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Restricted Three-Body Problem in Cylindrical Coordinates System

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ABSTRACT

In this paper, the equations of motion for spatial restricted circular three body problem will be established using the cylindrical coordinates. Initial value procedure that can be used to compute both the cylindrical and Cartesian coordinates and velocities is also developed.

Keywords: Spatial Restricted Circular Three Body Problem; Regularization

1. Introduction

Depending on the application, a certain coordinate system may be simpler to use than the Cartesian coordinate system. As an example, a physical problem with spherical symmetry defined in $\mathbf{R}^3$ (e.g., motion in the field of a point mass), is usually easier to solve in spherical polar coordinates than in Cartesian coordinates. Also, for instance, in the galactic rotation, cylindrical coordinates are usually adopted, while the spherical coordinates are suitable for the dynamics of globular clusters. In fact, the change of the dependent and/or independent variables for the differential equations of motion becomes of the focal point of researches in space dynamics. Some authors proposed successful methods to change of the dependent and/or independent variables so as to regularize the differential equations of motion. Of these, the method established by Stiefel and Scheifele, in 1971 [1]. Many studies on the applications of these devices for some orbital systems were done for the perturbed two body problem (e.g. [2-5]).

The great success of the these devices in regularizing the equations of motion for the perturbed two body problem ,and on the other hand, the importance of the three body problem in space dynamics (e.g [6]) and in stellar dynamics (e.g [7]), tempted us to develop the corresponding deceives for the three body problem.

The present paper represents the first phase of our studies towards establishing new differential equations for the different forms of the three body problem using some important coordinate systems.

The objective of the present paper, is to establish the equations of motion for spatial restricted circular three body problem in cylindrical coordinates system together with initial value procedure that can be used to compute both the cylindrical and Cartesian coordinates and velocities.

2. The Circular Restricted Three-Body Problem

If two of the bodies, say m_1 and m_2 in the three-body problem move in circular, coplanar orbits about their common center of mass and the mass of the third body is too small to affect the motion of the other bodies, the problem of the motion of the third body is called the circular ,restricted ,three body problem. The two revolving bodies are called the primaries, their masses are arbitrary but have such internal mass distributions that they may be considered point masses. The equations of motion of the third body in a dimensionless synodic rotating coordinate (x, y, z) system with the mean motion $n = 1$, are [8]

$$\ddot{x} - 2\dot{y} = \partial V/\partial x, \tag{1}$$

$$\ddot{y} + 2\dot{x} = \partial V/\partial y, \tag{2}$$

$$\ddot{z} = \partial V/\partial z, \tag{3}$$

where $V = V(x, y, z)$ is given as

$$V = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \tag{4}$$

μ denotes the mass of the smaller primary when the total mass of the primaries has been normalized to unity.

$$r_1^2 = \left(x + \mu\right)^2 + y^2 + z^2, \tag{5}$$

$$r_2^2 = (x + \mu - 1)^2 + y^2 + z^2. \tag{6}$$

3. The Equations of Motion in Cylindrical Coordinate System

In what follows we shall establish ,the differential equations for the spatial circular restricted three body problem in the cylindrical coordinate system (u_1, u_2, u_3) together with a list of the direct and the, inverse formulae of the transformation.

3.1. Coordinate and Velocity Transformations

$$x = u_1 \cos u_2 ; \quad y = u_1 \sin u_2 ; \quad z = u_3 \tag{7}$$

$$\dot{x} = \dot{u}_1 \cos u_2 - \dot{u}_2 u_1 \sin u_2;$$
$$\dot{y} = \dot{u}_1 \sin u_2 + \dot{u}_2 u_1 \cos u_2; \tag{8}$$
$$\dot{z} = \dot{u}_3 ,$$

where

$$0 \le u_1 < \infty, \quad -\pi < u_2 \le \pi, \quad -\infty < u_3 < \infty .$$

3.2. Inverse Transformations

From Equation (7) we have

$$u_1 = (x^2 + y^2)^{1/2}; \quad u_2 = \tan^{-1}\left(\frac{y}{x}\right); \quad u_3 = z . \tag{9}$$

From Equation (8) we get

$$\dot{u}_1 = \frac{(x\dot{x} + y\dot{y})}{u_1}; \quad \dot{u}_2 = \frac{(x\dot{y} - y\dot{x})}{u_1^2}; \quad \dot{u}_3 = \dot{z} . \tag{10}$$

where u_1 is given in terms of x and y by the first of Equation (9).

3.3. The Equations of Motion

The kinetic energy of a particle of unit mass in the cylindrical coordinate system is

$$T = \frac{1}{2}\left(\dot{u}_1^2 + u_1^2 \ \dot{u}_2^2 + \dot{u}_3^2 \right) \tag{11}$$

By using the transformation equations (Equation (7)), the gravitational potential V could be expressed in term of (u_1, u_2, u_3).

Using Lagrange's dynamical equations, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{u}_1}\right) - \frac{\partial T}{\partial u_1} = \frac{\partial V}{\partial u_1},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{u}_2}\right) - \frac{\partial T}{\partial u_2} = \frac{\partial V}{\partial u_2},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{u}_3}\right) - \frac{\partial T}{\partial u_3} = \frac{\partial V}{\partial u_3}.$$

Consequently, we deduce for the equations of motion in the cylindrical coordinate the forms

$$\ddot{u}_1 = u_1 \dot{u}_2^2 + \frac{\partial V}{\partial u_1} , \tag{12.1}$$

$$u_1^2 \ddot{u}_2 = -2u_1 \dot{u}_1 \dot{u}_2 + \frac{\partial V}{\partial u_2}, \tag{12.2}$$

$$\ddot{u}_3 = \frac{\partial V}{\partial u_3} . \tag{12.3}$$

where $\dfrac{\partial V}{\partial u_j}$ are given as

$$\frac{\partial V}{\partial u_j} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial u_j} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial u_j} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial u_j} ; j = 1,2,3, \tag{13}$$

$\partial x/\partial u_j$, $\partial y/\partial u_j$ and $\partial z/\partial u_j ; j = 1,2,3$ can be computed from Equation (7), while $\partial V/\partial x , \partial V/\partial y$ and $\partial V/\partial z$ can be computed from Equation (4), and we get

$$\frac{\partial V}{\partial u_1} = \frac{1}{(Q_2 Q_3)^{3/2}}\left((\mu-1)Q_3^{3/2}\left(Q_1 \cos u_2 + u_1 \sin^2 u_2\right)\right.$$
$$\left. - Q_2^{3/2}\left(\mu(Q_1-1)\cos u_2 + \left(\mu \sin^2 u_2 - Q_3^{3/2}\right)u_1\right)\right),$$

$$\frac{\partial V}{\partial u_2} = \frac{1}{(Q_2 Q_3)^{3/2}}\left(u_1 \sin u_2 \left(\mu Q_2^{3/2}\left(Q_1 - u_1 \cos u_2 - 1\right)\right.\right.$$
$$\left.\left. + (1-\mu)Q_3^{3/2}\left(Q_1 - u_1 \cos u_2\right)\right)\right),$$

$$\frac{\partial V}{\partial u_3} = \left(\frac{\mu-1}{Q_2^{3/2}} - \frac{\mu}{Q_3^{3/2}}\right).$$

where

$$Q_1 = \mu + u_1 \cos u_2,$$
$$Q_2 = Q_1^2 + u_1^2 \sin^2 u_2 + u_3^2,$$
$$Q_3 = 1 - 2Q_1 + Q_2.$$

4. Computational Developments

4.1. Initial Value Procedure

In what follows, we shall establish a procedure that can be used to compute $\forall t_0 \le t \le t_f$ (say) both:
1) the cylindrical coordinates and velocities $(u_1, u_2, u_3, \dot{u}_1, \dot{u}_2, \dot{u}_3)$, and
2) the Cartesian coordinates and velocities $(x, y, z, \dot{x}, \dot{y}, \dot{z})$.

So, such procedure is a double usefulness computational algorithm, for which a differential solver can be used for the cylindrical six order system to obtain $(u_1, u_2, u_3, \dot{u}_1, \dot{u}_2, \dot{u}_3)$. While the Cartesian coordinates and velocities $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ are obtained by the substitution in the direct transformation formulae (Equations (7) and (8)), rather than solving the six order system of

Equations (1), (2) and (3). By this way, great time can be saved.

This initial value procedure using sidereal cylindrical coordinate system will be described through its basic points, input, output and computational steps.

Input: 1) $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$ at $t = t_0$,

2) the final time $t = t_f$,

3) $\dfrac{\partial V}{\partial x} = F_1(x, y, z)$, $\dfrac{\partial V}{\partial y} = F_2(x, y, z)$,

$\dfrac{\partial V}{\partial z} = F_3(x, y, z)$.

Output: 1) $u_j; \dot{u}_j; j = 1, 2, 3$ from t_0 to t_f.

2) $x, y, z; \dot{x}, \dot{y}, \dot{z}$ from t_0 to t_f.

Computational Steps

1) Using the inverse transformation Equations (9) and (10) to find the initial values $u_{0j}; j = 1, 2, 3, 4, 5, 6$ using the given values $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$ at $t = t_0$.

2) Using the partial derivatives $\dfrac{\partial V}{\partial u_1}; \dfrac{\partial V}{\partial u_2}; \dfrac{\partial V}{\partial u_3}$ (functions of $u_j; j = 1, 2, 3$) to construct the analytical forms of Equation (12) as first order system in the form:

$$\dot{u}_1 = u_4$$
$$\dot{u}_2 = u_5$$
$$\dot{u}_3 = u_6$$
$$\dot{u}_4 = u_1 u_5^2 + \frac{\partial V}{\partial u_1}$$
$$\dot{u}_5 = \frac{-2u_4 u_5}{u_1} + \frac{1}{u_1^2} \frac{\partial V}{\partial u_2}$$
$$\dot{u}_6 = \frac{\partial V}{\partial u_3}$$

3) Using the initial conditions $u_{0j}; j = 1, 2, 3, 4, 5, 6$ from step 1 to solve numerically the above differential system for $u_j; j = 1, 2, \cdots, 6$ from t_0 to t_f, (note that $u_4 \equiv \dot{u}_1, u_5 \equiv \dot{u}_2, u_6 \equiv \dot{u}_3$).

4) Using $u_j; \dot{u}_j; j = 1, 2, 3$ from step 3 and the direct transformations Equations (7) and (8) to compute numerically x, y, z and $\dot{x}, \dot{y}, \dot{z}$ t_0 to t_f.

5) End.

5. Conclusion

In this paper, the equations of motion for spatial restricted circular three body problem was established in cylindrical coordinates system. Initial value procedure that can be used to compute both the cylindrical and Cartesian coordinates and velocities is also developed.

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