

Exact Traveling Wave Solutions of Nonlinear PDEs in Mathematical Physics

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ABSTRACT

In the present article, we construct the exact traveling wave solutions of nonlinear PDEs in mathematical physics via the variant Boussinesq equations and the coupled KdV equations by using the extended mapping method and auxiliary equation method. This method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear partial differential equations arising in mathematical physics.

Keywords: Extended Mapping Method; Auxiliary Equation Method; The Variant Boussinesq Equations; The Coupled KdV Equations; Traveling Wave Solutions

1. Introduction

The effort in finding exact solutions to nonlinear equations is important for the understanding of most nonlinear physical phenomena. For instance, the nonlinear wave phenomena observed in fluid dynamics, plasma and optical fibers are often modeled by the bell shaped *sech* solutions and the kink shaped *tanh* solutions. Many effective methods have been presented, such as inverse scattering transform method [1], Bäcklund transformation [2], Darboux transformation [3], Hirota bilinear method [4], variable separation approach [5], various tanh methods [6-9], homogeneous balance method [10], similarity reductions method [11,12], (G'/G) -expansion method [13], the reduction mKdV equation method [14], the tri-function method [15,16], the projective Riccati equation method [17], the Weierstrass elliptic function method [18], the Sine-Cosine method [19,20], the Jacobi elliptic function expansion [21,22], the complex hyperbolic function method [23], the truncated Painlevé expansion [24], the F-expansion method [25], the rank analysis method [26], the ansatz method [27,28], the exp-function expansion method [29], the sub-ODE method [30], and so on.

The main objective of this paper is used the extended mapping method and auxiliary equation method to construct the exact solutions for nonlinear evolution equations in the mathematical physics via the variant Boussinesq equations and the coupled KdV equations.

2. Description of the Extended Mapping Method

Suppose we have the following nonlinear PDE:

$$F(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0, \quad (1)$$

where $u = u(x, t)$ is an unknown function, F is a polynomial in $u = u(x, t)$ and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of a deformation method:

Step 1. The traveling wave variable

$$u(x, t) = u(\xi), \quad \xi = k(x - \omega t), \quad (2)$$

where k and ω are the wave number and the wave speed, respectively. Under the transformation (2), Equation (1) becomes an ordinary differential equation (ODE) as

$$P(u, -k\omega u', ku', k^2\omega^2 u'', k^2 u'', \dots) = 0, \quad (3)$$

where $' = \frac{d}{d\xi}$.

Step 2. Suppose that the solution Equation (3) has the following form:

$$\begin{aligned} u(\xi) = & a_0 + \sum_{i=1}^n (a_i f^i(\xi) + b_i f^{-i}(\xi)) \\ & + \sum_{i=2}^n c_i f^{i-2}(\xi) f'(\xi) \\ & + \sum_{i=1}^n d_i f^i(\xi) f'(\xi), \end{aligned} \quad (4)$$

where a_0, a_i, b_i, c_i and d_i are constants to be determined later, while $f(\xi)$ satisfies the auxiliary ordinary differential equation:

$$[f'(\xi)]^2 = pf^4(\xi) + qf^2(\xi) + r, \quad (5)$$

where p, q and r are arbitrary constants.

Step 3. The positive integer “ n ” can be determined by considering the homogeneous balance between the highest derivative term and the nonlinear terms appearing in Equation (3). Therefore, we can get the value of n in Equation (4).

Step 4. Substituting Equation (4) into Equation (3) with the condition (5), we obtain polynomial in $f^i(\xi)[f'(\xi)]^j$, ($i = \dots, -2, -1, 0, 1, 2, \dots; j = 0, 1$). Setting each coefficient of this polynomial to be zero, yields a set of algebraic equations for $a_0, a_i, b_i, c_i, d_i, \omega$ and k .

Step 5. Solving the algebraic equations by use of Maple or Mathematica, we have a_0, a_i, b_i, c_i, d_i and k expressed by p, q, r .

Step 6. Since the general solutions of auxiliary Equation (5) have been well known for us (see Appendix A), then substituting the obtained coefficients and the general solution of Equation (5) into Equation (4), we have the travelling wave solutions of the nonlinear PDE (1).

3. Applications of the Method

In this section, we apply the extended mapping method to construct the exact solutions for the variant Boussinesq equations and the coupled KdV equations. Which are very important nonlinear evolution equations in mathematical physics and have been paid attention by many researchers.

3.1. Example 1. The Variant Boussinesq Equations

We start with the variant Boussinesq equations [31] in the following form:

$$v_t = \alpha u_{xxx} - u_x - (uv)_x, \tag{6}$$

$$u_t = 3\alpha u_{xxt} - v_x - uu_x, \tag{7}$$

where α is a constant. As models for water waves, u is the velocity and v is the total depth. Wang [32] obtained their solitary wave solutions by using homogeneous balance method while Fan *et al.* [33] got a series of new traveling wave solutions of this system by using an algebraic method. The traveling wave variables below

$$u(x, t) = u(\xi), v(x, t) = v(\xi), \xi = k(x - \omega t), \tag{8}$$

permit us converting the Equations (6) and (7) into ODEs for $u(x, t) = u(\xi)$ and $v(x, t) = v(\xi)$ as follows:

$$-\omega v' + u' + (uv)' - \alpha k^2 u''' = 0, \tag{9}$$

and

$$-\omega u' + uu' + v' + 3\alpha k^2 \omega u'' = 0, \tag{10}$$

On integrating Equations (9) and (10) with respect to ξ once, we get

$$L_1 - \omega v + u + uv - \alpha k^2 u'' = 0, \tag{11}$$

and

$$L_2 - \omega u + \frac{u^2}{2} + v + 3\alpha k^2 \omega u' = 0, \tag{12}$$

where L_1 and L_2 are integration constants.

Suppose that the solutions of Equations (11) and (12) can be expressed by

$$u(\xi) = a_0 + \sum_{i=1}^n (a_i f^i(\xi) + b_i f^{-i}(\xi)) + \sum_{i=2}^n c_i f^{i-2}(\xi) f'(\xi) + \sum_{i=-1}^{-n} d_i f^i(\xi) f'(\xi), \tag{13}$$

$$v(\xi) = \alpha_0 + \sum_{i=1}^m (\alpha_i f^i(\xi) + \beta_i f^{-i}(\xi)) + \sum_{i=2}^m \gamma_i f^{i-2}(\xi) f'(\xi) + \sum_{i=-1}^{-m} e_i f^i(\xi) f'(\xi), \tag{14}$$

where $a_0, a_i, b_i, c_i, d_i, \alpha_0, \alpha_i, \beta_i, \gamma_i$, and e_i are constants to be determined later.

Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Equations (11) and (12), we get $n = m = 2$, hence the exact solutions of Equations (11) and (12) can be rewritten as:

$$u(\xi) = a_0 + a_1 f(\xi) + b_1 \frac{1}{f(\xi)} + a_2 f^2(\xi) + b_2 \frac{1}{f^2(\xi)} + c_2 f'(\xi) + d_1 \frac{f'(\xi)}{f(\xi)} + d_2 \frac{f'(\xi)}{f^2(\xi)}, \tag{15}$$

and

$$v(\xi) = \alpha_0 + \alpha_1 f(\xi) + \beta_1 \frac{1}{f(\xi)} + \alpha_2 f^2(\xi) + \beta_2 \frac{1}{f^2(\xi)} + \gamma_2 f'(\xi) + e_1 \frac{f'(\xi)}{f(\xi)} + e_2 \frac{f'(\xi)}{f^2(\xi)}, \tag{16}$$

where $a_0, a_1, a_2, b_1, b_2, c_2, d_1, d_2, \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_2, e_1$ and e_2 are constants to be determined later. Substituting Equations (15) and (16) with the condition (5) into Equations (11) and (12) and collecting all terms with the same power of $f^i(\xi)[f'(\xi)]^j$, ($i = \dots, -2, -1, 0, 1, 2, \dots; j = 0, 1$). Setting each coefficients of this polynomial to be zero, we get a system of algebraic equations which can be solved by Maple or Mathe-

matica. Thus we obtain the two sets of solutions as:

Case 1:

$$\begin{aligned}
 a_0 = a_1 = b_1 = b_2 = d_1 = d_2 = \alpha_1 = \beta_1 = \beta_2 = e_1 = e_2 = 0, \\
 a_2 = -\frac{p(1+6p\omega^2)}{q\omega}, c_2 = -\frac{\sqrt{p}(1+6\omega^2)}{q\omega}, \\
 \alpha_0 = \frac{1}{18}\left(-15 + \frac{1}{\omega^2}\right), \alpha_2 = \frac{p(1+6\omega^2)}{6q\omega^2}, \\
 \gamma_2 = \frac{\sqrt{p}(1+6\omega^2)}{6q\omega^2}, L_1 = \frac{q^2\omega^2(1-15\omega^2) + pr(1+6\omega^2)^2}{18q^2\omega^3}, \\
 L_2 = \frac{-3pr(1+6\omega^2)^2 + q^2(15\omega^2 - 1)}{18q^2\omega^2}, k = \pm \frac{1}{3\omega} \sqrt{\frac{1+6\omega^2}{2q\alpha}}.
 \end{aligned}
 \tag{17}$$

Case 2:

$$\begin{aligned}
 a_0 = a_1 = b_1 = a_2 = c_2 = d_1 = \alpha_1 = \beta_1 = \alpha_2 = e_1 = 0, \\
 b_2 = -\frac{r(1+6r\omega^2)}{q\omega}, d_2 = \frac{\sqrt{r}(1+6\omega^2)}{q\omega}, \\
 \alpha_0 = \frac{1}{18}\left(-15 + \frac{1}{\omega^2}\right), \beta_2 = \frac{r(1+6\omega^2)}{6q\omega^2}, \\
 e_2 = -\frac{\sqrt{r}(1+6\omega^2)}{6q\omega^2}, L_1 = \frac{q^2\omega^2(1-15\omega^2) + pr(1+6\omega^2)^2}{18q^2\omega^3}, \\
 L_2 = \frac{-3pr(1+6\omega^2)^2 + q^2(15\omega^2 - 1)}{18q^2\omega^2}, k = \pm \frac{1}{3\omega} \sqrt{\frac{1+6\omega^2}{2q\alpha}}.
 \end{aligned}
 \tag{18}$$

Note that, there are other cases which are omitted here. Since the solutions obtained here are so many, we just list some of the exact solutions corresponding to Case 2 to illustrate the effectiveness of the extended mapping method.

Substituting (18) into Equation (15) and Equation (16) yields

$$\begin{aligned}
 u(\xi) = & -\frac{r(1+6r\omega^2)}{q\omega} \left(\frac{1}{f^2(\xi)} \right) \\
 & + \frac{\sqrt{r}(1+6\omega^2)}{q\omega} \left(\frac{f'(\xi)}{f^2(\xi)} \right),
 \end{aligned}
 \tag{19}$$

and

$$\begin{aligned}
 v(\xi) = & \frac{1}{18}\left(-15 + \frac{1}{\omega^2}\right) + \frac{r(1+6\omega^2)}{6q\omega^2} \left(\frac{1}{f^2(\xi)} \right) \\
 & - \frac{\sqrt{r}(1+6\omega^2)}{6q\omega^2} \left(\frac{f'(\xi)}{f^2(\xi)} \right),
 \end{aligned}
 \tag{20}$$

where

$$\xi = \pm \frac{1}{3\omega} \sqrt{\frac{1+6\omega^2}{2q\alpha}} (x - \omega t).
 \tag{21}$$

According to the Appendix A, we have the following families of exact solutions:

Family 1. If $r = 1$, $q = -(1+m^2)$, $p = m^2$, $f(\xi) = sn(\xi)$, then we get

$$u(\xi) = \frac{(1+6\omega^2)}{(1+m^2)\omega} [ns^2(\xi) - cs(\xi)ds(\xi)],
 \tag{22}$$

and

$$\begin{aligned}
 v(\xi) = & \frac{1}{18}\left(-15 + \frac{1}{\omega^2}\right) \\
 & - \frac{(1+6\omega^2)}{6\omega^2(1+m^2)} [ns^2(\xi) - cs(\xi)ds(\xi)],
 \end{aligned}
 \tag{23}$$

where

$$\xi = \pm \frac{i}{3\omega} \sqrt{\frac{1+6\omega^2}{2\alpha(1+m^2)}} (x - \omega t).
 \tag{24}$$

Family 2. If $r = 1-m^2$, $q = 2m^2 - 1$, $p = -m^2$, $f(\xi) = cn(\xi)$, then we get

$$\begin{aligned}
 u(\xi) = & \frac{(m^2 - 1)[1+6\omega^2(1-m^2)]}{(2m^2 - 1)\omega} nc^2(\xi) \\
 & - \frac{\sqrt{1-m^2}(1+6\omega^2)}{(2m^2 - 1)\omega} dc(\xi)sc(\xi),
 \end{aligned}
 \tag{25}$$

and

$$\begin{aligned}
 v(\xi) = & \frac{1}{18}\left(-15 + \frac{1}{\omega^2}\right) + \frac{(1-m^2)(1+6\omega^2)}{6\omega^2(2m^2 - 1)} nc^2(\xi) \\
 & + \frac{\sqrt{1-m^2}(1+6\omega^2)}{6\omega^2(2m^2 - 1)} dc(\xi)sc(\xi),
 \end{aligned}
 \tag{26}$$

where

$$\xi = \pm \frac{1}{3\omega} \sqrt{\frac{1+6\omega^2}{2\alpha(2m^2 - 1)}} (x - \omega t).
 \tag{27}$$

Family 3. If $r = m^2 - 1$, $q = 2 - m^2$, $p = -1$, $f(\xi) = dn(\xi)$, then we get

$$\begin{aligned}
 u(\xi) = & \frac{(1-m^2)[1+6\omega^2(m^2 - 1)]}{(2-m^2)\omega} nd^2(\xi) \\
 & - \frac{m^2\sqrt{m^2 - 1}(1+6\omega^2)}{(2-m^2)\omega} sd(\xi)cd(\xi),
 \end{aligned}
 \tag{28}$$

and

$$v(\xi) = \frac{1}{18} \left(-15 + \frac{1}{\omega^2} \right) + \frac{(m^2 - 1)(1 + 6\omega^2)}{6\omega^2(2 - m^2)} nd^2(\xi) + \frac{m^2 \sqrt{m^2 - 1}(1 + 6\omega^2)}{6\omega^2(2 - m^2)} sd(\xi)cd(\xi), \tag{29}$$

where

$$\xi = \pm \frac{1}{3\omega} \sqrt{\frac{1 + 6\omega^2}{2\alpha(2 - m^2)}} (x - \omega t). \tag{30}$$

Family 4. If $r = m^2$, $q = -(1 + m^2)$, $p = 1$, $f(\xi) = dc(\xi)$, then we get

$$u(\xi) = \frac{m^2(1 + 6r\omega^2)}{\omega(1 + m^2)} cd^2(\xi) - \frac{m(1 - m^2)(1 + 6\omega^2)}{\omega(1 + m^2)} sd(\xi)nd(\xi), \tag{31}$$

and

$$v(\xi) = \frac{1}{18} \left(-15 + \frac{1}{\omega^2} \right) - \frac{m^2(1 + 6\omega^2)}{6\omega^2(1 + m^2)} cd^2(\xi) + \frac{m(1 - m^2)(1 + 6\omega^2)}{6\omega^2(1 + m^2)} sd(\xi)nd(\xi), \tag{32}$$

where

$$\xi = \pm \frac{i}{3\omega} \sqrt{\frac{1 + 6\omega^2}{2\alpha(1 + m^2)}} (x - \omega t). \tag{33}$$

Family 5. If $r = 1$, $q = 2 - m^2$, $p = 1 - m^2$, $f(\xi) = sc(\xi)$, then we get

$$u(\xi) = -\frac{(1 + 6\omega^2)}{\omega(2 - m^2)} [cs^2(\xi) - ds(\xi)ns(\xi)], \tag{34}$$

and

$$v(\xi) = \frac{1}{18} \left(-15 + \frac{1}{\omega^2} \right) + \frac{(1 + 6\omega^2)}{6\omega^2(2 - m^2)} [cs^2(\xi) - ds(\xi)ns(\xi)], \tag{35}$$

where

$$\xi = \pm \frac{1}{3\omega} \sqrt{\frac{1 + 6\omega^2}{2\alpha(2 - m^2)}} (x - \omega t). \tag{36}$$

Family 6. If $r = \frac{1}{4}$, $q = \frac{1}{2}(1 - 2m^2)$, $p = \frac{1}{4}$, $f(\xi) = ns(\xi) \pm cs(\xi)$, then we get

$$u(\xi) = -\frac{(2 + 3\omega^2)}{4\omega(1 - 2m^2)[ns(\xi) \pm cs(\xi)]^2} - \frac{(1 + 6\omega^2)dn(\xi)}{\omega(1 - 2m^2)[cn(\xi) \pm 1]}, \tag{37}$$

and

$$v(\xi) = \frac{1}{18} \left(-15 + \frac{1}{\omega^2} \right) + \frac{(1 + 6\omega^2)}{12\omega^2(1 - 2m^2)[ns(\xi) \pm cs(\xi)]^2} + \frac{(1 + 6\omega^2)dn(\xi)}{6\omega^2(1 - 2m^2)[cn(\xi) \pm 1]}, \tag{38}$$

where

$$\xi = \pm \frac{1}{3\omega} \sqrt{\frac{1 + 6\omega^2}{\alpha(1 - 2m^2)}} (x - \omega t). \tag{39}$$

Similarly, we can write down the other families of exact solutions of Equations (6) and (7) which are omitted for convenience.

3.2. Example 2. The Coupled KdV Equations

In this subsection, we consider the coupled KdV Equations [34]:

$$u_t = 6vv_x - 6\alpha uu_x - \alpha u_{xxx}, \tag{40}$$

$$v_t = -3\alpha uv_x - \alpha v_{xxx}, \tag{41}$$

where α is a constant. The traveling wave variables (8) permit us converting the Equations (40) and (41) into ODEs in the forms:

$$\alpha u' + 6vv' - 6\alpha uu' - \alpha k^2 u''' = 0, \tag{42}$$

$$\alpha v' - 3\alpha uv' - \alpha k^2 v''' = 0. \tag{43}$$

Suppose that the solutions of Equations (42) and (43) can be expressed by

$$u(\xi) = a_0 + \sum_{i=1}^n (a_i f^i(\xi) + b_i f^{-i}(\xi)) + \sum_{i=2}^n c_i f^{i-2}(\xi) f'(\xi) + \sum_{i=1}^{-n} d_i f^i(\xi) f'(\xi), \tag{44}$$

$$v(\xi) = \alpha_0 + \sum_{i=1}^m (\alpha_i f^i(\xi) + \beta_i f^{-i}(\xi)) + \sum_{i=2}^m \gamma_i f^{i-2}(\xi) f'(\xi) + \sum_{i=1}^{-m} e_i f^i(\xi) f'(\xi), \tag{45}$$

where $a_0, a_i, b_i, c_i, d_i, \alpha_0, \alpha_i, \beta_i, \gamma_i$ and e_i are constants to be determined later.

Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Equation (63) and Equation (64), we get $n = m = 2$, so Equations (44) and (45) can be rewritten as:

$$u(\xi) = a_0 + a_1 f(\xi) + b_1 \frac{1}{f(\xi)} + a_2 f^2(\xi) + b_2 \frac{1}{f^2(\xi)} + c_2 f'(\xi) + d_1 \frac{f'(\xi)}{f(\xi)} + d_2 \frac{f'(\xi)}{f^2(\xi)}, \tag{46}$$

$$v(\xi) = \alpha_0 + \alpha_1 f(\xi) + \beta_1 \frac{1}{f(\xi)} + \alpha_2 f^2(\xi) + \beta_2 \frac{1}{f^2(\xi)} + \gamma_2 f'(\xi) + e_1 \frac{f'(\xi)}{f(\xi)} + e_2 \frac{f'(\xi)}{f^2(\xi)}, \tag{47}$$

where $a_0, a_1, a_2, b_1, b_2, c_2, d_1, d_2, \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_2, e_1$ and e_2 are constants to be determined later. Substituting Equation (46) and Equation (47) with the condition (5) into Equation (42) and Equation (43) and collecting all terms with the same power of $f^i(\xi)[f'(\xi)]^j$, ($i = \dots, -2, -1, 0, 1, 2, \dots; j = 0, 1$). Setting each coefficients of this polynomial to be zero, we get a system of algebraic equations which can be solved by Maple or Mathematica to get the following sets of solutions:

Case 1:

$$\begin{aligned} a_0 = a_1 = b_1 = c_2 = d_1 = d_2 = \alpha_0 = \alpha_1 \\ = \beta_1 = \gamma_2 = e_1 = e_2 = 0, \\ a_2 = -\frac{\omega p}{q\alpha}, b_2 = -\frac{\omega r}{q\alpha}, \alpha_2 = \frac{\omega p}{q\sqrt{2\alpha}}, \\ \beta_2 = \frac{\omega r}{q\sqrt{2\alpha}}, k = \pm \frac{1}{2} \sqrt{\frac{\omega}{\alpha q}}. \end{aligned} \tag{48}$$

Case 2:

$$\begin{aligned} a_0 = a_1 = b_1 = b_2 = d_1 = d_2 = \alpha_0 = \alpha_1 \\ = \beta_1 = \beta_2 = e_1 = e_2 = 0, \\ a_2 = -\frac{2\omega p}{q\alpha}, c_2 = \frac{2\omega\sqrt{p}}{q\alpha}, \alpha_2 = -\frac{\omega p}{q} \sqrt{\frac{2}{\alpha}}, \\ \gamma_2 = \frac{\omega}{q} \sqrt{\frac{2p}{\alpha}}, k = \pm \sqrt{\frac{\omega}{q\alpha}}. \end{aligned} \tag{49}$$

Note that, there are other cases which are omitted here. Since the solutions obtained here are so many, we just list some of the exact solutions corresponding to Case 2 to illustrate the effectiveness of the extended mapping method.

Substituting (49) into Equation (46) and Equation (47) yields

$$u(\xi) = -\frac{2\omega p}{q\alpha} f^2(\xi) + \frac{2\omega\sqrt{p}}{q\alpha} f'(\xi), \tag{50}$$

and

$$v(\xi) = -\frac{\omega p}{q} \sqrt{\frac{2}{\alpha}} f^2(\xi) + \frac{\omega}{q} \sqrt{\frac{2p}{\alpha}} f'(\xi), \tag{51}$$

where

$$\xi = \pm \sqrt{\frac{\omega}{q\alpha}} (x - \omega t). \tag{52}$$

According to the Appendix A, we have the following families of exact solutions:

Family 1. If $r = 1, q = 2m^2 - 1, p = m^2(m^2 - 1), f(\xi) = sd(\xi)$, then we get

$$\begin{aligned} u(\xi) = \frac{2\omega m^2(1-m^2)}{\alpha(2m^2-1)} sd^2(\xi) \\ + \frac{2\omega m\sqrt{m^2-1}}{\alpha(2m^2-1)} nd(\xi)cd(\xi), \end{aligned} \tag{53}$$

and

$$\begin{aligned} v(\xi) = \frac{\omega m^2(1-m^2)}{(2m^2-1)} \sqrt{\frac{2}{\alpha}} sd^2(\xi) \\ + \frac{\omega m}{(2m^2-1)} \sqrt{\frac{2(m^2-1)}{\alpha}} nd(\xi)cd(\xi), \end{aligned} \tag{54}$$

where

$$\xi = \pm \sqrt{\frac{\omega}{\alpha(2m^2-1)}} (x - \omega t). \tag{55}$$

Family 2. If $r = m^2(m^2 - 1), q = 2m^2 - 1, p = 1, f(\xi) = ds(\xi)$, then we get

$$u(\xi) = -\frac{2\omega}{\alpha(2m^2-1)} [ds^2(\xi) + cs(\xi)ns(\xi)], \tag{56}$$

and

$$v(\xi) = -\frac{\omega}{2m^2-1} \sqrt{\frac{2}{\alpha}} [ds^2(\xi) + cs(\xi)ns(\xi)], \tag{57}$$

where

$$\xi = \pm \sqrt{\frac{\omega}{\alpha(2m^2-1)}} (x - \omega t). \tag{58}$$

Family 3. If $r = \frac{m^2}{4}, q = \frac{1}{2}(m^2 - 2), p = \frac{m^2}{4}, f(\xi) = sn(\xi) \pm icn(\xi)$, then we get

$$u(\xi) = -\frac{\omega m^2}{\alpha(m^2 - 2)} [sn(\xi) \pm icn(\xi)]^2 + \frac{2\omega m}{\alpha(m^2 - 2)} [cn(\xi) dn(\xi) \mp isn(\xi) dn(\xi)], \tag{59}$$

and

$$v(\xi) = -\frac{\omega m^2}{m^2 - 2} \sqrt{\frac{2}{\alpha}} [sn(\xi) \pm icn(\xi)]^2 + \frac{\omega m}{m^2 - 2} \sqrt{\frac{2}{\alpha}} [cn(\xi) dn(\xi) \mp isn(\xi) dn(\xi)], \tag{60}$$

where

$$\xi = \pm \sqrt{\frac{2\omega}{\alpha(m^2 - 2)}} (x - \omega t). \tag{61}$$

Family 4. If $r = 1$, $q = -(1 + m^2)$, $p = m^2$, $f(\xi) = sn(\xi)$, then we get

$$u(\xi) = \frac{2\omega m^2}{\alpha(1 + m^2)} sn^2(\xi) - \frac{2\omega m}{\alpha(1 + m^2)} cn(\xi) dn(\xi), \tag{62}$$

and

$$v(\xi) = \frac{\omega m^2}{1 + m^2} \sqrt{\frac{2}{\alpha}} sn^2(\xi) - \frac{\omega m}{1 + m^2} \sqrt{\frac{2}{\alpha}} cn(\xi) dn(\xi), \tag{63}$$

where

$$\xi = \pm i \sqrt{\frac{\omega}{\alpha(1 + m^2)}} (x - \omega t). \tag{64}$$

Family 5. If $r = 1 - m^2$, $q = 2m^2 - 1$, $p = -m^2$, $f(\xi) = cn(\xi)$, then we get

$$u(\xi) = \frac{2\omega m^2}{\alpha(2m^2 - 1)} cn^2(\xi) - \frac{2im\omega}{\alpha(2m^2 - 1)} sn(\xi) dn(\xi), \tag{65}$$

and

$$v(\xi) = \frac{\omega m^2}{2m^2 - 1} \sqrt{\frac{2}{\alpha}} cn^2(\xi) - \frac{im\omega}{2m^2 - 1} \sqrt{\frac{2}{\alpha}} sn(\xi) dn(\xi), \tag{66}$$

where

$$\xi = \pm \sqrt{\frac{\omega}{\alpha(2m^2 - 1)}} (x - \omega t). \tag{67}$$

Family 6. If $r = 1 - m^2$, $q = 2 - m^2$, $p = 1$, $f(\xi) = cs(\xi)$, then we get

$$u(\xi) = -\frac{2\omega}{\alpha(2 - m^2)} [cs^2(\xi) + ns(\xi) ds(\xi)], \tag{68}$$

and

$$v(\xi) = -\frac{\omega}{2 - m^2} \sqrt{\frac{2}{\alpha}} [cs^2(\xi) + ns(\xi) ds(\xi)], \tag{69}$$

where

$$\xi = \pm \sqrt{\frac{\omega}{\alpha(2 - m^2)}} (x - \omega t). \tag{70}$$

4. Conclusion and Discussion

In this article, we have found the exact solutions of the variant Boussinesq equations and the coupled KdV equations by using the extended mapping method and the auxiliary equation method. Also, we conclude according to the Appendix B that our results in terms of Jacobi elliptic functions generate into hyperbolic functions when $m \rightarrow 1$ and generate into trigonometric functions when $m \rightarrow 0$. This method provides a powerful mathematical tool to obtain more general exact solutions of a great many nonlinear PDEs in mathematical physics.

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Appendix A

The general solutions of the auxiliary Equation (5) and its derivatives [35] are listed as follows:

p	q	r	$f(\xi)$	$f'(\xi)$
m^2	$-(1+m^2)$	1	$sn(\xi)$	$cn(\xi)dn(\xi)$
$-m^2$	$2m^2-1$	$1-m^2$	$cn(\xi)$	$-sn(\xi)dn(\xi)$
-1	$2-m^2$	m^2-1	$dn(\xi)$	$-m^2sn(\xi)cn(\xi)$
1	$-(1+m^2)$	m^2	$dc(\xi)$	$(1-m^2)nc(\xi)sc(\xi)$
1	$2-m^2$	$1-m^2$	$cs(\xi)$	$-ns(\xi)ds(\xi)$
$1-m^2$	$2-m^2$	1	$sc(\xi)$	$nc(\xi)dc(\xi)$
$m^2(m^2-1)$	$2m^2-1$	1	$sd(\xi)$	$nd(\xi)cd(\xi)$
1	$2m^2-1$	$m^2(m^2-1)$	$ds(\xi)$	$-cs(\xi)ns(\xi)$
$\frac{1}{4}$	$\frac{1}{2}(1-2m^2)$	$\frac{1}{4}$	$ns(\xi) \pm cs(\xi)$	$-ds(\xi)cs(\xi) \mp ns(\xi)ds(\xi)$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{m^2}{4}$	$sn(\xi) \pm icn(\xi)$	$cn(\xi)dn(\xi) \mp i sn(\xi)dn(\xi)$

where $0 < m < 1$ is the modulus of the Jacobi elliptic functions and $i = \sqrt{-1}$.

Appendix B

The Jacobi elliptic functions $sn(\xi), cn(\xi), dn(\xi), ns(\xi), cs(\xi), ds(\xi), sc(\xi), sd(\xi)$ generate into hyperbolic functions when $m \rightarrow 1$ as follows:

$sn(\xi) \rightarrow \tanh(\xi)$	$cn(\xi) \rightarrow \operatorname{sech}(\xi)$	$dn(\xi) \rightarrow \operatorname{sech}(\xi)$	$ns(\xi) \rightarrow \operatorname{coth}(\xi)$
$cs(\xi) \rightarrow \operatorname{csch}(\xi)$	$ds(\xi) \rightarrow \operatorname{csch}(\xi)$	$sc(\xi) \rightarrow \sinh(\xi)$	$sd(\xi) \rightarrow \sinh(\xi)$

and into trigonometric functions when $m \rightarrow 0$ as follows:

$sn(\xi) \rightarrow \sin(\xi)$	$cn(\xi) \rightarrow \cos(\xi)$	$dn(\xi) \rightarrow 1$	$ns(\xi) \rightarrow \csc(\xi)$
$cs(\xi) \rightarrow \cot(\xi)$	$ds(\xi) \rightarrow \csc(\xi)$	$sc(\xi) \rightarrow \tan(\xi)$	$sd(\xi) \rightarrow \sin(\xi)$

Appendix C

$nc(\xi) = \frac{1}{cn(\xi)}$	$nd(\xi) = \frac{1}{dn(\xi)}$	$cd(\xi) = \frac{cn(\xi)}{dn(\xi)}$	$dc(\xi) = \frac{dn(\xi)}{cn(\xi)}$
$cs(\xi) = \frac{cn(\xi)}{sn(\xi)}$	$sc(\xi) = \frac{sn(\xi)}{cn(\xi)}$	$sd(\xi) = \frac{sn(\xi)}{dn(\xi)}$	$ds(\xi) = \frac{dn(\xi)}{sn(\xi)}$