On Certain Connected Resolving Parameters of Hypercube Networks[*]

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ABSTRACT

Given a graph $G = (V, E)$, a set $W \subset V$ is a resolving set if for each pair of distinct vertices $u, v \in V(G)$ there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for G. The cardinality of a minimum resolving set is called the *resolving number* or *dimension* of G and is denoted by $dim(G)$. A resolving set W is said to be a *star resolving set* if it induces a star, and a *path resolving set* if it induces a path. The minimum cardinality of these sets, denoted respectively by $sr(G)$ and $pr(G)$, are called the *star resolving number* and *path resolving number.* In this paper we investigate these resolving parameters for the hypercube networks.

Keywords: Resolving Set; Basis; Path Resolving Set; Star Resolving Set; Hypercube Network

1. Introduction

A query at a vertex v discovers or verifies all edges and non-edges whose endpoints have different distance from v. In the network verification problem [1], the graph is known in advance and the goal is to compute a minimum number of queries that verify all edges and non-edges. This problem has previously been studied as the problem of placing landmarks in graphs or determining the metric dimension of a graph [2]. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [3-5].

For an ordered set $W = \{w_1, w_2 \cdots w_k\}$ of vertices and a vertex v in a connected graph G, the *code* or *representation* of v with respect to W is the k-vector

$$C_W(v) = (d(v, w_1), d(v, w_2) \cdots d(v, w_k))$$

where $d(x, y)$ is the distance between the vertices x and y. The set W is a *resolving set* for G if distinct vertices of G have distinct codes with respect to W. Equivalently, for each pair of distinct vertices $u, v \in V(G)$ there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$. The minimum cardinality of a re-

solving set for G is called the *resolving number* or *dimension* and is denoted by $dim(G)$.

2. An Overview of the Paper

The concept of resolvability in graphs has previously appeared in literature. Slater [4,5] introduced this concept, under the name *locating sets*, motivated by its application to the placement of a minimum number of sonar detecting devices in a network so that the position of every vertex in the network can be uniquely determined in terms of its distance from the set of devices. He referred to a minimum resolving set as a reference set and called the cardinality of a minimum resolving set as the *location number*. Independently, Harary and Melter [3] discovered this concept, but used the term metric dimension, rather than location number. Later, Khuller *et al.* [2] also discovered these concepts independently and used the term metric dimension. These concepts were rediscovered by Chartrand *et al.* [6] and also by Johnson [7] while attempting to develop a capability of large datasets of chemical graphs.

It was noted in [8] that determining the metric dimension of a graph is *NP*-complete. It has been proved that the metric dimension problem is *NP*-hard [2] for general graphs. Manuel *et al.* [9] have shown that the problem remains *NP*-complete for bipartite graphs. There are many applications of resolving sets to problems of

network discovery and verification [1], pattern recognition, image processing and robot navigation [2], geometrical routing protocols [10], connected joins in graphs [11] and coin weighing problems [12]. This problem has been studied for trees, multi-dimensional grids [2], Petersen graphs [13], torus networks [14], Benes networks [9], honeycomb networks [15], enhanced hypercubes [16] and Illiac networks [17].

Many resolving parameters are formed by combining resolving property with another common graph-theoretic property such as being connected, independent, or acyclic. The generic nature of conditional resolvability in graphs provides various ways of defining new resolving parameters by considering different conditions. In general, a connected graph G can have many resolving sets. It is interesting to study those resolving set whose vertices are located close to one another. A resolving set W of G is connected if the subgraph induced by W is a nontrivial connected subgraph of G. The minimum cardinality of a connected resolving set is called *connected resolving number* and it is denoted by $cr(G)$ [18]. In this paper we introduce a new resolving parameter called *star resolving number*. A resolving set W is said to be a star resolving set if the subgraph induced by W is a star and a path resolving set [19] if W induces a path. In this paper we show the existence of *star* and *path* resolving sets in hypercube networks.

3. Topological Properties of Hypercube Networks

The hypercube is a very popular, versatile and vertex-transitive interconnection network. When the dimension of hypercube increases, the cardinality of its vertex set increases exponentially. The effectiveness of parallel computers is often determined by its communication network. The interconnection network is an important component of a parallel processing system. A good interconnection network should have less topological network cost and meanwhile keep the network diameter as shorter as possible [20].

Definition 3.1. *Let* Q^r *denote the graph of* r-*dimensional hypercube,* $r \geq 1$. *The vertex set*
$V(Q^r) = \{(x_0 x_1 \cdots x_{r-1}) : x_i = 0 \text{ or } 1\}$ *Two vertices*
$(x_0 x \cdots x_{r-1})$ *and* $(y_0 y_1 \cdots y_{r-1})$ *are adjacent if and only if they differ exactly in one position. See* **Figure 1**.

The hypercube Q^r has 2^r vertices and $r2^r$ edges. It is r-regular and its diameter is r. Further it is bipartite, Hamiltonian if $r \neq 1$ and Eulerian if r is even [21]. It has been proved in [22] that $dim(Q^r) \leq r$. The bound is tight for $r \leq 4$, and it is not tight for $r = 5$. A laborious calculation verifies that Q^5 is resolved by the 4-vertex set $\{00000, 00011, 00101, 01001\}$. Caceres
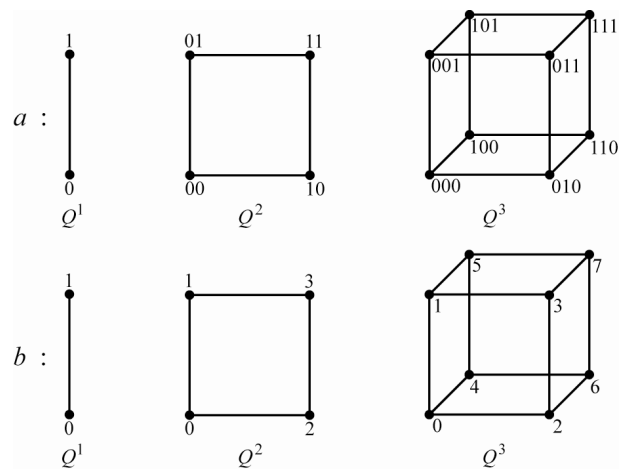

Figure 1. (a) Binary representation; (b) Decimal representation.

et al. [22] have determined $dim(Q^r)$ for small values of r by computer search; the values are shown in the following table:

r	2	3	4	5	6	7	8	9	10
$dim(Q^r)$	2	3	4	4	5	6	6	7	7

4. Star Resolving Number

We begin this section by defining a star and a star resolving set.

Definition 4.1. *An* r-*dimensional star, denoted by* S_r *is a graph with one vertex of degree* $r-1$ *and* $r-1$ *vertices of degree* 1. *The vertex of degree* $r-1$ *is called the hub of* S_r.

Definition 4.2. *A set* $W \subset V$ *is said to be a star resolving set if* W *resolves* G *and if it induces a star. The minimum cardinality of* W *is called the star resolving number and is denoted by* $sr(G)$.

Remark 1. It is clear that $1 \leq sr(G) \leq \Delta(G)+1$ for any graph G. In a star resolving set the maximum distance between any two locations (vertices) is 2.

We now proceed to identify a star resolving set in a hypercube network Q^r. It is clear that there are four copies of Q^{r-2} in Q^r. We denote them as Q_0^{r-2}, $Q_{1,1}^{r-2}$, $Q_{1,2}^{r-2}$ and Q_2^{r-2}. **Figure 2** exhibits the four copies of Q^3 in Q^5.

Let $x \in V(Q_0^{r-2})$. A vertex $x' \in V(Q_{1,1}^{r-2})$ or $V(Q_{1,2}^{r-2})$ is called the *image* of x if $d(x,x')=1$. Note that vertices in Q_0^{r-2}, at distance 1 from x are not considered as images of x. If x' is the image of x in $Q_{1,1}^{r-2}$ then x is called the *pre-image* of x'.

The next result which we state as Lemma 1 is crucial to our work. We omit the proof as this result has been proved in [16] for enhanced hypercubes.
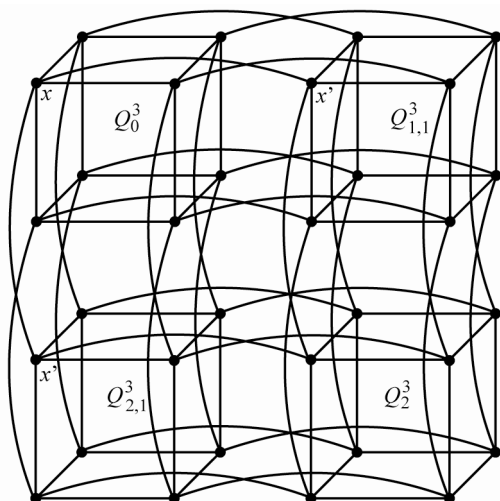
Figure 2. Four copies of Q_3 in Q_5.

Lemma 4.1. *Let* $x \in V(Q_0^{r-2})$ *and let* $x' \in V(Q_{1,1}^{r-2})$ *be the image of* x. *Let* w *be any vertex in* Q_0^{r-2}. *Then* $d(x',w) = 1 + d(x,w)$.

Lemma 4.2. *Let* $x \in V(Q_0^{r-2})$. *Let* $x_1' \in V(Q_{1,1}^{r-2})$ *and* $x_2' \in V(Q_{1,2}^{r-2})$ *be the images of* x. *Then* x_1' *and* x_2' *are equidistant from every vertex of* Q_0^{r-2}.

Proof. Since the shortest paths from x_1' and x_2' to any vertex of Q_0^{r-2} pass through x, the conclusion follows.

Lemma 4.3. *Let* $G = Q^r$, $r \geq 1$. *Then* $sr(G) \geq r$.

Proof. The subcube Q_0^{r-2} of Q^r is $r-2$ regular and hence it contains S_{r-1}. Now there exist vertices $x \in Q_{1,1}^{r-2}$ and $y \in Q_{1,2}^{r-2}$ such that x, y are equidistant from every vertex of Q_0^{r-2} and in particular from every vertex of S_{r-1}. This implies $sr(G) > r-1$.

Lemma 4.4. *Let* $G = Q^r$. *Then* $sr(G) \leq r$.

Proof. We prove the theorem by induction on r.

Base Case: Let $G = Q^3$ and $W_1 = \{w_0, w_1, w_2\}$, where $w_0 = 0, w_1 = 1$ and $w_2 = 2$. It follows from the definition of hypercube edges that w_0 is adjacent to both w_1 and w_2. It is easy to check that W_1 is a resolving set for Q^3. **Figure 3** shows the distinct codes of vertices in Q^3, with respect to $W_1 = \{w_0, w_1, w_2\}$. Since W_1 induces S_3, it is a star resolving set for G.

Now assume that the result is true for the hypercube Q^{r-1}. Let $W_1 = \{w_0\} \cup \{w_i : 1 \leq i \leq r-2\}$, where $w_i = 2^{i-1}$, be a star resolving set for Q^{r-1}. Here w_0 is the hub and it is adjacent to all $w_i, 1 \leq i \leq r-2$. Moreover $W_1 \subset V(Q_0^{r-2})$. Divide Q^r into four copies $Q_0^{r-2}, Q_{1,1}^{r-2}, Q_{1,2}^{r-2}$ and Q_2^{r-2}. There exist vertices $x \in Q_{1,1}^{r-2}$ and $y \in Q_{1,2}^{r-2}$ having the same codes with respect to every vertex of Q_0^{r-2} and in particular with respect to every vertex of W_1. Hence W_1 cannot resolve x and y. We exhibit a resolving set for Q^r. Define $W = W_1 \cup \{w_{r-1}\}$ where $w_{r-1} = 2^{r-2}$ is a vertex
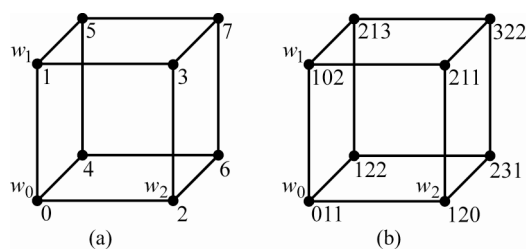

Figure 3. (a) Resolvingset W_1 in Q_3 ; (b) Codes of vertices of Q_3 with respect to W_1.

either in $Q_{1,1}^{r-2}$ or $Q_{1,2}^{r-2}$. Clearly w_{r-1} is adjacent to w_0. We claim that W is a resolving set for Q^r.

Case 1: $x, y \in V(Q_0^{r-2})$ or $V(Q_{1,1}^{r-2})$ or $V(Q_{1,2}^{r-2})$

Since $W_1 \subset V(Q_0^{r-2})$ and since $Q_0^{r-2} \cup Q_{1,1}^{r-2}$ and $Q_0^{r-2} \cup Q_{1,2}^{r-2}$ are isomorphic to Q^{r-1}, by induction hypothesis W_1 resolves x and y. The same argument applies to the following cases.

1) $x \in V(Q_0^{r-2})$ and $y \in V(Q_{1,1}^{r-2})$

2) $x \in V(Q_0^{r-2})$ and $y \in V(Q_{1,2}^{r-2})$

Case 2: $x \in V(Q_{1,1}^{r-2})$ and $y \in V(Q_{1,2}^{r-2})$

We need to prove that $d(x,w) \neq d(y,w)$ for some w in $W = \{w_0, w_1, w_2 \cdots w_{r-2}\} \cup \{w_{r-1}\}$. Let $x', y' \in V(Q_0^{r-2})$ be the images of x and y respectively.

Case 2.1: $x' \neq y'$

In this case

$$d(y, w_{r-1}) = d(y, y') + d(y', w_{r-1}) = 1 + d(x', w_{r-1})$$
$$= 1 + 1 + d(x, w_{r-1}) \neq d(x, w_{r-1}).$$

Case 2.2: $x' \neq y'$

Now x' and y' are resolved by some w in W_1. Hence $d(x', w) \neq d(y', w)$ and consequently $d(x, w) \neq d(y, w)$.

Case 3: $x \in V(Q_2^{r-2})$ and $y \in V(Q_2^{r-2})$

The proof is similar to Case 2.

Case 4: $x, y \in V(Q_2^{r-2})$

Let x' and y' be the images of x and y respectively. There are three possibilities $x', y' \in V(Q_{1,1}^{r-2})$ or $V(Q_{1,2}^{r-2})$ or $x' \in V(Q_{1,1}^{r-2})$ and $y' \in V(Q_{1,2}^{r-2})$. The conclusion will follow by Case 1 and Case 2.

Case 5: $x \in V(Q_{1,1}^{r-2})$ and $y \in V(Q_2^{r-2})$

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Let $x' \in V\left(Q_0^{r-2}\right)$ and $y' \in V\left(Q_{1,2}^{r-2}\right)$ be the images of x and y respectively. Since $Q_0^{r-2} \cup Q_{1,2}^{r-2}$ is resolved by W_1, there exist a $w \in W_1$ such that $d\left(x',w\right) \neq d\left(y',w\right)$. This implies that $d\left(x,w\right) \neq d\left(y,w\right)$.

Lemmas 3 and 4 imply the following result.

Theorem 4.1. *Let* $G = Q^r, r \geq 1$. *Then* $sr\left(G\right) = r$.

5. Path Resolving Number

In this section we determine a path resolving number for hypercube networks.

Definition 5.1. [19] A resolving set W of G is a path resolving set for G if the graph induced by W is a path. The minimum cardinality of W is called path resolving number and is denoted by $pr\left(G\right)$.

Lemma 5.1. *Let* $G = Q^r$, $r \geq 1$. *Then* $pr\left(G\right) \geq r$.

Proof. Let P be the path in Q_0^{r-2}. Now P cannot resolve Q^r as there are vertices $x \in Q_{1,1}^{r-2}$ and $y \in Q_{1,2}^{r-2}$ such that they are equidistant from every vertex of Q_0^{r-2} in particular from every vertex of P Since there exist a path P_r in Q_0^{r-2}, $pr\left(G\right) \geq r$.

Lemma 5.2. *Let* $G = Q^r$, $r \geq 1$. *Then* $pr\left(G\right) \leq r$.

Proof. Proceeding as in Lemma 4 we conclude that $W = \left\{2^i - 1, 1 \leq i \leq r\right\}$ is a path resolving set for Q^r.

Lemma 5 and Lemma 6 imply the following result.

Theorem 5.1. *Let* $G = Q^r, r \geq 1$. *Then* $pr\left(G\right) = r$.

6. Conclusion

In this paper we have introduced a new resolving parameter called a *star resolving number*. We have determined the star resolving number and path resolving number for hypercube networks. The problem is open for architectures like Benes and Butterfly networks.

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