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Some Properties on the Function Involving the Gamma Function

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ABSTRACT

We studied the monotonicity and Convexity properties of the new functions involving the gamma function, and get the general conclusion that Minc-Sathre and C. P. Chen-G. Wang's inequality are extended and *refined*.

Keywords: Gamma Function; Monotonicity; Convexity; Inequality

1. Introduction

The classical gamma function $\Gamma(x)=\int_{0}^{\infty} t^{x-1}e^{-t}\mathrm{d}t,\ (x>0)$ is one of the most important functions in analysis and its applications. The logarithmic derivative of the gamma function can be expressed in terms of the series

$$\omega(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{1+n}-\frac{1}{x+n}\right) \tag{1}$$

($x>0$; γ = 0.57721566490153286… is the Euler's constant), which is known in literature as psi or digamma function. We conclude from (1) by differentiation

$$\omega^{(k)}(x)=(-1)^{k+1}k!\sum_{n=0}^{\infty}\frac{1}{(x+n)^{k+1}},\ (x>0;k=1,2,3\cdots) \tag{2}$$

$\omega^{k}(x)$ are called polygamma functions.

H. Minc and L. Sathre [1] proved that the inequality

$$\frac{n}{n+1}<\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}<1,\ (n=1,2,\cdots) \tag{3}$$

is valid for all natural numbers n. The Inequality (3) can be refined and generalized as (see [2-4])

$$\frac{n+k+1}{n+m+k+1}<\left(\sum_{i=k+1}^{n+k} i\right)^{1/n}\Bigg/\left(\sum_{i=k+1}^{n++m+k} i\right)^{1/n+m}\leq\sqrt{\frac{n+k}{n+m+k}} \tag{4}$$

where k is a nonnegative integer, n and m are natural numbers. For $n=m=1$, the equality in (4) is valid. The Inequality (4) can be written as

$$\frac{n+k+1}{n+m+k+1}<\frac{\left[\Gamma(n+k+1)/\Gamma(k+1)\right]^{1/n}}{\left[\Gamma(n+m+k+1)/\Gamma(k+1)\right]^{1/n+m}}\leq\sqrt{\frac{n+k}{n+m+k}} \tag{5}$$

In 1985, D. Kershaw and A. Laforgia [5] showed the function $\left[\Gamma\left(1+\frac{1}{x}\right)\right]^{x}$ is strictly decreasing and $x\left[\Gamma\left(1+\frac{1}{x}\right)\right]^{x}$ strictly increasing on $(0,\infty)$, from which the Inequality (3) can be derived. In 2003, B.-N. Guo and F. Qi [2] proved that the function

$$f(x)=\frac{\left[\Gamma(x+y+1)/\Gamma(y+1)\right]^{1/x}}{(x+y+1)}$$

is decreasing in $x\geq1$ for fixed $y\geq0$, from which the left-hand side inequality of (5) can be obtained. In the 2009, C. P. Chen-G. Wang had obtained the extended inequality of the function above. They gave the limits of it and other results.

In this paper, our Theorem 1 considers the monotonicity and logarithmic convexity of the new function g on $(0,\infty)$. This extends and generalizes B.-N. Guo and F. Qi's [2] as well as C. P. Chen and G. Wang's [6] results.

Theorem 1. Let fixed $t\geq0$ and $s\geq0$ be real number, then the new function

$$g(x)=\frac{\left[\Gamma(x+s+1)/\Gamma(s+1)\right]^{1/(x+t)}}{x+s+1}$$

is strictly decreasing and strictly logarithmically convex on $(0,\infty)$, Moreover,

$$\lim_{x\to0}g(x)=e^{\omega(s+1)}\big/(s+1)\ \text{ and }\ \lim_{x\to\infty}g(x)=e^{-1}$$

Theorem 2. Let $k\geq2$ be an positive integer, $s\geq0$ be real number, then the function

$$h(x)=\frac{\left[\Gamma(x+s+1)/\Gamma(s+1)\right]^{1/(x+t)}}{(x+s+1)^{1/k}}$$

is strictly increasing on $(0,\infty)$.

2. Proof of the Theorems

Proof of Theorem 1. First, we define for fixed $t\ge 0$ and $s\ge 0$,

$$A(x)=(x+t)^2\frac{g'(x)}{g(x)}$$
$$=-\ln\frac{\Gamma(x+s+1)}{\Gamma(s+1)}+(x+t)\omega(x+s+1)-\frac{(x+t)^2}{x+s+1}$$

$$B(x)=(x+t)^3\frac{\mathrm{d}^2\left[\ln g(x)\right]}{\mathrm{d}^2(x)}$$
$$=2\ln\frac{\Gamma(x+s+1)}{\Gamma(s+1)}-2(x+t)\omega(x+s+1)$$
$$+(x+t)^2\omega'(x+s+1)+\frac{(x+t)^3}{(x+s+1)^2}$$

From the differentiation of $A(x)$, we should have

$$\frac{1}{(x+t)}A'(x)=\omega'(x+s+1)-\frac{1}{x+s+1}-\frac{1}{(x+s+1)^2}$$
$$=\sum_{n=1}^{\infty}\frac{1}{(x+s+n)^2}-\sum_{n=1}^{\infty}\left[\frac{1}{x+s+n}-\frac{1}{x+s+n+1}\right]$$
$$-\sum_{n=1}^{\infty}\left[\frac{s+1}{(x+s+n)^2}-\frac{s+1}{(x+s+n+1)^2}\right]$$
$$=-\sum_{n=1}^{\infty}\left[\frac{s}{(x+s+n)^2}+\frac{1}{(x+s+n)(x+s+n+1)}-\frac{s+1}{(x+s+n+1)^2}\right]$$
$$=\sum_{n=1}^{\infty}\frac{(2s+1)(x+s+n)+s}{(x+s+n)^2(x+s+n+1)^2}<0$$

Hence, the function $A(x)$ is strictly decreasing and $A(x)<A(0)$, for $x>0$, which yields the desired result that $g'(x)<0$ for $x>0$.

Using the asymptotic expansion [7, p. 257]

$$\ln\Gamma(x)=\left(x-\frac{1}{2}\right)\ln x-x+\ln\sqrt{2\pi}$$
$$+\frac{1}{12x}+o\left(x^{-3}\right)(x\to\infty),$$

and

$$\ln g(x)=\frac{1}{(x+t)}\left[\ln\Gamma(x+s+1)-\ln\Gamma(s+1)\right]-\ln(x+s+1)\tag{6}$$

we can conclude that $\lim\limits_{x\to\infty}g(x)=e^{-1}$.

By L'Hospital rule, we conclude from (6) that

$$\lim_{x\to 0}g(x)=\frac{e^{\omega(s+1)}}{(s+1)}$$

Then from the Differentiation of $B(x)$ yields

$$\frac{1}{(x+t)^2}B'(x)=\omega''(x+s+1)+\frac{1}{(x+s+1)^2}+\frac{2(s+1)}{(x+s+1)^3}$$
$$=-\sum_{n=1}^{\infty}\frac{2}{(x+s+n)^3}+\sum_{n=1}^{\infty}\left[\frac{1}{(x+s+n)^2}-\frac{1}{(x+s+n+1)^2}\right]$$
$$+\sum_{n=1}^{\infty}\left[\frac{2(s+1)}{(x+s+n)^3}-\frac{2(s+1)}{(x+s+n+1)^3}\right]$$
$$=\sum_{n=1}^{\infty}\frac{3(2s+1)(x+s+n)^2+(6s+1)(x+s+n)+2s}{(x+s+n)^3(x+s+n+1)^3}>0.$$

Hence, the function $B(x)$ is strictly increasing and $B(x)>B(0)$ for $x>0$, which yields the desired result that $\frac{\mathrm{d}^2\left[\ln g(x)\right]}{\mathrm{d}x^2}>0$ for $x>0$.

Proof of Theorem 2. Define for $k\ge 0$ be an positive integer and $x>0$,

$$C(x)=(x+t)^2\frac{h'(x)}{h(x)}$$
$$=-\ln\frac{\Gamma(x+s+1)}{\Gamma(s+1)}+(x+t)\omega(x+s+1)$$
$$-\frac{(x+t)^2}{k(x+s+1)}$$

Differentiation of $C(x)$ gives

$$\frac{1}{(x+t)}C'(x)$$
$$=\omega'(x+s+1)-\frac{1}{k(x+s+1)}-\frac{s+1}{k(x+s+1)^2}$$
$$=\sum_{n=1}^{\infty}\frac{1}{(x+s+n)^2}-\frac{1}{k(x+s+1)}-\frac{s+1}{k(x+s+1)^2}$$
$$>\int_1^{\infty}\frac{\mathrm{d}t}{(x+s+1)^2}-\frac{1}{k(x+s+1)}-\frac{s+1}{k(x+s+1)^2}$$
$$=\frac{x}{k(x+s+1)^2}>0.$$

Hence, the function $C(x)$ is strictly increasing and $C(x) > C(0)$ for $x > 0$ which yields the desired result that $h'(x) > 0$ for $x > 0$.

3. Use the Theorem

From the proof above the following corollaries are obvious.

Corollary 1. Let fixed $t \ge 0$ and $s \ge 0$ be a real number, then for all real numbers $x > 0$,

$$e^{-1} < \frac{\left[\Gamma(x+s+1)/\Gamma(s+1)\right]^{1/(x+t)}}{x+s+1} < \frac{e^{\omega(s+1)}}{(s+1)} \tag{7}$$

Both bounds in (7) are best possible.

Corollary 2. Let fixed $t \ge 0$, $\alpha > 0$ and $s \ge 0$ be real numbers, $k \ge 2$ be an positive integer, then for all real numbers $x > 0$,

$$\frac{x+s+1}{x+\alpha+s+1} < \frac{\left[\Gamma(x+s+1)/\Gamma(s+1)\right]^{1/x+t}}{\left[\Gamma(x+\alpha+s+1)/\Gamma(s+1)\right]^{1/(x+t+\alpha)}} < \sqrt[k]{\frac{x+s+1}{x+\alpha+s+1}} \tag{8}$$

In particular, taking in (8) $t = 0$, $k = 2$, we obtain the result that Minc-Sathre and C. P. Chen-G. Wang got

$$\frac{x+s+1}{x+\alpha+s+1} < \frac{\left[\Gamma(x+s+1)/\Gamma(s+1)\right]^{1/x}}{\left[\Gamma(x+\alpha+s+1)/\Gamma(s+1)\right]^{1/(x+\alpha)}} < \sqrt{\frac{x+s+1}{x+\alpha+s+1}} \tag{9}$$

The inequality is an improvement of above, and we can extend it as the below form.

Corollary 3. Let $k \ge 2$, we have

$$\frac{x+s+1}{x+\alpha+s+1} < \frac{\left[\Gamma(x+s+1)/\Gamma(s+1)\right]^{1/x}}{\left[\Gamma(x+\alpha+s+1)/\Gamma(s+1)\right]^{1/(x+\alpha)}} < \sqrt[k]{\frac{x+s+1}{x+\alpha+s+1}} \tag{10}$$

In most particular, weobtain

Corollary 4. Let *t* be an positive integer, we get

$$\frac{n+1}{n+2} < \frac{\sqrt[n+t]{n!}}{\sqrt[n+t+1]{(n+1)!}} < \sqrt[k]{\frac{n+1}{n+2}} \tag{11}$$

and for $k > 2$,

$$\frac{n+1}{n+2} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < \sqrt[k]{\frac{n+1}{n+2}} \tag{12}$$

Corollary 5. Let *t* be an positive integer, we get

$$\frac{n+1}{n+2} < \frac{\sqrt[n+t]{n!}}{\sqrt[n+t+1]{(n+1)!}} < \sqrt{\frac{n+1}{n+2}} \tag{13}$$

The Inequality (13) is an improvement of (3).

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REFERENCES

[1] H. Minc and L. Sathre, "Some Inequalities Involving $(r!)^{1/r}$," *Proceedings of the Edinburgh Mathematical Society*, Vol. 14, No. 65, 1964, pp. 41-46. doi:10.1017/S0013091500011214

[2] B.-N. Guo and F. Qi, "Inequalities and Monotonicity for the Ratio of Gamma Functions," *Taiwanese Journal of Mathematics*, Vol. 7, No. 2, 2003, pp. 239-247.

[3] F. Qi, "Inequalities and Monotonicity of Sequences Involving $\sqrt[n]{(n+k)!/k!}$," *Soochow Journal of Mathematics*, Vol. 29, No. 4, 2003, pp. 353-361.

[4] F. Qi and Q.-M. Luo, "Generalization of H. Minc and J. Sathre's Inequality," *Tamkang Journal of Mathematics*, Vol. 31, No. 2, 2000, pp. 145-148.

[5] D. Kershaw and A. Laforgia, "Monotonicity Results for the Gamma Function," *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, Vol. 119, 1985, pp. 127-133.

[6] C.-P. Chen and G. Wang, "Monotonicity and Logarithmic Convexity Properties for the Gamma Function," *Scientia*, Vol. 5, No. 1, 2009, pp. 51-54.

[7] M. Abramowitz and I. A. Stegun (Eds.), "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards," *Applied Mathematics Series*, 4th Printing, Washington, Vol. 55, 1965.

