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On the Derivative of a Polynomial

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ABSTRACT

Certain refinements and generalizations of some well known inequalities concerning the polynomials and their derivatives are obtained.

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1. Introduction to the Statement of Results

Let $P_n(z)$ denote the space of all complex polynomials $P(z)=\sum_{j=1}^{n} a_j z^j$ of degree *n*. If $P\in P_n$, then

$$\max_{|z|=1}|P'(z)|\le n\max_{|z|=1}|P(z)| \tag{1}$$

and

$$\max_{|z|=R>1}|P(z)|\le R^n\max_{|z|=1}|P(z)|. \tag{2}$$

Inequality (1) is an immediate consequence of S.Bernstein's theorem (see [1]) on the derivative of a trigonometric polynomial. Inequality (2) is a simple deduction from the maximum modulus principle (see [2, p. 346] or [3, p. 137]).

Both the inequalities (1) and (2) are sharp and the equality in (1) and (2) holds if and only if $P(z)$ has all its zeros at the origin. It was shown by Frappier, Rahman and Ruscheweyh [4, Theorem 8] that if $P\in P_n$, then

$$\max_{|z|=1}|P'(z)|\le n\max_{1\le k\le 2n}\left|P\left(e^{ik\pi/n}\right)\right|. \tag{3}$$

Clearly (3) represents a refinement of (1), since the maximum of $|P(z)|$ on $|z|=1$ may be larger than the maximum of $|P(z)|$ taken over $(2n)^{th}$ roots of unity, as is shown by the simple example $P(z)=z^n+ia$, $a>0$.

A. Aziz [5] showed that the bound in (3) can be considerably improved. In fact proved that if $P\in P_n$, then for every given real α,

$$\max_{|z|=1}|P'(z)|\le\frac{n}{2}\left(M_\alpha+M_{\alpha+\pi}\right) \tag{4}$$

where

$$M_\alpha=\max_{1\le k\le n}\left|P\left(e^{i(\alpha+2k\pi)/n}\right)\right| \tag{5}$$

and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha+\pi$. The result is best possible and equality in (4) holds for $P(z)=z^n+re^{i\alpha}, -1\le r\le 1$.

Clearly inequality (4) is an interesting refinement of inequality (3) and hence of Bernstein inequality (1) as well.

If we restrict ourselves to the class of polynomials $P\in P_n$ having no zero in $|z|<1$, then the inequality (1) can be sharpened. In fact, P. Erdös conjectured and later P. D. Lax [6] (see also [7]) verified that if $P(z)\ne 0$ for $|z|<1$, then (1) can be replaced by

$$\max_{|z|=1}|P'(z)|\le\frac{n}{2}\max_{|z|=1}|P(z)|. \tag{6}$$

In this connection A. Aziz [5], improved the inequality (4) by showing that if $P\in P_n$ and $P(z)$ does not vanish in $|z|<1$, then for every real α,

$$\max_{|z|=1}|P'(z)|\le\frac{n}{2}\left(M_\alpha^2+M_{\alpha+\pi}^2\right)^{1/2} \tag{7}$$

where M_α is defined by (5). The result is best possible and equality in (7) holds for $P(z)=z^n+e^{i\alpha}$.

A. Aziz [5] also proved that if $P\in P_n$ and $P(z)\ne 0$ in $|z|<1$, then for every real α and $R>1$,

$$\max_{|z|=1}|P(Rz)-P(z)|\le\frac{R^n-1}{2}\left(M_\alpha^2+M_{\alpha+\pi}^2\right)^{1/2} \tag{8}$$

In this paper, we first present the following result which is a refinement of inequality (7).

Theorem 1. *If* $P\in P_n$, $P(z)$ *does not vanish in* $|z|<1$ *and* $m=\min_{|z|=1}|P(z)|$, *then for every real* α,

$$\max_{|z|=1}|P'(z)|\le\frac{n}{2}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m^2\right)^{1/2}. \tag{9}$$

where M_α *is defined by* (5). *The result is best possible and equality in* (9) *holds for* $P(z)=z^n+e^{i\alpha}$.

As an application of Theorem 1, we mention the corresponding improvement of (8).

Theorem 2. *If* $P \in P_n$, *and* $P(z) \neq 0$ *for* $|z|<1$ *and* $m = \min_{|z|=1}|P(z)|$ *then for every real* α *and* $R>1$,

$$\left|P(Rz)-P(z)\right| \leq \frac{R^n-1}{2}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m^2\right)^{1/2} \quad (10)$$

where M_α *is defined by* (5). *The result is best possible and equality in* (10) *holds for* $P(z)=z^n+e^{i\alpha}$.

Here we also consider the class of polynomials $P \in P_n$ having no zero in $|z|<k$, $k>0$ and present some generalizations of the inequalities (9) and (10). First we consider the case $k \geq 1$ and prove the following result which is a generalization of inequality (9).

Theorem 3. *If* $P \in P_n$ *does not vanish in* $|z|<k$, $k \geq 1$ *and* $m=\min_{|z|=k}|P(z)|$, *then for every real* α,

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{\sqrt{2(1+k^2)}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m^2\right)^{1/2} \quad (11)$$

where M_α *is defined by* (5).

Next result is a corresponding generalization of the inequality (10).

Theorem 4. *If* $P \in P_n$ *does not vanish in* $|z|<k$, $k \geq 1$ *and* $m=\min_{|z|=k}|P(z)|$, *then for every real* α *and* $R>1$,

$$\left|P(Rz)-P(z)\right| \leq \frac{R^n-1}{\sqrt{2(1+k^2)}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m^2\right)^{1/2} \quad (12)$$

where M_α *is defined by* (5).

Remark 1. For $k=1$, Theorem 3 and Theorem 4 reduces to the Theorem 1 and Theorem 2 respectively.

For the case $k \leq 1$, we have been able to prove:

Theorem 5. *If* $P \in P_n$, $P(z)$ *has no zero in* $|z|<k$, $k \leq 1$ *and* $m=\min_{|z|=k}|P(z)|$, *then for every real* α,

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{\sqrt{2(1+k^{2n})}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m^2\right)^{1/2}, \quad (13)$$

provided $|P'(z)|$ *and* $|Q'(z)|$ *attain maximum at the same point on* $|z|=1$ *where* $Q(z)=z^n\overline{P(1/\overline{z})}$. *The result is best possible and equality in* (13) *holds for* $P(z)=z^n+k^n$.

Theorem 6. *If* $P \in P_n$, $P(z)$ *has no zero in* $|z|<k$, $k \leq 1$ *and* $m=\min_{|z|=k}|P(z)|$, *then for every real* α *and* $R>1$,

$$\left|P(Rz)-P(z)\right| \leq \frac{R^n-1}{\sqrt{2(1+k^{2n})}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m^2\right)^{1/2}, \quad (14)$$

provided $|P'(z)|$ *and* $|Q'(z)|$ *attain maximum at the same point on* $|z|=1$ *where* $Q(z)=z^n\overline{P(1/\overline{z})}$. *The result is best possible and equality in* (14) *holds for* $P(z)=z^n+k^n$.

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first Lemma is due to A. Aziz [5].

Lemma 1. *If* $P \in P_n$, *then for* $|z|=1$ *and for every real* α,

$$|P'(z)|^2+|nP(z)-zP'(z)|^2 \leq \frac{n^2}{2}\left(M_\alpha^2+M_{\alpha+\pi}^2\right) \quad (15)$$

where M_α *is defined by* (5).

Lemma 2. *If* $P \in P_n$ *and* $P(z) \neq 0$ *for* $|z|<k$, $k \geq 1$, *then for* $|z|=1$,

$$k|P'(z)| \leq |nP(z)-zP'(z)|-nm$$

where $m=\min_{|z|=k}|P(z)|$.

Lemma 2 is a special cases of a result due to A. Aziz and N. A. Rather [8, Lemma 5].

Lemma 3. *If* $P \in P_n$ *does not vanish in* $|z|<k$, $k \leq 1$, *then*

$$k^n|P'(z)| \leq \max_{|z|=1}|Q'(z)| \text{ for } |z|=1$$

where $Q(z)=z^n\overline{P(1/\overline{z})}$.

This Lemma is due to N. K. Govil [9].

Lemma 4. *If* $P(z)$ *is a polynomial of degree* n *which does not vanish in* $|z|<k$, $k \leq 1$, *then* for $|z|=1$

$$k^n|P'(z)|+n\min_{|z|=k}|P(z)| \leq \max_{|z|=1}|Q'(z)|$$

where $Q(z)=z^n\overline{P(1/\overline{z})}$.

Proof of Lemma 4. Let $m=\min_{|z|=k}|P(z)|$. If $P(z)$ has a zero on $|z|=k$, then $m=0$ and the result follows from Lemma 3. Henceforth we assume that $P(z)$ has no zero on $|z|=k$, therefore $m>0$ and

$$m \leq |P(z)| \text{ for } |z|=k.$$

If α is any real or complex number with $|\alpha|<1$, then for $|z|=k$,

$$\left|\alpha m z^n/k^n\right| < |P(z)|.$$

By Rouche's Theorem, it follows that the polynomial $F(z)=P(z)-\alpha m z^n/k^n$ does not vanish in $|z|<k$, for every real or complex number α with $|\alpha|<1$. Applying Lemma 3 to the polynomial $F(z)$, we get

$$k^n |F'(z)| \le \max_{|z|=1} |G'(z)| \text{ for } |z|=1. \quad (16)$$

where

$$G(z) = z^n \overline{P(1/\overline{z})} = z^n \overline{P(1/\overline{z})} - \overline{\alpha} m/k^n = Q(z) - \overline{\alpha} m/k^n.$$

Replacing $F(z)$ by $P(z) - \alpha m z^n/k^n$ and $G(z)$ by $Q(z) - \overline{\alpha} m/k^n$, we obtain from (16) for $|z|=1$,

$$k^n \left| P'(z) - n\alpha m z^{n-1}/k^n \right| \le \max_{|z|=1} |Q'(z)|. \quad (17)$$

Now choosing the argument of α in the left hand side of (17) such that

$$\left| P'(z) - n\alpha m z^{n-1}/k^n \right| = |P'(z)| + |nm\alpha/k^n|$$

we obtain for $|z|=1$,

$$k^n |P'(z)| + |\alpha| nm \le \max_{|z|=1} |Q'(z)|.$$

Letting $|\alpha| \to 1$, we get the desired result. This proves Lemma 4.

3. Proof of the Theorems

Proof of Theorem 1. By hypothesis $P(z)$ does not vanish in $|z|<1$ and $m = \min_{|z|=k} |P(z)|$, therefore, by Lemma 2 with $k=1$, we have

$$\left(|P'(z)| + nm\right)^2 \le |nP(z) - zP'(z)|^2 \text{ for } |z|=1.$$

This gives with the help of Lemma 1

$$|P'(z)|^2 + \left(|P'(z)| + nm\right)^2 \le |P'(z)|^2 + |nP(z) - zP'(z)|^2 \le \frac{n^2}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right).$$

Since

$$\left(|P'(z)| + nm\right)^2 = |P'(z)|^2 + n^2m^2 + 2nm|P'(z)| \ge |P'(z)|^2 + n^2m^2,$$

it follows that

$$2|P'(z)|^2 + n^2m^2 \le \frac{n^2}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right),$$

which implies for $|z|=1$

$$|P'(z)| \le \frac{n}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)^{1/2}$$

and hence

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)^{1/2}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Applying (2) to the polynomial $P'(z)$ which is of degree $n-1$ and using Theorem 1, we obtain for $t \ge 1$ and $0 \le \theta < 2\pi$,

$$\left|P'\left(te^{i\theta}\right)\right| \le t^{n-1} \max_{|z|=1} |P'(z)| \le \frac{n}{2} t^{n-1} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)^{1/2}.$$

Hence for each $\theta, 0 \le \theta < 2\pi$ and $R > 1$, we have

$$\left|P\left(\mathrm{Re}^{i\theta}\right) - P\left(e^{i\theta}\right)\right| = \left|\int_1^R e^{i\theta} P'\left(te^{i\theta}\right) \mathrm{d}t\right| \le \int_1^R \left|P'\left(te^{i\theta}\right)\right| \mathrm{d}t$$
$$\le \frac{1}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)^{1/2} \int_1^R nt^{n-1} \mathrm{d}t.$$
$$= \frac{1}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)^{1/2} \left(R^n - 1\right).$$

This implies for $|z|=1$ and $R>1$,

$$|P(Rz) - P(z)| \le \frac{R^n - 1}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)^{1/2},$$

which proves Theorem 2.

The proof of the Theorem 3 and 4 follows on the same lines as that of Theorems 1 and 2, so we omit the details.

Proof of Theorem 5. Since all the zeros of $P(z)$ lie in $|z| \ge k$, where $k \le 1$, $m = \min_{|z|=k} |P(z)|$, by Lemma 4, we have

$$k^n \max_{|z|=1} |P'(z)| + nm \le \max_{|z|=1} |Q'(z)|, \quad (18)$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$. Also by hypothesis $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z|=1$, if

$$\max_{|z|=1} |P'(z)| = \left|P'\left(e^{i\alpha}\right)\right|, 0 \le \alpha < 2\pi, \quad (19)$$

then

$$\max_{|z|=1} |Q'(z)| = \left|Q'\left(e^{i\alpha}\right)\right|, 0 \le \alpha < 2\pi \quad (20)$$

and it can be easily verified that

$$|Q'(z)| = |nP(z) - zP'(z)| \text{ for } |z|=1.$$

Therefore, by Lemma 1

$$\left|P'\left(e^{i\alpha}\right)\right|^2 + \left|Q'\left(e^{i\alpha}\right)\right|^2 = \left|P'\left(e^{i\alpha}\right)\right|^2 + \left|nP\left(e^{i\alpha}\right) - e^{i\alpha}P'\left(e^{i\alpha}\right)\right|^2 \le \frac{n^2}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right).$$

This gives with the help of (18), (19) and (20) that

$$\left|P'\left(e^{i\alpha}\right)\right|^2 + \left(k^n \left|P'\left(e^{i\alpha}\right)\right| + nm\right)^2 \le \left|P'\left(e^{i\alpha}\right)\right|^2 + \left|Q'\left(e^{i\alpha}\right)\right|^2 \le \frac{n^2}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right),$$

which implies,

$$\left|P'\left(e^{i\alpha}\right)\right|^2 + k^{2n}\left|P'\left(e^{i\alpha}\right)\right|^2 + n^2m^2 \le \frac{n^2}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right).$$

Equivalently,

$$\left|P'\left(e^{i\alpha}\right)\right|^2 \le \frac{n^2}{2\left(1+k^{2n}\right)}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)$$

and hence

$$\max_{|z|=1}\left|P'(z)\right| \le \frac{n}{\sqrt{2\left(1+k^{2n}\right)}}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)^{1/2}.$$

This completes the proof of Theorem 5.

Theorem 6 follows on the same lines as that of Theorem 2, so we omit the details.

REFERENCES

[1] A. C. Schaffer, "Inequalities of A. Markoff and S. Bernstein for Polynomials and Related Functions," *Bulletin of the American Mathematical Society*, Vol. 47, 1941, pp. 565-579. doi:10.1090/S0002-9904-1941-07510-5

[2] M. Riesz, "Uber Einen Satz des Herrn Serge Bernstein," *Acta Mathematica*, Vol. 40, 1916, pp. 337-347. doi:10.1007/BF02418550

[3] G. Pólya and G. Szegö, "Aufgaben und lehrsätze aus der Analysis," Springer-Verlag, Berlin, 1925.

[4] C. Frappier, Q. I. Rahman and St. Ruscheweyh, "New Inequalities for Polynomials," *Transactions of the American Mathematical Society*, Vol. 288, 1985, pp. 69-99. doi:10.1090/S0002-9947-1985-0773048-1

[5] A. Aziz, "A Refinement of an Inequality of S.Bernstein," *Journal of Mathematical Analysis and Applications*, Vol. 142, No. 1, 1989, pp. 226-235. doi:10.1016/0022-247X(89)90370-3

[6] P. D. Lax, "Proof of a Conjecture of P.Erdös on the Derivative of a Polynomial," *Bulletin of the American Mathematical Society*, Vol. 50, 1944, pp. 509-513. doi:10.1090/S0002-9904-1944-08177-9

[7] A. Aziz and Q. G. Mohammad, "Simple Proof of a Theorem of Erdos and Lax," *Proceedings of the American Mathematical Society*, Vol. 80, 1980, pp. 119-122.

[8] A. Aziz and N. A. Rather, "New L^q Inequalities for Polynomials," *Mathematical Inequalities and Applications*, Vol. 2, 1998, pp. 177-191. doi:10.7153/mia-01-16

[9] N. K. Govil and Q. I. Rahman, "Functions of Exponential Type Not Vanishing in a Half Plane and Related Polynomials," *Transactions of the American Mathematical Society*, Vol. 137, 1969, pp. 501-517. doi:10.1090/S0002-9947-1969-0236385-6

