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Inverse Shadowing and Weak Inverse Shadowing Property

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ABSTRACT

In this paper we show that an Ω-stable diffeomorphism f has the weak inverse shadowing property with respect to classes of continuous method θ_s and θ_c and some of the Ω-stable diffeomorphisms have weak inverse shadowing property with respect to classes $\mathcal{T}_0$. In addition we study relation between minimality and weak inverse shadowing property with respect to class $\mathcal{T}_0$ and relation between expansivity and inverse shadowing property with respect to class $\mathcal{T}_0$.

Keywords: Inverse Shadowing Property; Minimal Homeomorphism; δ-Method; Positive Expansive

1. Introduction

Inverse shadowing was introduced by Corless and Pilyugin [1] and also as a part of the concept of bishadowing by Diamond *et al.* [2]. Kloeden, Ombach and Pokroskii [3] defined this property using the concept of δ-method. One can also see [4-7] for more information about the concept of δ-method. Authors in [8] studied on locally genericity of weak inverse shadowing with respect to class $\mathcal{T}_0$. For flows, there are lots of existing work on finding the minimal sets in a systems with shadowing property. See for example [9-12]. In this paper we study diffeomorphisms with weak inverse shadowing property with respect to class as θ_s, θ_c and $\mathcal{T}_0$. First we show that an Ω-stable diffeomorphism f has weak inverse shadowing property with respect to classes of continuous method θ_s and θ_c (Theorem 1) and some Ω-stable diffeomorphisms have weak inverse shadowing property with respect to classes $\mathcal{T}_0$ (Theorem 2). In addition we study relation between minimality and weak inverse shadowing property with respect to class $\mathcal{T}_0$ and show that a chain transitive homeomorphism f on compact metric space X is minimal if and only if it has weak inverse shadowing property with respect to class $\mathcal{T}_0$ (Theorem 3). Finally we study relation between positively expansive and inverse shadowing property with respect to class $\mathcal{T}_0$ and show that if f has inverse shadowing property with respect to class $\mathcal{T}_0$, then f is not positive expansive (Theorem 4).

Let (X,d) be a compact metric space and let $f: X \to X$ be a homeomorphism (a discrete dynamical system on X). A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called an orbit of f, denote by $o(x,f)$, if for each $n \in \mathbb{Z}$, $x_{n+1} = f(x_n)$ and is called a δ-pseudo-orbit of f if

$$d\left(f(x_n), x_{n+1}\right) \le \delta, \forall n \in \mathbb{Z}.$$

Denote the set of all homeomorphisms of X by $Z(X)$. In $Z(X)$ consider the complete metric

$$d_0(f,g) = \max\left\{\max_{x\in X} d\left(f(x), g(x)\right), \max_{x\in X} d\left(f^{-1}(x), g^{-1}(x)\right)\right\},$$

which generates the C^0-topology.

Let $X^{\mathbb{Z}}$ be the space of all two sided sequence $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in X$, endowed with the product topology. For $\delta > 0$ let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f.

A mapping $\varphi: X \to \Phi_f(\delta) \subset X^{\mathbb{Z}}$ is said to be a δ-method for f if $\varphi(x)_0 = x$, where $\varphi(x)_0$ is the 0-component of $\varphi(x)$. If φ is a δ-method which is continuous then it is called a continuous δ-method. The set of all δ-methods (resp. continuous δ-methods) for f will be denoted by $\mathcal{T}_0(f,\delta)$ (resp. $\mathcal{T}_c(f,\delta)$). If $g: X \to X$ is a homeomorphism with $d_0(f,g) < \delta$, then g induces a continuous δ-method φ_g for f defined by

$$\varphi_g(x) = \left\{g^n(x) : n \in \mathbb{Z}\right\}.$$

Let $\mathcal{T}_h(f,\delta)$ denote the set of all continuous δ-

methods φ_g for f which are induced by $g \in Z(M)$ with $d_0(f,g) < \delta$.

Let $A \subseteq M$ and $A \neq \phi$, a homeomorphism f is said to have the inverse shadowing property with respect to the class $\mathcal{T}_\alpha$, $\alpha = 0$, c, h, in A if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\mathcal{T}_\alpha(f,\delta)$ and any point $x \in A$ there exists a point $y \in M$ for which

$$d\left(f^n(x), \varphi(y)_n\right) < \varepsilon, n \in \mathbb{Z}.$$

A homeomorphiosm f is said to have weak inverse shadowing property with respect to the class $\mathcal{T}_\alpha$, $\alpha = 0$, c, h, in A if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\mathcal{T}_\alpha(f,\delta)$ and any point $x \in A$ there exists a point $y \in M$ for which

$$\varphi(y) \subset N_\varepsilon\left(o(x,f)\right).$$

Fix $\delta > 0$. A continuous δ-method of class θ_s for the diffeomorphism f is a sequence $\Psi = \{\psi_k : k \in Z\}$, where any ψ_k is a continuous mapping $\psi_k : M \to M$ such that

$$\max_{x \in M} d\left(\psi_k(x), f(x)\right) < d, k \in Z.$$

A sequence $\xi = \{x_k \in M : k \in Z\}$ is a pseudo-orbit generated by a continuous d-method $\Psi = \{\psi_k\}$ of a class θ_s if

$$x_{k+1} = \psi_k(x_k), k \in Z.$$

Fix $\delta > 0$. A continuous δ-method of class θ_c for the diffeomophism f is a sequence $\Psi = \{\psi_k : k \in Z\}$, with $\psi_0(x) = x$ for $x \in M$ and such that any ψ_k is a continuous mapping $\psi_k : M \to M$ with the property

$$\max_{x \in M} d\left(f(\psi_k(x)), \psi_{k+1}(x)\right) < \delta, k \in Z.$$

A sequence $\xi = \{x_k \in M : k \in Z\}$ is a pseudo-orbit generated by a continuous δ-method $\Psi = \{\psi_k\}$ of class θ_c if

$$x_k = \psi_k(x_0), k \in Z.$$

If a sequence is generated by θ_c or θ_s we briefly write $\xi \in G\Psi$.

A diffeomorphism f is said to have (weak) inverse shadowing property if for any $x \in M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that, for any continuous δ-method Ψ, we can find a pseudo-orbit $\xi \in G\Psi$ satisfying the inequalities

$$d\left(f^k(x), x_k\right) < \varepsilon, k \in Z.$$

$$\left(\{x_k\} \subset N_\varepsilon\left(O(x,f)\right).\right)$$

Pilyugin [5] showed that a structurally stable diffeomoriphism has the inverse shadowing property with respect to classes of continuous method, θ_c and θ_s. He also showed that any diffeomorphism belonging to the C^1-interior of the set of diffeomorphisms having the inverse shadowing property with respect to classes of continuous method, θ_c and θ_s is structurally stable.

2. Diffeomorphisms with Weak Inverse Shadowing Property with Respect to Class θ_s, θ_c and $\mathcal{T}_0$

In this section we show that an Ω-stable diffeomorphism f has the weak inverse shadowing property with respect to classes of continuous methods θ_s and θ_c and if we impose some condition on an Ω-stable diffeomorphism, then it has weak inverse shadowing property with respect to classes $\mathcal{T}_0$.

Theorem 1 *If a diffeomorphism f is Ω-stable, then it has the weak inverse shadowing property with respect to both classes θ_c and θ_s.*

Before proving this main result, let us briefly recall some definitions. A diffeomorphism $f : M \to M$ is called Ω-stable if there is a C^1-neighborhood U of f such that for any $g \in U$, $g|_{\Omega(g)}$ is topologically conjugate to $f|_{\Omega(f)}$.

A diffeomorfphism f is called an Axiom A system if $\Omega(f)$ is hyperbolic and if $\Omega(f) = \overline{pref}$. Axiom A and no-cycle systems are Ω-stable [13].

Let f be an Axiom A diffeomorphism of M. By the Smale spectral Decomposition Theorem, the nonwandering set $\Omega(f)$ an e represented as a finite union of basic sets Λ_i.

$$\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_k.$$

In the proof of theorem 1 in [5], Pilyugin has used the following statement.

If a C^1-diffeomorphism f satisfies Axiom A and the strong transversality condition, then there exist constants $C > 0$ and $\lambda \in (0,1)$ and linear subspace $S(p)$, $U(p)$ of T_pM for $p \in M$ such that

$$T_pM = S(p) \oplus U(p),$$
$$Df(p)S(p) \subset S(f(p)),$$
$$Df^{-1}(p)U(p) \subset U\left(f^{-1}(p)\right),$$

and

$$\left|Df^k(p)V\right| \le C\lambda^k |V| \text{ for } k \ge 0 \text{ and } V \in S(p), \quad (1)$$

$$\left|Df^k(p)V\right| \le C\lambda^k |V| \text{ for } k \ge 0 \text{ and } V \in U(p), \quad (2)$$

if $P(p)$ and $Q(p)$ are the projectors in T_pM onto $S(p)$ parallel to $U(p)$ and onto $U(p)$ parallel to $S(p)$, respectively, then

$$\|P(p)\| \le C \text{ and } \|Q(p)\| \le C \qquad (3)$$

(here $\|.\|$ is the operator norm).

Conditions (1), (2) and (3) play a basic role in the proof of theorem 1 in [5]. If Λ_i is a basic set then we can see for every $x \in \Lambda_i$, conditions (1), (2) hold. Since $\|f(x)\|$ is bounded for $x \in \Lambda_i$, standard reasening shows (see, for example, Lemma 12.1 in [14]) that there exists a constant C for which inequalities (3) hold. Hence similar to the proof of theorem 1 in [5], f has the inverse shadowing property with respect to classes θ_s and θ_c on Λ_i. The following two propositions are well known (proposition 1 is the classical Birkhoff theorem [13], for proofs of statements similar to proposition 2, see [15], for example).

proposition 1 *Let f be a homeomorphism of a compact topological space X and U be a neighborhood of its nonwandering set. Then there exists a positive integer N such that*

$$\text{Card}\left(k : f^k(x) \notin U\right) \le N$$

for every $x \in X$, where *Card A* is the cardinality of a set A.

In the following proposition, we assume that f is an Ω-stable diffeomorphism of a closed smooth manifold.

proposition 2 *If Λ_i is a basic set, then for any neighborhood U of Λ_i there exists neighborhood V with the following property: if for some $x \in V$ and $m > 0$, $f^m(x) \notin U$, then $f^{m+k}(x) \notin V$ for $k \ge 0$.*

Lemma 1 *Let f be an Ω-stable diffeomorphism and $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_k$ be the Smale Spectral Decomposition. Let U_i be a neighborhood of Λ_i for $i = 1, \cdots, k$. Then for any $x \in M$ there exists $N_0 \in N$ and U_i for some $1 \le i \le k$, such that*

$$\left\{f^{N_0+k}(x)\right\}_{k\ge 0} \subset U_i,$$

and similarly there exists U_j

$$\left\{f^{-N_0+k}(x)\right\}_{k\ge 0} \subset U_j,$$

Proof. Suppose that the lemma is not true for some $x \in M$. Let V_i be a neighborhood of Λ_i as in proposition 2. Proposition 1 shows that there exists $n_i \in N$ such that $f^{n_i}(x) \in V_i$ for some $1 \le i \le k$. By assumption there exists $m_i \in N, m_i > n_i$ such that $f^{m_i}(x) \notin U_i$. By proposition 2, $f^{m_i+n}(x) \notin V_i$ for $n \ge 0$. Thus using proposition 1, there exists $n_j \in N, n_j > m_i$ such that $f^{n_j}(x) \in V_j$ for some $1 \le j \le k$ $j \ne i$ and there exists $m_j \in N$, $m_j > n_j$ such that $f^{m_j}(x) \notin U_j$. By proposition 2, $f^{m_j+n}(x) \notin V_j$ for $n \ge 0$. This process show that

$$\text{Card}\left\{n : f^n(x) \notin U := \bigcup_{i=1}^{k} V_i\right\} = \infty$$

contradicting proposition 1. Proof of

$$\left\{f^{-N_0+k}(x)\right\}_{k\ge 0} \subset U_j,$$

is similar.

Proof of theorem 1. Let $x \in M$ and $\varepsilon > 0$ be arbitrary. Let U_i be a neighborhood of Λ_i for $i = 1, \cdots, k$, such that shadowing property hold for them. By lemma 1 there exists a positive number N_0, such that $\left\{f^n(x)\right\}_{n \ge N_0} \subset U_i$ for some $1 \le i \le k$. Since M is compact, there exist $l_2 > l_1 \ge N_0$, such that

$$d\left(f^{l_2}(x), f^{l_1}(x)\right) < \frac{\delta'}{2},$$

where $\delta' = \delta\left(\frac{\varepsilon}{2}\right)$ is as in the shadowing theorem for hyperbolic set. So $\bar{\xi} = \left\{f^{l_1}(x), f^{l_1+1}(x), \cdots, f^{l_2-1}(x)\right\}$ is a periodic δ'-pseudo-orbit of f in U_i. By shadowing theorem for hyperbolic sets, there is $z \in \Lambda_i$, which $\frac{\varepsilon}{2}$-shadows $\bar{\xi}$. This shows that

$$o(z, f) \subset N_{\frac{\varepsilon}{2}}\left(o(x, f)\right). \quad (*)$$

But Λ_i has the inverse shadowing property with respect to classes θ_s and θ_c. Thus there exists $\delta > 0$ such that for any continuous δ-method φ, we can find a pseudo-orbit $\xi = \{x_k\}_{k\in Z} \in G\Psi$ satisfying

$$d\left(f^k(z), x_k\right) < \frac{\varepsilon}{2}, k \in Z. \quad (**)$$

Inequalities $(*)$ and $(**)$ show that $\xi \subset N_\varepsilon\left(o(x, f)\right)$. This complete the proof of theorem 1.

Theorem 2 *Let f be an Ω-stable diffeomorphism and $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k$ be the Smale Spectral Decomposition such that $\{\Lambda_i, \ i = 1, \cdots, k\}$ be fix point sources or sinks. Then f has the weak inverse shadowing property with respect to class $\mathcal{T}_0$ in $M - Fix(f)$, where $Fix(f)$ is set of fix points of f.*

Proof. Let $\varepsilon > 0$ and $x \in M$ be arbitrary that is not fix point and $\{U_i, i = 1, \cdots, k\}$ be open neighborhoods of $\{\Lambda_i, i = 1, \cdots, k\}$ respectively with diameter less than ε. Lemma 1 shows that there exists $N_0' \in N$ and U_i and U_j for some $1 \le i, j \le k$, such that

$$\left\{f^{N_0'+k}(x)\right\}_{k\ge 0} \subset U_i,$$

and

$$\left\{f^{-N_0'+k}(x)\right\}_{k\ge 0} \subset U_j,$$

Note that U_i is a neighborhood of fix point sink and U_j is a neighborhood of fix point source. Choose

$$0 < \delta_0 < \frac{\min\left\{\dim\left(N_{\varepsilon'}(\Lambda_i) - f\left(N_{\varepsilon'}(\Lambda_i)\right)\right) \text{ and } \dim\left(N_{\varepsilon'}(\Lambda_j) - f^{-1}\left(N_{\varepsilon'}(\Lambda_j)\right)\right)\right\}}{2}$$

such that

$$d\left(f(x), f(z)\right) < \frac{\varepsilon'}{4}$$

for every $x, z \in M$ with $d(x,z) < \delta$, where

$$0 < \varepsilon' < \min\left\{\left\{\operatorname{diam}(U_i), i = 1,\cdots,k\right\}, \frac{\varepsilon}{2}\right\}$$
$$\text{and } f\left(N_{\varepsilon'}(\Lambda_i)\right) \subset \left(N_{\varepsilon'}(\Lambda_i)\right)$$
$$\text{and } f^{-1}\left(N_{\varepsilon'}(\Lambda_j)\right) \subset N_{\varepsilon'}(\Lambda_j).$$

This shows that if $\xi = \{x_k\} \subset M$ is a δ_0-pseudo orbit and $x_l \in N_{\varepsilon'}(\Lambda_i)\left(x_l \in N_{\varepsilon'}(\Lambda_j)\right)$ then $\{x_n\}_{n\geqslant l} \subset N_{\varepsilon'}(\Lambda_i)\left(\{x_n\}_{n\leqslant l} \subset N_{\varepsilon'}(\Lambda_j)\right)$. there exists $N_0 \in N$ such that

$$\left\{f^{N_0+k}(x)\right\}_{k\geq 0} \subset N_{\varepsilon'}(\Lambda_i) \quad (1)$$

and

$$\left\{f^{-N_0-k}(x)\right\}_{k\geq 0} \subset N_{\varepsilon'}(\Lambda_j). \quad (2)$$

Choose $\delta_{N_0+1} < \delta_{N_0} < \cdots < \delta_1 < \delta_0$ such that if $d(x,y) < \delta_i$ for $i = N_0+1, N_0, \cdots, 1$ then

$$d\left(f(x), f(y)\right) < \frac{\delta_{i-1}}{N_0+1}$$
$$\text{and } d\left(f^{-1}(x), f^{-1}(y)\right) < \frac{\delta_{i-1}}{N_0+1}$$

And also $\frac{\delta_i}{N_0+1} + \delta_{N_0+1} < \delta_i$ for $i = N_0+1, N_0, \cdots, 1$. So for any δ_{N_0+1}-pseudo orbit

$$\left\{x_{-N_0-1}, x_{-N_0}, \cdots, x, x_1, \cdots, x_{N_0+1}\right\}$$

we have

$$d\left(f^i(x), x_i\right) < \frac{\varepsilon'}{2}, \quad i = -N_0-1, \cdots, N_0+1. \quad (3)$$

Now for any (δ_{N_0+1})-method φ, by regarding the process of choosing δ_{N_0+1} and (4), (5), (6) we have $\varphi(x) \subset N_\varepsilon\left(o(x,f)\right)$, and this completes the proof of theorem 2.

The following example shows that an Ω-stable may-be has not the weak inverse shadowing property with respect to class $\mathcal{T}_0$ in its fix point.

Example. Represent $\mathbb{T}^2$ as the sqare $[-2,2]\times[-2,2]$, with identified opposite sides. Let $g: \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism with the following properties:

The nonwandering set $\Omega(g)$ of g is the union of 4 hyperbolic fixed points, that is, $\Omega(g) = \{p_1, p_2, p_3, p_4\}$, where p_1 is a source, p_2 is a sink, and p_3, p_4 are saddles;

$$\begin{aligned} W^u\{p_4\} \cup \{p_3\} &= W^s(p_3) \cup \{p_4\} \\ &= [-2,2]\times\{0\}, W^s(p_4) \\ &= \{1\}\times(-2,2), W^u(p_3) = \{-1\}\times(-2,2), \end{aligned}$$

where $W^s(p_i)$ and $W^u(p_i)$ are the stable and unstable manifolds, respectively.

There exist neighborhoods U_3, U_4 of p_3, p_4 such that $g(x) = p_i + D_{p_i}g(x - p_i)$ for $x \in U_i$.

The eigenvalues of $D_{p_3}g$ are $-\mu, \nu$ with $0 < \nu < 1 < \mu$, and the eigenvalues of $D_{p_4}g$ are $-\lambda, \kappa$ with $0 < \lambda < 1 < \kappa$.

Plamenevskaya [16] showed that g has the weak shadowing property if and only if the number $\frac{\log\lambda}{\log\mu}$ is irrational. Note that g does not have the shadowing property. We can see that g does not have the weak inverse shadowing property with respect to class $\mathcal{T}_0$ as well (Note that the number $\frac{\log\lambda}{\log\mu}$ is not necessary irrational). For any $0 < \varepsilon < \frac{1}{4}$, let $0 < \delta < \varepsilon$ be the number of the weak inverse shadowing property of g. Construct a δ-method as following:

$$\begin{aligned} &\varphi(p_4) \\ &= \left\{\cdots, f^{-2}(p_4), f^{-1}(p_4), p_4, x_0, f(x_0), f^2(x_0), \cdots\right\}, \end{aligned}$$

where $x_0 \in (-1,1)\times\{0\} \subset W^u(p_4)$ and $d(p_4, x_0) < \delta$. For every $x \neq p_4$, define

$$\varphi(x) = \left\{\cdots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \cdots\right\},$$

3. Relation between Minimality and Weak Inverse Shadowing Property with Respect to Class $\mathcal{T}_0$

A homeomorphism $f: X \to X$ is called minimal if $f(A) = A$, A closed, implies either $A = M$ or $A = \phi$. It is easy to see that f is minimal if and only if $\overline{o(x,f)} = X$ for every $x \in X$.

A homeomorphiosm f is said to be chain transitive if for every $x, y \in X$ and $\delta > 0$ there are δ-pseudo-orbits from x to y and from y to x.

The following example shows that there exists homeomorphiosms f with inverse shadowing property with respect to class $\mathcal{T}_0$ which is not minimal.

Example. Let $X := \left\{x = \{x_n\}_{-\infty}^{\infty} : x_n \in \{0,1\}\right\}$ with metric

$$d(x,y) = \begin{cases} 2 & \text{if } x_0 \neq y_0 \\ \dfrac{1}{\min\{|k| : x_k \neq y_k\}} & \text{if } x_0 = y_0 \end{cases}$$

Let $\pi : \{-n,-n+1,\cdots,n\} \to \{-n,-n+1,\cdots,n\}$ be a permutation of the set $\{-n,-n+1,\cdots,n\}$ for some $n \in \mathbb{N}$. Let $f_n(x) = x_{\pi(i)}$ if $-n \le i \le n$, and x_i otherwise.

f_n is a homeomorphism and every point of X is a periodic point for f_n. We claim f_n has weak inverse shadowing property with respect to class $\mathcal{T}_0$.

Proof of claim. Given $\varepsilon > 0$ choose $N_0 > n$ such that $\frac{1}{N_0} < \frac{\varepsilon}{2}$. $d(x,y) < \frac{1}{N_0}$ if and only if $x_i = y_i$ where $-N_0 \le i \le N_0$. Let $\delta = \frac{1}{N_0}$ and φ be a δ-method. Let $x, y, z \in X$, then $d(f_n(x), y) < \delta$ implies $f_n(x)_i = y_i$ for $-N_0 \le i \le N_0$ and hence by definition of f_n, $f_n^2(x)_i = f_n(y)_i$ for $-N_0 \le i \le N_0$. Also $d(f_n(y), z) < \delta$ implies $f_n(y)_i = z_i$ for $-N_0 \le i \le N_0$. Hence if $d(f_n(x), y) < \delta$ and $d(f_n(y), z) < \delta$ then $f_n^2(x)_i = z_i$ for $-N_0 \le i \le N_0$, and so $d(f_n^2(x), z) < \delta$. Using this procedure we will get $d(f_n^i(x), \varphi(x)_i) < \delta$ for $i \ge 0$. A similar reasoning with having in mind that f_n is a homeomorphism proves that $d(f_n^i(x), \varphi(x)_i) < \delta$ for $i \le 0$. Hence $d(f_n^i(x), \varphi(x)_i) < \delta$ for $i \in \mathbb{Z}$ and f_n has inverse shadowing property with respect to $\mathcal{T}_0$. It is easy to see that f_n is not minimal.

Theorem 3 *Let* f *be a chain transitive homeomorphism on compact metric space* X. *Then* f *is minimal if and only if* f *has weak inverse shadowing property with respect to class* $\mathcal{T}_0$.

Proof. Suppose that f has weak inverse shadowing property with respect to class $\mathcal{T}_0$ and $z \in X$. Let U be an open set in X. Choose $x_0 \in U$ and $\varepsilon > 0$ such that $N_\varepsilon(x_0) \subset U$. There is $\delta > 0$ such that for each δ-method φ, there is $y \in X$ such that

$$\varphi(y) \subset N_{\frac{\varepsilon}{2}}(o(x, f))$$

For every $x \in X$, there is a δ-chain, $(x, x, \cdots, x_{l_n}, x_0)$ from x to x_0. Consider

$$\delta_x = \left\{\cdots, f^{-2}(x), f^{-1}(x), x, x_1, x_2, \cdots, x_{l_n}, x_0, f(x_0), \cdots\right\}$$

as a δ-pseudo-orbit, such that it's 0-component be x. Construct a δ-method φ_{x_0} such that $\varphi_{x_0}(x) = \delta_x$. Hence there is $y \in X$ such that $\varphi_{x_0}(y) \subset N_{\frac{\varepsilon}{2}}(o(z, f))$, and so $d(x_0, f^l(z)) < \frac{\varepsilon}{2}$ for some $l \in \mathbb{Z}$. Therefore $o(z, f) \cap U \neq \varnothing$. This shows that each orbit of X is dense in X and so f is minimal. The converse *i.e.* to see that each minimal homeomorphism has weak inverse shadowing property with respect to class $\mathcal{T}_0$, is obvious.

4. Relation between Expansivity and Inverse Shadowing Property with Respect to Class $\mathcal{T}_0$

A homeomorphism f on metric space (X, d) is said expansive if there exists constant $e > 0$ such that for every $x, y \in X, (x \neq y)$ there exists integer number N_0 such that $d(f^{N_0}(x), f^{N_0}(y)) > e$.

Theorem 4 *If homeomorphism* f *on metric space* (X, d) *has the inverse shadowing property with respect to class* $\mathcal{T}_0$, *then* f *is not expansive*.

Proof. Suppose that f is expansive and has the inverse shadowing property with respect to class $\mathcal{T}_0$. Let $e > 0$ be as in definition of expansivity and $0 < \delta < e$ be such that for any δ-method φ in $\mathcal{T}_0$ and any point $x \in X$ there exists a point $y \in X$ for which $d(f^n(x), \varphi(y)_n) < e, n \in \mathbb{Z}.$

Let $x_0 \in X$ be arbitrary. Choose $y_0 \neq x_0$ such that $d(x_0, y_0) < \delta$ and $d(f(x_0), f(y_0)) < \delta$. Construct a δ-method φ as following.

For any $x \neq x_0$ define

$$\varphi(x) = \left\{\cdots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \cdots\right\}$$

and

$$\varphi(x_0) = \left\{\cdots, f^{-2}(y_0), f^{-1}(y_0), x, f(y_0), f^2(y_0), \cdots\right\}$$

Since f has the inverse shadowing property with respect to class $\mathcal{T}_0$, for x_0 there exists $y \in X$ such that

$$d(f^n(x_0), \varphi(y)_n) < e, n \in \mathbb{Z}.$$

By regarding to choose of δ-method φ, we have

$$d(f^n(x_0), f^n(y)) < e \text{ for } n \in \mathbb{Z}$$

for some $y \neq x_0$, that contradicts the expansivity of f. This completes the proof of theorem.

5. Conclusion

In this paper we showed that an Ω-stable diffeomorphism f has the weak inverse shadowing property with respect to classes of continuous method θ_s and θ_c

and some of the Ω-stable diffeomorphisms have weak inverse shadowing property with respect to classes $\mathcal{T}_0$. In addition we studied relation between minimality and weak inverse shadowing property with respect to class $\mathcal{T}_0$ and relation between expansivity and inverse shadowing property with respect to class $\mathcal{T}_0$.

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