Applied Mathematics, 2012, 3, 997-1000
http://dx.doi.org/10.4236/am.2012.39147 Published Online September 2012 (http://www.SciRP.org/journal/am)

An Instability Result to a Certain Vector Differential Equation of the Sixth Order

Cemil Tunç
Department of Mathematics, Faculty of Sciences, Yüzüncü Yıl University, Van, Turkey
Email: cemtunc@yahoo.com

Received August 12, 2012; revised September 12, 2012; accepted September 17, 2012

ABSTRACT

The nonlinear vector differential equation of the sixth order with constant delay is considered in this article. New criteria for instability of the zero solution are established using the Lyapunov-Krasovskii functional approach and the differential inequality techniques. The result of this article improves previously known results.

Keywords: Vector; Nonlinear Differential Equation; Sixth Order; Lyapunov-Krasovskii Functional; Instability; Delay

1. Introduction

In 2008, E. Tunç and C. Tunç [1] proved a theorem on the instability of the zero solution of the sixth order nonlinear vector differential equation

$$\begin{aligned}&X^{(6)}(t)+AX^{(5)}(t)+BX^{(4)}(t)\\&+E\left(X(t),\dot{X}(t),\cdots,X^{(5)}(t)\right)\dddot{X}(t)+F\left(\dot{X}(t)\right)\ddot{X}(t)\\&+G\left(X(t),\dot{X}(t),\cdots,X^{(5)}(t)\right)\dot{X}(t)+H\left(X(t)\right)=0.\end{aligned}\tag{1}$$

The objective of this article is to investigate the instability of the zero solution of the sixth order nonlinear vector differential equation with constant delay, $\tau>0$,

$$\begin{aligned}&X^{(6)}(t)+AX^{(5)}(t)+BX^{(4)}(t)\\&+E\left(X(t),X(t-\tau),\dot{X}(t),\cdots,X^{(5)}(t-\tau)\right)\dddot{X}(t)\\&+F\left(\dot{X}(t)\right)\ddot{X}(t)\\&+G\left(X(t),X(t-\tau),\dot{X}(t),\cdots,X^{(5)}(t-\tau)\right)\dot{X}(t)\\&+H\left(X(t-\tau)\right)=0,\end{aligned}\tag{2}$$

by the Lyapunov-Krasovskii functional approach under assumptions $X\in\Re^n$; A and B are constant $n\times n$-symmetric matrices; E, F and G are continuous $n\times n$-symmetric matrix functions depending, in each case, on the arguments shown; $H:\Re^n\to\Re^n$, $H(0)=0$ and H is continuous. Let $J_H(X)$ denote the Jacobian matrix corresponding to $H(X)$, that is,

$$J_H(X)=\left(\frac{\partial h_i}{\partial x_j}\right),\quad (i,j=1,2,\cdots,n),$$

where $(x_1,x_2,\cdots,x_n)$ and $(h_1,h_2,\cdots,h_n)$ are the components of X and H, respectively. We also assume that the Jacobian matrix $J_H(X)$ exists and is continuous.

It should be noted that Equation (2) is the vector version for systems of real nonlinear differential equations of the sixth order

$$\begin{aligned}&x_i^{(6)}+\sum_{k=1}^{n}a_{ik}x_k^{(5)}+\sum_{k=1}^{n}b_{ik}x_k^{(4)}\\&+\sum_{k=1}^{n}e_{ik}\left(x_1,x_1(t-\tau),\cdots,x_k(t-\tau),\cdots,x_1^{(5)},\cdots,x_k^{(5)}(t-\tau)\right)\dddot{x}_k\\&+\sum_{k=1}^{n}f_{ik}\left(x_1',\cdots,x_k'\right)x_k''\\&+\sum_{k=1}^{n}g_{ik}\left(x_1,x_1(t-\tau),\cdots,x_1^{(5)},\cdots,x_k^{(5)}(t-\tau)\right)x_k'\\&+h_i\left(x_1(t-\tau),\cdots,x_n(t-\tau)\right)=0,\ (i=1,2,\cdots,n).\end{aligned}$$

We can write Equation (2) in the system form

$$\begin{aligned}\dot{S}=&-AS-BU\\&-E\left(X,X(t-\tau),Y,\cdots,S(t-\tau)\right)W-F(Y)Z\\&-G\left(X,X(t-\tau),Y,\cdots,U,S(t-\tau)\right)Y\\&-H(X)+\int_{t-\tau}^{t}J_H\left(X(s)\right)Y(s)\,\mathrm{d}s,\end{aligned}\tag{3}$$

which is obtained from (2) by setting $\dot{X}=Y$, $\ddot{X}=Z$, $\dddot{X}=W$, $X^{(4)}=U$ and $X^{(5)}=S$. Throughout what follows $X(t),Y(t),\cdots,S(t)$ are abbreviated as $X,Y,\cdots,S$, respectively.

Consider, in the case $n=1$, the linear differential equation of the sixth order:

$$x^{(6)}+a_1x^{(5)}+a_2x^{(4)}+a_3\dddot{x}+a_4\ddot{x}+a_5\dot{x}+a_6x=0,\tag{4}$$

where $a_1,a_2,\cdots,a_6$ are real constants.

It is known from the qualitative properties of solutions of Equation (4) that the zero solution of this equation is unstable if and only if the associated auxiliary equation

$$\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0 \quad (5)$$

has at the least one root with a positive real part. The existence of such a root depends on (though not always all of) the coefficients $a_1, a_2, \cdots, a_6$ in Equation (5). Basing on the relations between the roots and the coefficients of Equation (5) it can be said that if

$$a_1 > 0,\, a_5 > \frac{1}{4}a_3^2 a_1^{-1}$$

or

$$a_4 > \frac{1}{2}a_2^2,\, a_6 < 0, \quad (6)$$

then at the least one root of Equation (5) has a positive real part for arbitrary values of a_1, a_3 and a_5 or a_2, a_4 and a_6, respectively.

It should be noted that Equation (2) is an *n*-dimensional generalization of Equation (4), and when we establish our assumptions, we will take into consideration the estimates in (6). The symbol $\langle X, Y\rangle$ corresponding to any pair *X*, *Y* in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, and $\lambda_i(A)$, $(i = 1, 2, \cdots, n)$, are the eigenvalues of the $n \times n$-matrix *A*.

It is worth mentioning that using the Lyapunov functions or Lyapunov-Krasovskii functionals and based on the Krasovskii properties [2], the instability of the solutions of the sixth order nonlinear scalar differential equations and the sixth order vector differential equations without delay were discussed by Ezeilo [3], Tejumola [4], Tiryaki [5] and Tunç [6-13]. The aim of this paper is to improve the results of ([1,3]) form the scalar and vector differential equations without delay to the sixth order nonlinear vector differential equation with delay, Equation (2).

2. Main Result

First, we give an algebraic result.

Lemma. Let *D* be a real symmetric $n \times n$-matrix. Then for any $X \in \Re^n$

$$\delta_d \|X\|^2 \le \langle DX, X\rangle \le \Delta_d \|X\|^2,$$

where δ_d and Δ_d are the least and greatest eigenvalues of D, respectively (Bellman [14]).

Let $r \ge 0$ be given, and let $C = C([-r,0], \Re^n)$ with

$$\|\phi\| = \max_{-r\le s\le 0} |\phi(s)|,\ \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r, A) \to \Re^n$ is continuous, $0 < A \le \infty$, then, for each *t* in $[0, A)$, x_t in *C* is defined by

$$x_t(s) = x(t+s),\, -r \le s \le 0,\, t \ge 0.$$

Let *G* be an open subset of *C* and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t),\, F(0) = 0,\, x_t = x(t+\theta),$$
$$-r \le \theta \le 0,\, t \ge 0,$$

where $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on *F* that each initial value problem

$$\dot{x} = F(x_t), \quad x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \le \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \ge 0$. The zero solution is said to be unstable if it is not stable.

The result to be proved is the following theorem.

Theorem. In addition to the basic assumptions imposed on *A, B, E, F, G* and *H* that appear in Equation (2), we suppose that there are constants a_1, a_2, a_6 and δ such that the following conditions hold:

The matrices *A, B, E, F, G* and $J_H(X)$ are symmetric and $\lambda_i(A)\operatorname{sgn} a_1 \ge a_1 \operatorname{sgn} a_1$, $\lambda_i(B) \ge a_2$, $H(0) = 0$, $H(X) \ne 0$, when $X \ne 0$, $-a_6 \le \lambda_i(J_H(.)) < 0$ and

$$\lambda_i(G(.))\operatorname{sgn} a_1 - \frac{1}{4|a_1|}(\lambda_i(E(.))^2 \ge \delta > 0,$$
$$(i = 1, 2, \cdots, n).$$

If

$$\tau < \frac{\delta}{\sqrt{n}a_6},$$

then the zero solution of Equation (2) is unstable.

Remark. It is worth mentioning that there is no sign restriction on eigenvalues of *F*, and it is obvious that for the delay case our assumptions also have a very simple form and their applicability can be easily verified.

Proof. Define a Lyapunov-Krasovskii functional

$$V_1(.) = V_1(X_t, Y_t, Z_t, W_t, U_t, S_t):$$

$$V_1 = V_0 \operatorname{sgn} a_1 - \lambda \int_{-\tau}^{0}\int_{t+s}^{t} \|Y(\theta)\|^2 \mathrm{d}\theta \mathrm{d}s,$$

where

$$V_0 = -\int_0^1 \langle \sigma F(\sigma Y)Y, Y\rangle \mathrm{d}\sigma - \int_0^1 \langle H(\sigma X), X\rangle \mathrm{d}\sigma$$
$$-\langle S, Y\rangle - \langle AU, Y\rangle - \langle BW, Y\rangle$$
$$+\frac{1}{2}\langle BZ, Z\rangle + \langle Z, U\rangle + \langle AW, Z\rangle$$
$$-\frac{1}{2}\langle W, W\rangle - \lambda \int_{-\tau}^{0}\int_{t+s}^{t} \|Y(\theta)\|^2 \mathrm{d}\theta \mathrm{d}s,$$

where λ is a certain positive constant and will be determined later in the proof.

It follows that

$$V_1(0,0,0,0,0,0)=0$$

and

$$\begin{aligned}&V_1\left(0,0,\varepsilon\operatorname{sgn}a_1,0,\left(1+|a_2|\right)\varepsilon,0\right)\\&=\left\{\frac{1}{2}\left\langle B\varepsilon\operatorname{sgn}a_1,\varepsilon\operatorname{sgn}a_1\right\rangle+\left\langle\varepsilon\operatorname{sgn}a_1,\left(1+|a_2|\right)\varepsilon\right\rangle\right\}\operatorname{sgn}a_1\\&=\frac{1}{2}\left\langle B\varepsilon,\varepsilon\right\rangle\operatorname{sgn}a_1+\left\langle\varepsilon,\left(1+|a_2|\right)\varepsilon\right\rangle\\&\ge-\frac{1}{2}|a_2|\|\varepsilon\|^2+\left(1+|a_2|\right)\|\varepsilon\|^2=\left(1+\frac{1}{2}|a_2|\right)\|\varepsilon\|^2>0\end{aligned}$$

for all arbitrary $\varepsilon\neq0$, $\varepsilon\in\Re^n$ so that the property (P_1) of Krasovskii [2] holds.

Using a basic calculation, the time derivative of V_1 along solutions of (3) results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_1(.)=&\left\{\left\langle AW,W\right\rangle+\left\langle F(Y)Z,Y\right\rangle\right.\\&+\left\langle H(X),Y\right\rangle+\left\langle E(.)W,Y\right\rangle+\left\langle G(.)Y,Y\right\rangle\\&-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left\langle\sigma F(\sigma Y)Y,Y\right\rangle\mathrm{d}\sigma\\&\left.-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left\langle H(\sigma X),X\right\rangle d\sigma\right\}\operatorname{sgn}a_1\\&-\left\langle\int_{t-\tau}^{t}J_H\left(X(s)\right)Y(s)\mathrm{d}s,Y\right\rangle\\&-\lambda\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\tau}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2\mathrm{d}\theta\mathrm{d}s.\end{aligned}$$

The following estimates can be easily calculated:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left\langle H(\sigma X),X\right\rangle\mathrm{d}\sigma=\left\langle H(X),Y\right\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left\langle\sigma F(\sigma Y)Y,Y\right\rangle\mathrm{d}\sigma=\left\langle F(Y)Z,Y\right\rangle,$$

$$\begin{aligned}&\left\langle\int_{t-\tau}^{t}J_H\left(X(s)\right)Y(s)\mathrm{d}s,Y\right\rangle\\&\ge-\|Y\|\left\|\int_{t-\tau}^{t}J_H\left(X(s)\right)Y(s)\mathrm{d}s\right\|\\&\ge-\sqrt{n}a_6\|Y\|\left\|\int_{t-\tau}^{t}Y(s)\right\|\mathrm{d}s\ge-\sqrt{n}a_6\|Y\|\int_{t-\tau}^{t}\|Y(s)\|\mathrm{d}s\\&\ge-\frac{1}{2}\sqrt{n}a_6\tau\|Y\|^2-\frac{1}{2}\sqrt{n}a_6\int_{t-\tau}^{t}\|Y(s)\|^2\mathrm{d}s\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\tau}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2\mathrm{d}\theta\mathrm{d}s=\|Y\|^2\tau-\int_{t-\tau}^{t}\|Y(\theta)\|^2\mathrm{d}\theta$$

so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_1(.)\ge&\left\{\left\langle AW,W\right\rangle+\left\langle E(.)W,Y\right\rangle+\left\langle G(.)Y,Y\right\rangle\right\}\operatorname{sgn}a_1\\&-\left(\lambda+\frac{1}{2}\sqrt{n}a_6\right)\tau\|Y\|^2+\left(\lambda-\frac{1}{2}\sqrt{n}a_6\right)\int_{t-\tau}^{t}\|Y(\theta)\|^2\mathrm{d}\theta.\end{aligned}$$

Using the assumptions of the theorem, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_1(.)\ge&\left\{\left\langle a_1W,W\right\rangle+\left\langle E(.)W,Y\right\rangle+\left\langle G(.)Y,Y\right\rangle\right\}\operatorname{sgn}a_1\\&-\left(\lambda+\frac{1}{2}\sqrt{n}a_6\right)\tau\|Y\|^2+\left(\lambda-\frac{1}{2}\sqrt{n}a_6\right)\int_{t-\tau}^{t}\|Y(\theta)\|^2\mathrm{d}\theta\\=&\,a_1\left\|W+\frac{1}{2|a_1|}E(.)Y\operatorname{sgn}a_1\right\|^2-\frac{1}{4|a_1|}\left\langle E(.)Y,E(.)Y\right\rangle\\&+\left\langle G(.)Y,Y\right\rangle\operatorname{sgn}a_1-\left(\lambda+\frac{1}{2}\sqrt{n}a_6\right)\tau\|Y\|^2\\&+\left(\lambda-\frac{1}{2}\sqrt{n}a_6\right)\int_{t-\tau}^{t}\|Y(\theta)\|^2\mathrm{d}\theta\\\ge&-\frac{1}{4|a_1|}\left\langle E(.)Y,E(.)Y\right\rangle+\left\langle G(.)Y,Y\right\rangle\operatorname{sgn}a_1\\&-\left(\lambda+\frac{1}{2}\sqrt{n}a_6\right)\tau\|Y\|^2+\left(\lambda-\frac{1}{2}\sqrt{n}a_6\right)\int_{t-\tau}^{t}\|Y(\theta)\|^2\mathrm{d}\theta\\\ge&\left\{\delta-\left(\lambda+\frac{1}{2}\sqrt{n}a_6\right)\tau\right\}\|Y\|^2+\left(\lambda-\frac{1}{2}\sqrt{n}a_6\right)\int_{t-\tau}^{t}\|Y(\theta)\|^2\mathrm{d}\theta.\end{aligned}$$

Let

$$\lambda=\frac{1}{2}\sqrt{n}a_6$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_1(.)\ge\left(\delta-\sqrt{n}a_6\tau\right)\|Y\|^2\ .$$

If $\tau<\dfrac{\delta}{\sqrt{n}a_6}$, then, for a positive constant k_1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_1(.)\ge k_1\|Y\|^2\ge0$$

so that the property (P_2) of Krasovskii [2] holds.

It is seen that

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}V_1(.)=0\Leftrightarrow Y=\dot{X},Z=\dot{Y}=0,W=\dot{Z}=0,\\U=\dot{W}=0,S=\dot{U}=0\text{ for all }t\ge0\end{gathered}$$

so that

$$X=\xi,Y=Z=W=U=S=0.$$

Using these estimates in (3) and the assumptions of the theorem, we get $H(\xi)=0\Leftrightarrow\xi=0$. Thus, we have

$X = Y = Z = W = U = T = 0$ for all $t \geq 0$. So that the property (P_3) of Krasovskii [2] holds.

The proof of the theorem is complete.

Example. For the particular case $n = 2$ in Equation (2), we have

$$A = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix},$$

$$\lambda_1(A) = 9, \quad \lambda_2(A) = 11,$$

$$\lambda_i(A) \operatorname{sgn} a_1 \geq 9 = a_1 \operatorname{sgn} a_1,$$

$$E(.) = \begin{bmatrix} 2 + \dfrac{4}{1 + x_1^2 + \cdots + s_1^2} & 0 \\ 0 & 2 + \dfrac{4}{1 + x_2^2 + \cdots + s_2^2} \end{bmatrix},$$

$$\lambda_1(E(.)) = 2 + \frac{4}{1 + x_1^2 + \cdots + s_1^2},$$

$$\lambda_2(E(.)) = 2 + \frac{4}{1 + x_2^2 + \cdots + s_2^2},$$

$$\lambda_i(E(.))^2 \leq 36,$$

$$G(.) = \begin{bmatrix} 6 + x_1^2 + \cdots + s_1^2 & 0 \\ 0 & 6 + x_2^2 + \cdots + s_2^2 \end{bmatrix},$$

$$\lambda_1(G(.)) = 6 + x_1^2 + \cdots + s_1^2,$$

$$\lambda_2(G(.)) = 6 + x_2^2 + \cdots + s_2^2,$$

$$\lambda_i(G(.)) \operatorname{sgn} a_1 - \frac{1}{4|a_1|}\left(\lambda_i(E(.))\right)^2 \geq 5 = \delta > 0,$$

$$H(X(t-\tau)) = \begin{bmatrix} -9x_1(t-\tau) \\ -9x_2(t-\tau) \end{bmatrix},$$

$$J_H(X) = \begin{bmatrix} -9 & 0 \\ 0 & -9 \end{bmatrix},$$

$$\lambda_i(J_H(.)) = 9,$$

$$-a_6 = -9 \leq \lambda_i(J_H(.)) < 0, \quad (i = 1, 2).$$

If

$$\tau < \frac{5}{9\sqrt{2}},$$

then all the assumptions of the theorem hold.

REFERENCES

[1] E. Tunç and C. Tunç, "On the Instability of Solutions of Certain Sixth-Order Nonlinear Differential Equations," *Nonlinear Stud*, Vol. 15, No. 3, 2008, pp. 207-213.

[2] N. N. Krasovskii, "Stability of Motion. Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay," Stanford University Press, Stanford, 1963.

[3] J. O. C. Ezeilo, "An Instability Theorem for a Certain Sixth Order Differential Equation," *Journal of the Australian Mathematical Society*, Vol. 32, No. 1, 1982, pp. 129-133. doi:10.1017/S1446788700024460

[4] H. O. Tejumola, "Instability and Periodic Solutions of Certain Nonlinear Differential Equations of Orders Six and Seven," *Proceedings of the National Mathematical Centre*, National Mathematical Center, Abuja, 2000.

[5] A. Tiryaki, "An Instability Theorem for a Certain Sixth Order Differential Equation," *Indian Journal of Pure and Applied Mathematics*, Vol. 21, No. 4, 1990, pp. 330-333.

[6] C. Tunç, "An Instability Result for Certain System of Sixth Order Differential Equations," *Applied Mathematics and Computation*, Vol. 157, No. 2, 2004, pp. 477-481. doi:10.1016/j.amc.2003.08.046

[7] C. Tunç, "On the Instability of Certain Sixth-Order Nonlinear Differential Equations," *Electronic Journal of Differential Equations*, No. 117, 2004, p. 6.

[8] C. Tunç, "On the Instability of Solutions to a Certain Class of Non-Autonomous and Non-Linear Ordinary Vector Differential Equations of Sixth Order," *Albanian Journal of Mathematics*, Vol. 2, No. 1, 2008, pp. 7-13.

[9] C. Tunç, "A Further Result on the Instability of Solutions to a Class of Non-Autonomous Ordinary Differential Equations of Sixth Order," *Applications & Applied Mathematics*, Vol. 3, No. 1, 2008, pp. 69-76.

[10] C. Tunç, "New Results about Instability of Nonlinear Ordinary Vector Differential Equations of Sixth and Seventh Orders," *Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis*, Vol. 14, No. 1, 2007, pp. 123-136.

[11] C. Tunç, "Instability for a Certain Functional Differential Equation of Sixth Order," *Journal of the Indonesian Mathematical Society*, Vol. 17, No. 2, 2011, pp. 123-128.

[12] C. Tunç, "Instability Criteria for Solutions of a Delay Differential Equation of Sixth Order," *Journal of Advanced Research in Applied Mathematics*, Vol. 4, No. 2, 2012, pp. 1-7.

[13] C. Tunç, "An Instability Theorem for a Certain Sixth Order Nonlinear Delay Differential Equation," *Journal of the Egyptian Mathematical Society*, 2012, in Press.

[14] R. Bellman, "Introduction to Matrix Analysis," 2nd Edition, In: G. Golub, Ed., *Classics in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1997.

