Scientific Research

A Similarity Technique for Solving Two-Layer Shallow-Water Equations

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ABSTRACT

This paper is devoted to the analysis of the two-layer shallow-water equations representing gravity currents. A similarity technique which is the characteristic function method is applied for this study. The application of the characteristic function method makes it possible to obtain the similarity forms depending on a group of infinitesimal transformations. Thus, the number of independent variables is reduced by one and the governing partial differential equations with the auxiliary conditions reduce to a system of ordinary differential equations with the appropriate auxiliary conditions. Numeric solutions are presented and discussed.

Keywords: The Characteristic Function Method; The Two-Layer Shallow-Water Equations; Gravity Currents

1. Introduction

The present study focuses on a two-layer shallow-water system of incompressible, immiscible and inviscid fluids with a free surface. When two fluids of differing densities interact in such a way that a vertical interface exists between the fluids, the resulting motion consists of the heavier fluid flowing horizontally beneath the lighter fluid. Such a flow is said to form gravity current, an overall view of many of the phenomena associated with the gravity currents is presented by Simpson [1]. Various numerical methods have been employed to solve these model equations such as finite difference, finite element and spectral methods [2]. Leveque use MacCromick's method [3] and Godunov's method used by Godlewski *et al.* [4] which gives numerical standard schemes to solve the systems of conservation laws. Jin *et al.* [5] have presented finite difference methods, called relaxation schemes. Thereby, Montogomery *et al.* [6] have used these relaxation schemes for systems of conservation laws associated with the gravity currents for a two-layer model. D'Alesio *et al.* [7]. Gravity currents considering Lie symmetry groups has been investigated by Glaister [8]. He did apply Lie symmetry groups of two-dimensional shallow-water equations with cylindrical symmetry numerically in conjunction with the Rankine_Hugoniot shock relations. Velan *et al.* [9] studied Lie symmetries and found the invariant solutions of the dispersive shallow-water equation which is in the single equation form. Özer [10,11]. Also there are several solution techniques to deal with the determining equations in the Lie group

analysis of differential equations [12,13]. The main purpose of this paper is to find similarity solutions of two-layer shallow-water equations representing gravity currents by using the characteristic function method. As the characteristic function method is not based on linear operators, it is applicable to both linear and nonlinear differential models [14-16].

2. Mathematical Formulation of the Problem

In this study, we consider a two-layer shallow water resting on a horizontal surface with respective densities ρ_1, ρ_2. We neglect the friction between the fluids and the bottom and we also assume that the effect of viscosity is negligible. According to the shallow-water theory, we shall assume that the length of the current is much larger than its depth. By using this assumption we neglect the vertical accelerations and we can say that the pressure is hydrostatic. Assuming that the depth of an ambient fluid is much larger than the thickness of the current. The horizontal velocities u1 in the upper layer and u2 in the lower layer are independent of height and pressure field by using the assumption that the pressure is hydrostatic [17]. We also assume that Reynolds number of the flow is sufficiently high so that viscous forces are negligible, as well as the surface tension. We also assume that there is no mixing between layers. By employing these assumptions and using the kinematics and dynamic boundary conditions at the interface, the two-layer shallow-water equations are modeled as follows [18].

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial \tau}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial h}{\partial t} - \gamma \frac{\partial \tau}{\partial t} + h \frac{\partial u_1}{\partial x} + u_1 \frac{\partial h}{\partial x} - H \frac{\partial u_1}{\partial x} - \gamma \tau \frac{\partial u_1}{\partial x} - \gamma u_1 \frac{\partial \tau}{\partial x} = 0$$
$$(2)$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \frac{\partial h}{\partial x} + (1-\gamma) \frac{\partial \tau}{\partial x} = 0 \qquad (3)$$

$$\frac{\partial h}{\partial t} + h \frac{\partial u_2}{\partial x} + u_2 \frac{\partial h}{\partial x} = 0 \qquad (4)$$

This model is illustrated in **Figure 1**, where $\tau(x,t)$ represents the displacement of the upper layer, $h(x,t)$ is the thickness of the lower layer, H is the mean total depth, x is an coordinate system with the *x-axis* along the bottom. $u_i(x,t)$, $(i = 1, 2)$, denotes the horizontal velocity components for the upper and the lower layer, t is the time and $\gamma \equiv (g'/g)$, g' is the combined gravity defined by $g' = (\rho_2 - \rho_1)g/\rho_2$, g is the gravity.

We study the system under the following conditions;

$$u_1(0,t) = u_{10} f_1(t)$$
$$u_2(0,t) = u_{20} f_2(t)$$
$$h(x,t) = h_0 f_2(t) \qquad (5)$$
$$\tau(x,t) = \tau_0 f_2(t) - H/\gamma$$

2.1. Invariance Analysis

The infinitesimal transformation of the system variables $(t, x; u_1, u_2, h, \tau)$ is defined as follows;

$$\bar{t} = t + \varepsilon A(t,x,u_1,u_2,h,\tau) + O(\varepsilon^2)$$
$$\bar{x} = x + \varepsilon B(t,x,u_1,u_2,h,\tau) + O(\varepsilon^2)$$
$$\bar{u}_1 = u_1 + \varepsilon M_1(t,x,u_1,u_2,h,\tau)) + O(\varepsilon^2)$$
$$\bar{u}_2 = u_2 + \varepsilon M_2(t,x,u_1,u_2,h,\tau) + O(\varepsilon^2) \qquad (6)$$
$$\bar{h} = h + \varepsilon M_3(t,x,u_1,u_2,h,\tau) + O(\varepsilon^2)$$
$$\bar{\tau} = \tau + \varepsilon M_4(t,x,u_1,u_2,h,\tau) + O(\varepsilon^2)$$
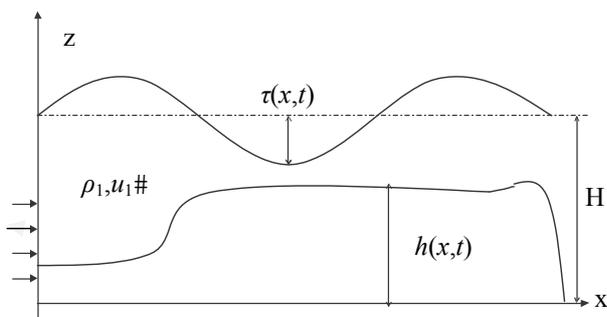
Figure 1. Two-layer shallow-water representation.

The transformation of u_1, u_2, h and τ derivatives p's, q's, r's and s's are defined as;

$$p_1 = \frac{\partial u_1}{\partial t}, p_2 = \frac{\partial u_1}{\partial x}, p_{12} = \frac{\partial^2 u_1}{\partial t \partial x},$$

$$q_1 = \frac{\partial u_2}{\partial t}, q_2 = \frac{\partial u_2}{\partial x}, q_{12} = \frac{\partial^2 u_2}{\partial t \partial x},$$

$$r_1 = \frac{\partial h}{\partial t}, r_2 = \frac{\partial h}{\partial x}, r_{12} = \frac{\partial^2 \tau}{\partial t \partial x}, \qquad (7)$$

$$s_2 = \frac{\partial \tau}{\partial x}, s_{12} = \frac{\partial^2 \tau}{\partial t \partial x}, s_1 = \frac{\partial \tau}{\partial t},$$

$$\bar{p}_i = p_i + \varepsilon P_i(t,x,u_1,u_2,h,\tau,p_1,p_2,p_{12},$$
$$q_1,q_2,q_{12},r_1,r_2,r_{12},s_1,s_2,s_{12}) + O(\varepsilon^2)$$

$$\bar{q}_i = q_i + \varepsilon Q_i(t,x,u_1,u_2,h,\tau,p_1,p_2,p_{12},$$
$$q_1,q_2,q_{12},r_1,r_2,r_{12},s_1,s_2,s_{12}) + O(\varepsilon^2)$$

$$\bar{r}_i = r_i + \varepsilon R_i(t,x,u_1,u_2,h,\tau,p_1,p_2,p_{12},$$
$$q_1,q_2,q_{12},r_1,r_2,r_{12},s_1,s_2,s_{12}) + O(\varepsilon^2)$$

$$\bar{s}_i = s_i + \varepsilon S_i(t,x,u_1,u_2,h,\tau,p_1,p_2,p_{12},$$
$$q_1,q_2,q_{12},r_1,r_2,r_{12},s_1,s_2,s_{12}) + O(\varepsilon^2)$$

$$\bar{p}_{ji} = p_{ji} + \varepsilon P_{ji}(t,x,u_1,u_2,h,\tau,p_1,p_2,p_{12},$$
$$q_1,q_2,q_{12},r_1,r_2,r_{12},s_1,s_2,s_{12}) + O(\varepsilon^2)$$

$$\bar{q}_{ji} = q_{ji} + \varepsilon Q_{ji}(t,x,u_1,u_2,h,\tau,p_1,p_2,p_{12},$$
$$q_1,q_2,q_{12},r_1,r_2,r_{12},s_1,s_2,s_{12}) + O(\varepsilon^2)$$

$$\bar{r}_{ji} = r_{ji} + \varepsilon R_{ji}(t,x,u_1,u_2,h,\tau,p_1,p_2,p_{12},$$
$$q_1,q_2,q_{12},r_1,r_2,r_{12},s_1,s_2,s_{12}) + O(\varepsilon^2)$$

$$\bar{s}_{ji} = s_{ji} + \varepsilon S_{ji}(t,x,u_1,u_2,h,\tau,p_1,p_2,p_{12},$$
$$q_1,q_2,q_{12},r_1,r_2,r_{12},s_1,s_2,s_{12}) + O(\varepsilon^2) \qquad (8)$$

where the subscript i and j stand for derivative with respect to t, x.

A, B, M's, P's, Q's, S's and S's are the infinitesimal prolongations. According to these Definitions (1)-(4) reduce to;

$$G_1 = p_1 + u_1 p_2 + s_2 = 0 \qquad (9)$$

$$G_2 = r_1 - \gamma s_1 + h p_2 + u_1 r_2 - H p_2 - \gamma \tau p_2 - \gamma u_1 s_2 = 0$$
$$(10)$$

$$G_3 = q_1 + u_2 q_2 + r_2 + (1-\gamma) s_2 = 0 \qquad (11)$$

$$G_4 = r_1 + h q_2 + u_2 r_2 = 0 \qquad (12)$$

the system of differential Equations (1)-(4) of the form $G_i = 0$, $(i = 1 - 4)$, will be invariant if $DG_i = 0$, Where the

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total operator D is written as:

$$
\begin{aligned}
D = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + M_1\frac{\partial}{\partial u_1} + M_2\frac{\partial}{\partial u_2} + M_3\frac{\partial}{\partial h} \\
+ M_4\frac{\partial}{\partial \tau} + P_1\frac{\partial}{\partial p_1} + P_2\frac{\partial}{\partial p_2} + P_{12}\frac{\partial}{\partial p_{12}} + Q_1\frac{\partial}{\partial q_1} \\
+ Q_2\frac{\partial}{\partial q_2} + Q_{12}\frac{\partial}{\partial q_{12}} + R_1\frac{\partial}{\partial r_1} + R_2\frac{\partial}{\partial r_2} + R_{12}\frac{\partial}{\partial r_{12}} \\
+ S_1\frac{\partial}{\partial s_1} + S_2\frac{\partial}{\partial s_2} + S_{12}\frac{\partial}{\partial s_{12}}
\end{aligned}
\tag{13}
$$

The applications of the total operator D to Equations (7)-(10) gives:

$$
DG_1 = M_1\frac{\partial u_1}{\partial x} + P_1 + u_1 P_2 + S_2 = 0
\tag{14}
$$

$$
\begin{aligned}
DG_2 = R_1 - \gamma S_1 + h P_2 + M_3\frac{\partial u_1}{\partial x} \\
+ M_1\frac{\partial h}{\partial x} + u_1 R_2 - H P_2 - \gamma \tau P_2 \\
- \gamma\frac{\partial u_1}{\partial x}M_4 - \gamma M_1\frac{\partial \tau}{\partial x} - \gamma u_1 S_2 = 0
\end{aligned}
\tag{15}
$$

$$
DG_3 = Q_1 + M_2\frac{\partial u_2}{\partial x} + u_2 Q_2 + R_2 + (1-\gamma)S_2 = 0
\tag{16}
$$

$$
DG_4 = R_1 + M_3\frac{\partial u_2}{\partial x} + h Q_2 + M_2\frac{\partial h}{\partial x} + u_2 R_2 = 0
\tag{17}
$$

2.2. Evaluation of Infinitesimals

Here we will find the explicit solutions of the infinitesimal functions A, B and $M's$, by solving the Equations (14)-(17). "The power-series solution form" is one of the most effective techniques for finding the solutions of determining equations in the symmetry group analysis of differential equations [14,19]. So we consider the following power-series forms for the infinitesimal functions:

$$
\begin{aligned}
A = A_1 + A_2 t + A_3 x + A_4 t^2 + A_5 x^2 + A_6 t x \\
+ A_7 t^2 x + A_8 x^2 + \cdots +
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
B = B_1 + B_2 t + B_3 x + B_4 t^2 + B_5 x^2 \\
+ B_6 t x + B_7 t^2 x + B_8 t x^2 + \cdots + \cdots
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
M_1 = M_{11} + M_{12} t + M_{13} x + M_{14} u_1 + M_{15} u_2 \\
+ M_{16} h + M_{17}\tau + M_{18} xt + \cdots +
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
M_2 = M_{21} + M_{22} t + M_{23} x + M_{24} u_1 + M_{25} u_2 \\
+ M_{26} h + M_{27}\tau + M_{28} xt + \cdots +
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
M_3 = M_{31} + M_{32} t + M_{33} x + M_{34} u_1 + M_{35} u_2 \\
+ M_{36} h + M_{37}\tau + M_{38} xt + \cdots +
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
M_4 = M_{41} + M_{42} t + M_{43} x + M_{44} u_1 + M_{45} u_2 \\
+ M_{46} h + M_{47}\tau + M_{48} xt + \cdots +
\end{aligned}
\tag{23}
$$

where A_{ij}, B_{ij} and M_{ij} $i, j = 1, 2$ are constant coefficients. Then substituting the power-series forms (18)-(23) into the determining Equations (14)-(17) we equate powers of the variables x, t, u_1, u_2, h and τ and calculate the constant coefficients of the power-series forms by equating each coefficient of various powers to zero, which gives the general characteristic function of two-layer shallow-water equations. After the straightforward calculations for any finite integer order of power-series forms, we find that:

$$
A = (A_1 - A_2)t + A_3
\tag{24}
$$

$$
B = (2A_1 - A_2)x + B_1
\tag{25}
$$

$$
M_1 = A_1 u_1
\tag{26}
$$

$$
M_2 = A_1 u_2
\tag{27}
$$

$$
M_3 = 2A_1 h
\tag{28}
$$

$$
M_4 = 2A_1(\tau + H/\gamma)
\tag{29}
$$

2.3. Reduction to Ordinary Differential Equations

In this section, we will obtain the reduced forms of the two-layer shallow-water equations by using infinitesimal group transformations obtained in the previous section. Here we will try to reduce for each sub algebra in the optimal system to obtain the reduced forms of the system (1)-(4). For this purpose, we need to write the characteristic equation in the following form:

$$
\begin{aligned}
\frac{dt}{(A_1 - A_2)t + A_3} = \frac{dx}{(2A_1 - A_2)x + B_1} = \frac{du_1}{A_1 u_1} \\
= \frac{du_2}{A_1 u_2} = \frac{dh}{2A_1 h} = \frac{d\tau}{2A_1(\tau + H/\gamma)}
\end{aligned}
\tag{30}
$$

Case (1). $A_3 = 0$ and $A_1 = B_1 = 1$, $A_2 = 2A_1$. The characteristic equation can be rewritten as

$$
\frac{dt}{-t} = \frac{dx}{1} = \frac{du_1}{u_1} = \frac{du_2}{u_2} = \frac{dh}{2h} = \frac{d\tau}{2(\tau + H/\gamma)}
\tag{31}
$$

Solving Equation (31) yields the similarity variable,

$$
\eta = t e^x
\tag{32}
$$

And the similarity forms are obtained by the integration of equations in the characteristic Equation (31) giving

$$
u_1(x,t) = \frac{F_1(\eta)}{t}, u_2(x,t) = \frac{F_2(\eta)}{t}
\tag{33}
$$

$$h(x,t) = \frac{F_3(\eta)}{t^2}, \tau(x,t) = \frac{F_4(\eta)}{t^2} - \frac{H}{\gamma} \qquad (34)$$

with the corresponding boundary conditions

$$u_1(0,t) = u_{10}/t, \ u_2(0,t) = u_{20}/t \qquad (35)$$

$$h(x,t) = h_0/t^2, \tau(x,t) = \tau_0/t^2 - H/\gamma \qquad (36)$$

Now, we have the reduced system of ordinary differential equations

$$-F_1 + \eta F_1' + \eta F_1 F_1' + \eta F_4' = 0 \qquad (37)$$

$$\eta F_3' - \gamma \eta F_4' + 2\gamma F_4 + \eta F_3 F_1' + \eta F_1 F_3' - \gamma \eta F_1' - \gamma \eta F_1 F_4' = 0 \qquad (38)$$

$$-F_2 + \eta F_2' + \eta F_2 F_2' + \eta F_3' + (1-\gamma)\eta F_4' = 0 \qquad (39)$$

$$-2F_3 + \eta F_3' + \eta F_3 F_2' + \eta F_2 F_3' = 0 \qquad (40)$$

with the reduced associated boundary conditions

$$F_1(0) = F_{10}, F_2(0) = F_{20} \qquad (41)$$

$$F_3(x,t) = F_{30}, F_4(0) = F_{40} \qquad (42)$$

Case (2). $A_3 = B_1 = 0$ and $A_1 = 1, A_2 = 2A_1$

We can rewrite the characteristic equation in the following form as:

$$\frac{dt}{-A_1 t} = \frac{dx}{0} = \frac{du_1}{A_1 u_1} = \frac{du_2}{A_1 u_2} = \frac{dh}{2A_1 h} = \frac{d\tau}{2A_1(\tau + H/\gamma)} \qquad (43)$$

Giving the similarity variable,

$$\eta = x \qquad (44)$$

Integrating Equation (43), the similarity forms are obtained as

$$u_1(x,t) = \frac{F_1(\eta)}{t}, u_2(x,t) = \frac{F_2(\eta)}{t} \qquad (45)$$

$$h(x,t) = \frac{F_3(\eta)}{t^2}, \tau(x,t) = \frac{F_4(\eta)}{t^2} - H/\gamma \qquad (46)$$

with the corresponding boundary conditions

$$u_1(0,t) = u_{10}/t, u_2(0,t) = u_{20}/t \qquad (47)$$

$$h(x,t) = h_0/t^2, \tau(x,t) = \tau_0/t^2 - H/\gamma \qquad (48)$$

and the reduced system of ordinary differential equations is

$$F_1 - F_1 F_1' - F_4' = 0 \qquad (49)$$

$$-2F_3 + 2\gamma F_4 + F_3 F_1' + F_1 F_3' - \gamma F_4 F_1' - \gamma F_1 F_4' = 0 \qquad (50)$$

$$-F_2 + F_2 F_2' + F_3' + (1-\gamma) F_4' = 0 \qquad (51)$$

$$-2F_3 + F_3 F_2' + F_2 F_3' = 0 \qquad (52)$$

with the reduced boundary conditions

$$F_1(0) = F_{10}, F_2(0) = F_{20} \qquad (53)$$

$$F_3(x,t) = F_{30}, F_4(0) = F_{40} \qquad (54)$$

Case (3). $A_1 = 2, A_2 = A_3 = B_1 = 1$

The characteristic equation is

$$\frac{dt}{t+1} = \frac{dx}{3x+1} = \frac{du_1}{2u_1} = \frac{du_2}{2u_2} = \frac{dh}{4h} = \frac{d\tau}{4(\tau + H/\gamma)} \qquad (55)$$

Giving the similarity variable

$$\eta = \frac{(t+1)^3}{3x+1} \qquad (56)$$

and the similarity forms

$$u_1(x,t) = (t+1)^2 F_1(\eta), \ u_2(x,t) = (t+1)^2 F_2(\eta) \qquad (57)$$

$$h(x,t) = (t+1)^4 F_3(\eta), \tau(x,t) = (t+1)^4 F_4(\eta) - H/\gamma \qquad (58)$$

with the boundary conditions

$$u_1(0,t) = (t+1)^2 u_{10}, u_2(0,t) = (t+1)^2 u_{20} \qquad (59)$$

$$h(0,t) = (t+1)^4 h_0, \tau(0,t) = (t+1)^4 \tau_0 - H/\gamma \qquad (60)$$

The reduced system of ordinary differential equations is

$$2F_1 + 3\eta F_1' - 3\eta^2 F_1 F_1' - 3\eta^2 F_4' = 0 \qquad (61)$$

$$4F_3 + 3\eta F_3' - 4\gamma F_4 - 3\gamma \eta F_4' - 3\eta^2 F_3 F_1' \\ -3\eta^2 F_1 F_3' + 3\eta^2 \gamma F_4 F_1' + 3\gamma \eta^2 F_1 F_4' = 0 \qquad (62)$$

$$2F_2 + 3\eta F_2' - 3\eta^2 F_2 F_2' - 3\eta^2 F_3' - 3(1-\gamma)\eta^2 F_4' = 0 \qquad (63)$$

$$4F_3 + 3\eta F_3' - 3\eta^2 F_3 F_2' - 3\eta^2 F_2 F_3' = 0 \qquad (64)$$

with the reduced boundary conditions

$$F_1(0) = F_{10}, F_2(0) = F_{20} \qquad (65)$$

$$F_3(x,t) = F_{30}, F_4(0) = F_{40} \qquad (66)$$

3. Numerical Results and Discussion

As the analytic solution of systems of ordinary differential Equations (37)-(40), (49)-(52) and (61)-(64), we solve these systems of ordinary differential equations numerically by using fourth order-Runge-Kutta method coupled with the shooting method by employing the corresponding boundary conditions in terms of the similarity variables given above, the results are illustrated in the following figures.

Figure 2 shows the behavior of the dimensionless velocities of the two layers $F_1(\eta)$ and $F_2(\eta)$, where $F_2(\eta)$ increases with respect to the similarity variable similar to log-shape, at the same time $F_1(\eta)$ increases up to $\eta \cong 0.03$ after that the velocity is nearly constant. On the other hand **Figure 3** shows the behavior of the height of the lower layer $F_3(\eta)$ and the free surface of
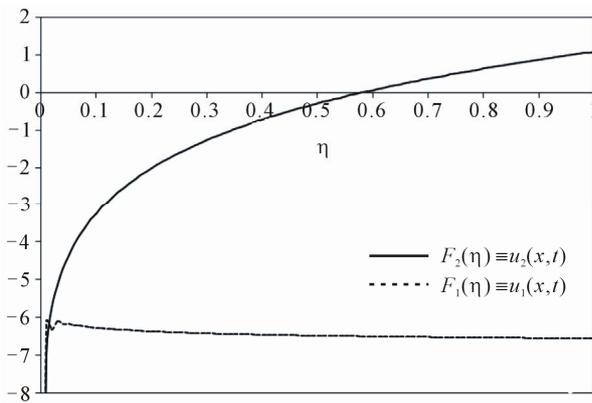
Figure 2. Singular wave solutions for the flow velocities $F_1(\eta)$, $F_2(\eta)$ **corresponding to Case (1) for large value of** η **and** $\gamma = 1.0$**, where** $F_1(\eta)$ **refer to** $u_1(x,t)$ **and** $F_2(\eta)$ **refer to** $u_2(x,t)$.
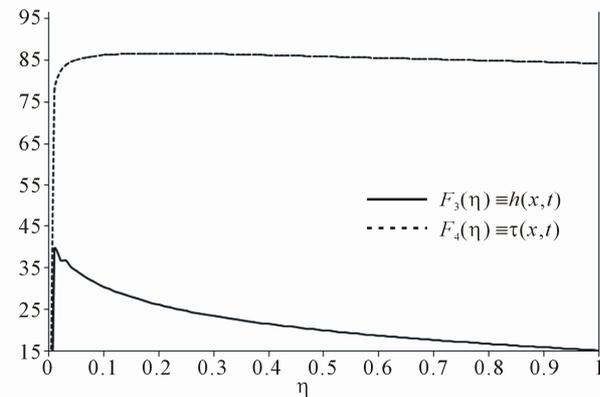
Figure 3. Singular wave solutions of the height of the lower layer $F_3(\eta)$ **and the free surface of the upper layer** $F_4(\eta)$ **of Case (I) for large value of** η **and** $\gamma = 1.0$**, where** $F_3(\eta)$ **refer to** $h(x,t)$ **and** $F_4(\eta)$ **refer to** $\tau(x,t)$.

the upper layer $F_4(\eta)$ they increase until $\eta \cong 0.03$ after that $F_4(\eta)$ decrease with low rate, at the same time $F_3(\eta)$ decrease with high rate similar to exponential-shape.

In **Figures 4-7**, there is a similarity in the shape between the two velocities $F_1(\eta)$ and $F_2(\eta)$ from one hand, and between the profile depth of the lower layer $F_3(\eta)$ against the free displacement of the upper layer $F_4(\eta)$ from another hand, because we have paid the two fluids in the same direction and velocities close to each other. These velocities and the profile depth of the lower layer and the free displacement of the upper layer increase abruptly up to $\eta \cong 0.025$, after this value, the velocities $F_4(\eta)$ and $F_2(\eta)$ and the profile depth of the lower layer $F_3(\eta)$ and the free displacement of the upper layer $F_4(\eta)$ linearly increase in a low rate with respect to the value of η.

According to **Figures 8** and **9**, under a specific condition there are disturbances on the dimensionless veloci-
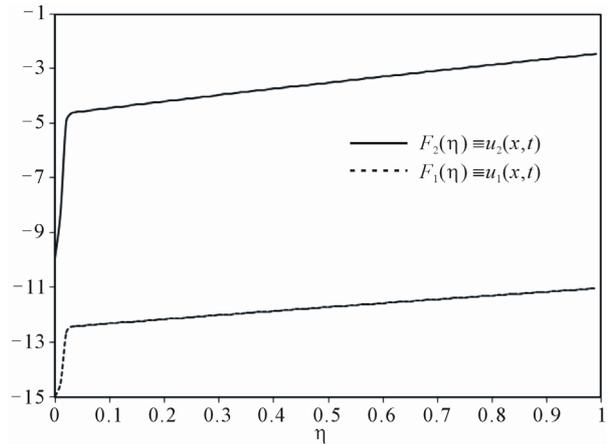
Figure 4. Plot of similarity solutions for the flow velocities $F_1(\eta)$, $F_2(\eta)$ **Corresponding to Case (2) for constants:** $F_{10} = -15$, $F_{20} = -10$, $F_{30} = 25$, $F_{40} = 10$, **and** $\gamma = 1.0$**, where** $F_1(\eta)$ **refer to** $u_1(x,t)$ **and** $F_2(\eta)$ **refer to** $u_2(x,t)$.
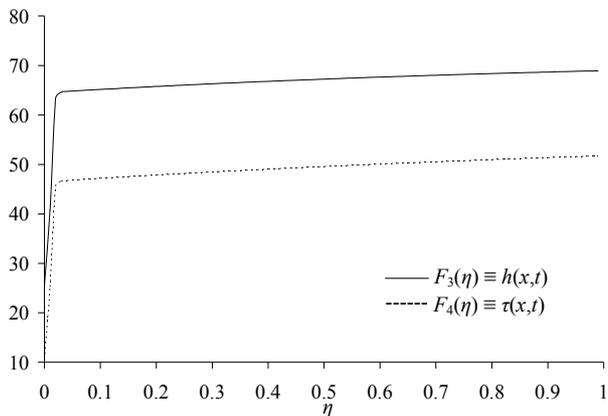
Figure 5. Plot of similarity solutions for the height and the free surface $F_3(\eta)$, $F_4(\eta)$ **respectively corresponding to Case (2) for constants:** $F_{10} = -15$, $F_{20} = -10$, $F_{30} = 25$, $F_{40} = 10$, **and** $\gamma = 1.0$**, where** $F_3(\eta)$ **refer to** $h(x,t)$ **and** $F_4(\eta)$ **refer to** $\tau(x,t)$.
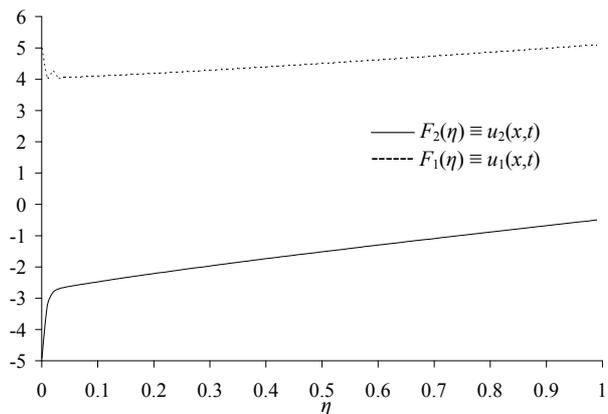
Figure 6. Plot of similarity solutions for the flow velocities $F_1(\eta)$, $F_2(\eta)$ **corresponding to Case (2) for constants:** $F_{10} = 5$, $F_{20} = -5$, $F_{30} = 5$, $F_{40} = -5$, **and** $\gamma = 1.0$**, where** $F_1(\eta)$, **refer to** $u_1(x,t)$ **and** $F_2(\eta)$ **refer to** $u_2(x,t)$.
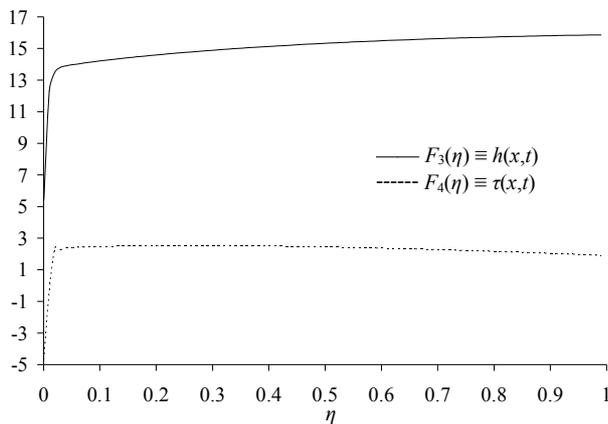
Figure 7. Plot of similarity solutions for the height and the free surface, $F_3(\eta)$, $F_4(\eta)$ respectively corresponding to Case (2) for constants: $F_{10} = 5$, $F_{20} = -5$, $F_{30} = 5$, $F_{40} = -5$, and $\gamma = 1.0$, where $F_3(\eta)$ refer to $h(x,t)$ and $F_4(\eta)$ refer to $\tau(x,t)$.
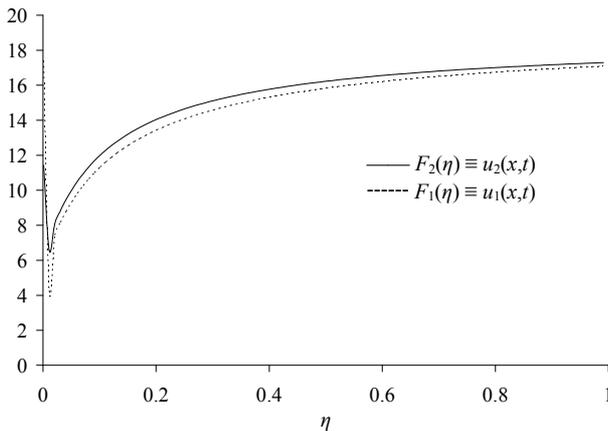
Figure 8. Singular wave solutions for the flow velocities $F_1(\eta)$, $F_2(\eta)$ corresponding to Case (3) for large value of η and $\gamma = 1.0$, where $F_1(\eta)$ refer to $u_1(x,t)$ and $F_2(\eta)$ refer to $u_2(x,t)$.
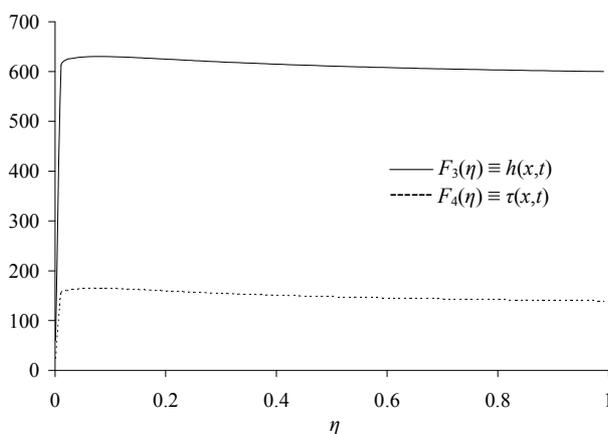
Figure 9. Singular wave solutions of the height of the lower layer $F_3(\eta)$ and the free surface of the upper layer $F_4(\eta)$ of Case (3) for large value of η and $\gamma = 1.0$, where $F_3(\eta)$ refer to $h(x,t)$ and $F_4(\eta)$ refer to $\tau(x,t)$.

ties profiles of the two layers up to $\eta \cong 0.01$, these disturbances occurring at the same time which hydraulic jump occurs in the dimensionless depth of the lower layer and the displacement of the upper layer, after this value the two layers velocities profiles increase with respect to the similarity variable η similar to log-shape, at the same time $F_3(\eta)$ and $F_4(\eta)$ decreases with low rate.

4. Conclusion

The present analysis employed the characteristic function method to solve the equations representing the two-layer shallow-water equations. Different reduction forms were obtained, illustrated and discussed. Although the analytic solutions of the reduced forms were not available, the obtained numeric solution well presented the behavior of the two layers of the problems. For specific values of the group parameters, the solutions were obtained and presented and it's also available to obtain and present the solutions for other cases. In our analysis a new similarity variable was obtained in Case (2), also other new similarity variables were obtained in Cases (1) and (3) giving critical singular wave solutions which have not been reported in [12].

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