) + u(1, y) + u^{(1,0)}(-1, y) + u^{(1,0)}(0, y) + u^{(1,0)}(1, y) = g(x) \tag{5.2}$$

$$u(-1, -1) + u(-1, 0) + u(-1, 1) + u(0, -1) + u(0, 0) + u(0, 1) + u(1, -1) + u(1, 0) + u(1, 1) = \lambda \tag{5.3}$$

where  $f$  is a function of  $x$ ,  $g$  is a function of  $y$  and  $\lambda$  is constant.

Then, there are the following matrix forms at  $x = -1, 0, 1$  and similar way for  $y = -1, 0, 1$ ;

$$T_x(-1) = [1 \ -1 \ 1 \ -1 \ \dots], \\ T_x(0) = [1 \ 0 \ -1 \ 0 \ 1 \ \dots], \\ T_x(1) = [1 \ 1 \ 1 \ 1 \ \dots]$$

Derivative of  $T_x$  at  $x = -1, 0, 1$  and similar way for  $y = -1, 0, 1$ ;

$$T_x^{(1)}(-1) = [0 \ 1 \ -2^2 \ 3^2 \ -4^2 \ 5^2 \ \dots], \\ T_x^{(1)}(0) = [0 \ 1 \ 0 \ -3 \ 0 \ 5 \ 0 \ -7 \ \dots], \\ T_x^{(1)}(1) = [0 \ 1^2 \ 2^2 \ 3^2 \ 4^2 \ 5^2 \ \dots]$$

We assume that the functions  $f(x)$  and  $g(y)$  can be expanded as

$$f(x) = \sum_{r=0}^N f_r T_r(x)$$

and

where  $W_{\sigma} = [w_{m,n}]$ ,  $Y_{\sigma} = [y_{m,n}]$ ,  $\sigma = 1, 2, \dots, 6$

Matrix Equation (4.9) can be reduce to new matrix equation by making use of

$$x_i(N+1) + \ell j(N+1) + k = w_{i,j} y_{k,\ell}; \ k = 0, 1, 2, \dots, N, \\ j = 0, 1, 2, \dots, N, \ \ell = 0, 1, 2, \dots, N, \ i = 0, 1, 2, \dots, N$$

Then the new matrix Equation (4.7) becomes

$$\sum_{\sigma=1}^6 X_{\sigma} C = \bar{S} \tag{4.10}$$

where

$$X_{\sigma} = [x_{z,q}], \ z, q = 0, 1, 2, \dots, (N+1)(N+1), \\ \sigma = 1, 2, \dots, 6$$

and

$$g(y) = \sum_{s=0}^N g_s T_s(y)$$

or in the matrix form

$$f(x) = T_x f$$

and

$$g(y) = g T_y$$

where

$$f = \left[ \frac{1}{2} f_0 \ f_1 \ \dots \ f_N \right]^t, \\ g = \left[ \frac{1}{2} g_0 \ g_1 \ \dots \ g_N \right]^t$$

In addition, at  $x = -1, 0, 1$  and  $y = -1, 0, 1$ , we obtain the matrix forms

$$u(x, -1) = T_x \cdot A \cdot T_y(-1), \\ u(x, 0) = T_x \cdot A \cdot T_y(0), \\ u(x, 1) = T_x \cdot A \cdot T_y(1),$$

$$\begin{aligned}
 u^{(0,1)}(x,-1) &= T_x \cdot A \cdot T_y^{(1)}(-1), \\
 u^{(0,1)}(x,0) &= T_x \cdot A \cdot T_y^{(1)}(0), \\
 u^{(0,1)}(x,1) &= T_x \cdot A \cdot T_y^{(1)}(1), \\
 u(1,-1) &= T_x(1) \cdot A \cdot T_y(-1), \\
 u(-1,y) &= T_x(-1) \cdot A \cdot T_y, \\
 u(0,y) &= T_x(0) \cdot A \cdot T_y, \\
 u(1,y) &= T_x(1) \cdot A \cdot T_y, \\
 u^{(1,0)}(-1,y) &= T_x^{(1)}(-1) \cdot A \cdot T_y, \\
 u^{(1,0)}(0,y) &= T_x^{(1)}(0) \cdot A \cdot T_y, \\
 u^{(1,0)}(1,y) &= T_x^{(1)}(1) \cdot A \cdot T_y, \\
 u(1,0) &= T_x^{(1)}(1) \cdot A \cdot T_y(0), \\
 u(-1,-1) &= T_x(-1) \cdot A \cdot T_y(-1), \\
 u(-1,0) &= T_x(-1) \cdot A \cdot T_y(0), \\
 u(-1,1) &= T_x(-1) \cdot A \cdot T_y(1), \\
 u(0,-1) &= T_x(0) \cdot A \cdot T_y(-1), \\
 u(0,0) &= T_x(0) \cdot A \cdot T_y(0), \\
 u(0,1) &= T_x(0) \cdot A \cdot T_y(1),
 \end{aligned}$$

Substituting these matrices forms into conditions (5.1)-(5.3), and then simplifying, we get the fundamental matrix equations of conditions as follows:

$$AU = f, VA = g \text{ and } QAZ = \lambda \tag{5.4}$$

where

$$\begin{aligned}
 U &= T_y(-1) + T_y(0) + T_y(1) + T_y^{(1)}(-1) + T_y^{(1)}(0) + T_y^{(1)}(1) \\
 V &= T_x(-1) + T_x(0) + T_x(1) + T_x^{(1)}(-1) + T_x^{(1)}(0) + T_x^{(1)}(1) \\
 Q &= T_x(-1) + T_x(0) + T_x(1), \\
 Z &= T_y^{(1)}(-1) + T_y^{(1)}(0) + T_y^{(1)}(1)
 \end{aligned}$$

### 6. Former Method for the Solution

We can assume that Equation (6.1) is in the form

$$\bar{X} \cdot C = \bar{S} \tag{6.1}$$

where  $\bar{X} = \sum_{\sigma=1}^6 X_{\sigma}$ .

Then the augmented matrix of Equation (6.1) becomes  $[\bar{X}; \bar{S}]$  or

$$\left[ \begin{array}{cccc|c}
 \overline{x_{0,0}} & \overline{x_{0,1}} & \cdots & \overline{x_{0,N(N+2)}} & s_{0,0} \\
 \overline{x_{1,0}} & \overline{x_{1,1}} & \cdots & \overline{x_{1,N(N+2)}} & s_{0,1} \\
 \vdots & \vdots & \cdots & \vdots & \vdots \\
 \overline{x_{N(N+2),0}} & \overline{x_{N(N+2),1}} & \cdots & \overline{x_{N(N+2),N(N+2)}} & s_{N,N}
 \end{array} \right] \tag{6.2}$$

If we take the new matrix forms of the conditions as

$\bar{U}C = f$ ,  $\bar{V}C = g'$  and  $\bar{Q}C = \lambda$ , respectively, the augmented matrices of them become  $[\bar{U}; f]$ ,  $[\bar{V}; g']$  and  $[\bar{Q}; \lambda]$  or more clearly

$$\left[ \begin{array}{cccc|c}
 \overline{u_{0,0}} & \overline{u_{0,1}} & \cdots & \overline{u_{0,N(N+2)}} & \frac{1}{2}f_0 \\
 \overline{u_{1,0}} & \overline{u_{1,1}} & \cdots & \overline{u_{1,N(N+2)}} & f_1 \\
 \vdots & \vdots & \cdots & \vdots & \vdots \\
 \overline{u_{N,0}} & \overline{u_{N,1}} & \cdots & \overline{u_{N,N(N+2)}} & f_N
 \end{array} \right] \tag{6.3}$$

$$\left[ \begin{array}{cccc|c}
 \overline{v_{0,0}} & \overline{v_{0,1}} & \cdots & \overline{v_{0,N(N+2)}} & \frac{1}{2}g_0 \\
 \overline{v_{1,0}} & \overline{v_{1,1}} & \cdots & \overline{v_{1,N(N+2)}} & g_1 \\
 \vdots & \vdots & \cdots & \vdots & \vdots \\
 \overline{v_{N,0}} & \overline{v_{N,1}} & \cdots & \overline{v_{N,N(N+2)}} & g_N
 \end{array} \right] \tag{6.4}$$

and

$$\left[ \overline{q_{0,0}} \quad \overline{q_{0,1}} \quad \cdots \quad \overline{q_{0,N(N+2)}}; \quad \lambda \right] \tag{6.5}$$

Consequently, by replacing Equations (6.3)-(6.5) by the last  $2N + 1$  rows of Equation (6.2), we have the new augment matrix

$$\left[ \begin{array}{cccc|c}
 \overline{x_{0,0}} & \overline{x_{0,1}} & \cdots & \overline{x_{0,N(N+2)}} & s_{0,0} \\
 \overline{x_{1,0}} & \overline{x_{1,1}} & \cdots & \overline{x_{1,N(N+2)}} & s_{0,1} \\
 \vdots & \vdots & \cdots & \vdots & \vdots \\
 \overline{x_{N^2-1,0}} & \overline{x_{N^2-1,1}} & \cdots & \overline{x_{N^2-1,N(N+2)}} & s_{N-2,N-1} \\
 \overline{u_{0,0}} & \overline{u_{0,1}} & \cdots & \overline{u_{0,N(N+2)}} & \frac{1}{2}f_0 \\
 \vdots & \vdots & \cdots & \vdots & \vdots \\
 \overline{u_{N,0}} & \overline{u_{N,1}} & \cdots & \overline{u_{N,N(N+2)}} & f_N \\
 \overline{v_{0,0}} & \overline{v_{0,1}} & \cdots & \overline{v_{0,N(N+2)}} & \frac{1}{2}g_0 \\
 \vdots & \vdots & \cdots & \vdots & \vdots \\
 \overline{v_{N,0}} & \overline{v_{N,1}} & \cdots & \overline{v_{N,N(N+2)}} & g_N \\
 \overline{q_{0,0}} & \overline{q_{0,1}} & \cdots & \overline{q_{0,N(N+2)}} & \lambda
 \end{array} \right]$$

From the solution of this system we can find matrix C or matrix A.

### 7. Applications

The Chebyshev matrix method applied in this study is useful in finding approximate solutions of second-order linear partial differential equations in both homogeneous and non-homogeneous cases, in terms of Chebyshev polynomials. We illustrate it by the following examples.

**Example 1.** We now consider the problem [7]:

$$\begin{aligned}
 u_{tt} &= u_{xx} + 6; \\
 u(x,0) &= x^2, \\
 u_t(x,0) &= 4x
 \end{aligned} \tag{1}$$

And seek the solution in the form

$$u_t(x,t) = \sum_{r=0}^2 \sum_{s=0}^2 a_{r,s} T_{r,s}(x,t), \quad (2)$$

$$(x,t) \in [-1,1]$$

Then we obtain the matrix equation

$$A(2^2 M^2)^t - 2^2 M^2 A = 6R, \quad (3)$$

where

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And the condition matrices are

$$A \cdot T_t(0) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}^t, \quad (4)$$

$$A \cdot T_t^{(1)}(0) = [0 \ 4 \ 0]^t, \quad (5)$$

By replacing the new matrix form of Equations (4) and (5) in the new matrix form of Equation (3), we have the matrix equation under given conditions as follows:

$$\begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} a_{0,0} \\ \frac{1}{2} a_{0,1} \\ \frac{1}{2} a_{0,2} \\ \frac{1}{2} a_{1,0} \\ a_{1,1} \\ a_{1,2} \\ \frac{1}{2} a_{2,0} \\ a_{2,1} \\ a_{2,2} \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Hence, we obtain the augmented matrix

$$\begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 & -4 & 0 & 0; & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4; & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0; & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1; & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0; & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0; & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0; & 0 \end{bmatrix}$$

The solution of this system is

$$A = \begin{bmatrix} \frac{5}{2} & 0 & 2 \\ 0 & 4 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

and thereby the solution of the problem (1) becomes

$$u(x,t) = \frac{5}{2} T_{0,0}(x,t) + 2 T_{0,2}(x,t) + 4 T_{1,1}(x,t) + \frac{1}{2} T_{2,0}(x,t)$$

or

$$u(x,t) = (x+2t)^2$$

This is exact solution [7].

**Example 2**

Let us now study the equation

$$x^2 u_{xx} - y^2 u_{yy} = 0$$

with conditions which are

$$u(0,y) = 1, \quad u(1,y) = e^y, \\ u(x,0) = 1, \quad u(x,1) = e^x,$$

The first four terms of the series expansions:

$$u(0,y) = 1, \quad u(1,y) = e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6}, \\ u(x,0) = 1, \quad u(x,1) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6},$$

Chebyshev matrix forms of the conditions,

$$[1 \ 0 \ -1 \ 0 \ 1 \ 0]A = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$[1 \ 1 \ 1 \ 1 \ 1 \ 1]A = \begin{bmatrix} \frac{5}{4} & \frac{9}{8} & \frac{1}{4} & \frac{1}{24} & 0 & 0 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ \frac{9}{8} \\ \frac{1}{4} \\ \frac{1}{24} \\ 0 \\ 0 \end{bmatrix}$$

Matrix form of the equation is

$$M_2 4M^2 A - A(M_2 4M^2)^t = 0$$

$$\begin{bmatrix}
 0 & 0 & \frac{1}{2} & 0 & 7 & 0 \\
 0 & 0 & 0 & \frac{9}{2} & 0 & \frac{55}{2} \\
 0 & 0 & \frac{1}{2} & 0 & 10 & 0 \\
 0 & 0 & 0 & \frac{3}{2} & 0 & \frac{35}{2} \\
 0 & 0 & 0 & 0 & 3 & 0 \\
 0 & 0 & 0 & 0 & 0 & 5
 \end{bmatrix} A$$

$$-A \begin{bmatrix}
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
 0 & \frac{9}{2} & 0 & \frac{3}{2} & 0 & 0 \\
 7 & 0 & 10 & 0 & 3 & 0 \\
 0 & \frac{55}{2} & 0 & \frac{35}{2} & 0 & 5
 \end{bmatrix} = 0$$

From the solution of this matrix equation under the given conditions, we get the Chebyshev coefficients matrix as

$$A = \begin{bmatrix}
 \frac{9}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 \\
 0 & \frac{35}{32} & 0 & \frac{1}{32} & 0 & 0 \\
 \frac{1}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 \\
 0 & \frac{1}{32} & 0 & \frac{1}{96} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}$$

The solution of problem is obtained as

$$u(x, y) = 1 + xy + \frac{(x, y)^2}{2} + \frac{(x, y)^3}{6},$$

Which is the first four terms of  $e^{(xy)}$ .

### 8. Conclusions

Analytic solutions of the second order linear partial differential equations with variable coefficients are usually

difficult. In many cases, it is required to approximate solutions. For this purpose, the Chebyshev matrix method can be proposed.

In this study, the usefulness of the Chebyshev matrix method presented for the approximate solution of the second order linear partial differential equations is discussed. Also, the method can be applied to both the non-homogeneous and homogeneous cases.

A considerable advantage of the method is that the solution is expressed as a truncated Chebyshev series and thereby a Taylor polynomial. Furthermore, after calculation of the series coefficients, the solution  $u(x, y)$  of the equations can be easily evaluated for arbitrary values of  $(x, y)$  at low computation effort.

An interesting feature of the Chebyshev matrix method is that the method can be used in finding exact solutions in much cases. The method can be also extending to the solution of the higher order linear partial differential equations.

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