Applied Mathematics, **2012, 3, 276-282**
http://dx.doi.org/10.4236/am.2012.33043 Published Online March 2012 (http://www.SciRP.org/journal/am)

A Direct Derivation of the Exact Fisher Information Matrix for Bivariate Bessel Distribution of Type I

Mohammad Reza Kazemi[1], Alireza Nematollahi[2]
[1]Department of Statistics, Fasa University, Fasa, Iran
[2]Department of Statistics, College of Sciences, Shiraz University, Shiraz, Iran
Email: kazemimr88@gmail.com, nematollahi@susc.ac.ir

Received December 10, 2011; revised February 6, 2012; accepted February 14, 2012

ABSTRACT

This paper deals with a direct derivation of Fisher's information matrix for bivariate Bessel distribution of type I. Some tools for the numerical computation and some tabulations of the Fisher's information matrix are provided.

Keywords: Bessel Distribution; Fisher Information Matrix; Digamma Function

1. Introduction

The role of the Bessel functions in probability distributions can be trace back to [1,2]. The application of the univariate Bessel distribution in creating a robust alternative to the normal distribution is investigated by [3]. Univariate Bessel function distributions have been also used to the signal processing, [4]. Basic properties of these distributions with their links with some well-known distributions are described in [5]. More discussion on Bessel Distributions can be found in [6]. The bivariate Bessel distribution of type one (BB1) is specified by the following joint density function

$$\begin{aligned} &f\left(x,y;a_1,a_2,b_1,b_2,v\right) \\ &= c\left\{\left|\left(a_1y-a_2x\right)\left(b_2x-b_1y\right)\right|\right\}^{v-2} \\ &\cdot\exp\left\{-\frac{\left(a_1y-a_2x\right)^2+\left(b_2x-b_1y\right)^2}{2\left(a_1b_2-a_2b_1\right)^2}\right\}, \end{aligned} \tag{1.1}$$

for $x>0,\ y>0,\ v\in R,\ a_1>b_1>0,\ a_2>b_2>0,$ where the constant $c=c\left(a_1,a_2,b_1,b_2,v\right)$ is given by

$$\frac{1}{c}=\left|a_1b_2-a_2b_1\right|\left(a_1b_2-a_2b_1\right)^{2(v-2)}2^v\Gamma^2\left(\frac{v}{2}\right).$$

The density (1.1) is introduced by [7], by using a characterization of Bessel distribution due to the [8]. Reference [8] showed that if U and V are independent chi-squared random variables with common degrees of freedom v, then the distribution of $X=a_1U+b_1V$ subject to $\left(a_1>b_1>0\right)$ is a Bessel distribution. In a nice generalization, Reference [7] have proved that the joint pdf of $X=a_1U+b_1V,\left(a_1>b_1>0\right)$ and $Y=a_2U+b_2V,$ $\left(a_2>b_2>0\right)$ is given by (1.1). They have called this distribution as bivariate Bessel distribution of type I. For more information about Bessel distribution, see [7,8]. It is well-known that under certain condition, the inverse of Fisher Information Matrix (FIM) is the covariance matrix of the estimate of the parameters. The FIM has many application in statistics and other sciences. For an excellent recent references on applications of the FIM see [9]. The aim of this paper is to compute FIM for the bivariate density function (1.1) and is organized as follow: an explicit expression for the FIM for Bivariate Bessel distribution of type I is given in Section 2. Computing FIM for a special case of bivariate density function is given in Section 3. In Section 4, we provide some tools for the numerical computation of the FIM. Some tables of the FIM are also given.

2. An Explicit Expression for the FIM

For a given observation $\left(x,y\right)$, the FIM has the form

$$\left[I_{j,k}\right]_{j,k=1}^{p}=\left[E\left(-\frac{\partial^2\ln L\left(\boldsymbol{\theta}\right)}{\partial\theta_j\partial\theta_k}\right)\right]_{j,k=1}^{p}, \tag{2.1}$$

where $L\left(\boldsymbol{\theta}\right)=f\left(x,y,\boldsymbol{\theta}\right)$ and $\boldsymbol{\theta}=\left(\theta_1,\cdots,\theta_p\right)$ are the parameters of the density. According to (1.1), the unknown vector of parameters is $\boldsymbol{\theta}=\left(a_1,a_2,b_1,b_2,v\right)$. The log likelihood of (1.1) for observation $\left(x,y\right)$, is Finally,

$$\begin{aligned} \ln L\left(\boldsymbol{\theta}\right) &= \ln\left(c\right)+\left(v-2\right)\ln\left\{\left|\left(a_1y-a_2x\right)\left(b_2x-b_1y\right)\right|\right\} \\ &-\frac{\left(a_1y-a_2x\right)^2+\left(b_2x-b_1y\right)^2}{2\left(a_1b_2-a_2b_1\right)^2} \end{aligned} \tag{2.2}$$

Take $m=a_1b_2-a_2b_1$ and $n=a_1b_2+a_2b_1$. In the following, we compute all the second derivatives of log likelihood (2.2) subject to parameters.

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1^2} = -\frac{(v-2)y^2}{(a_1y-a_2x)^2} - \frac{y^2}{m^2} + 4\frac{(a_1y-a_2x)yb_2}{m^3} - 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)b_2^2}{m^4} + \frac{(2v-3)b_2^2}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1 \partial a_2} = \frac{(v-2)yx}{(a_1y-a_2x)^2} + \frac{xy}{m^2} - 2\frac{(a_1y-a_2x)yb_1}{m^3} - 2\frac{(a_1y-a_2x)xb_2}{m^3} + 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)b_2b_1}{m^4} - \frac{(2v-3)b_2b_1}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1 \partial b_1} = -2\frac{(a_1y-a_2x)ya_2}{m^3} - 2\frac{y(b_2x-b_1y)b_2}{m^3} + 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)b_2a_2}{m^4} - \frac{(2v-3)b_2a_2}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1 \partial b_2} = 2\frac{(a_1y-a_2x)ya_1}{m^3} + 2\frac{(b_2x-b_1y)xb_2}{m^3} - 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)b_2a_1}{m^4}$$
$$+\frac{(a_1y-a_2x)^2+(b_2x-b_1y)^2}{m^3} + \frac{(2v-3)b_2a_1}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1 \partial v} = \frac{y}{a_1y-a_2x} - 2\frac{b_2}{m},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_2^2} = -\frac{(v-2)x^2}{(a_1y-a_2x)^2} - \frac{x^2}{m^2} + 4\frac{(a_1y-a_2x)xb_1}{m^3} - 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)b_1^2}{m^4} + \frac{(2v-3)b_1^2}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_2 \partial b_1} = 2\frac{(a_1y-a_2x)xa_2}{m^3} + 2\frac{(b_2x-b_1y)yb_1}{m^3} - 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)b_1a_2}{m^4}$$
$$-\frac{(a_1y-a_2x)^2+(b_2x-b_1y)^2}{m^3} + \frac{(2v-3)b_1a_2}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_2 \partial b_2} = -2\frac{(a_1y-a_2x)xa_1}{m^3} - 2\frac{x(b_2x-b_1y)b_1}{m^3} + 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)b_1a_1}{m^4} - \frac{(2v-3)b_1a_1}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_2 \partial b_2} = -\frac{x}{a_1y-a_2x} + 2\frac{b_1}{m},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_1^2} = -\frac{(v-2)y^2}{(b_2x-b_1y)^2} - \frac{y^2}{m^2} + 4\frac{(b_2x-b_1y)ya_2}{m^3} - 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)a_2^2}{m^4} + \frac{(2v-3)a_2^2}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_1 \partial b_2} = \frac{(v-2)yx}{(b_2x-b_1y)^2} + \frac{xy}{m^2} - 2\frac{(b_2x-b_1y)ya_1}{m^3} - 2\frac{(b_2x-b_1y)xa_2}{m^3} + 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)a_2a_1}{m^4} - \frac{(2v-3)a_2a_1}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_1 \partial v} = -\frac{y}{b_2x-b_1y} + 2\frac{a_2}{m},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_2^2} = -\frac{(v-2)x^2}{(b_2x-b_1y)^2} - \frac{x^2}{m^2} + 4\frac{(b_2x-b_1y)xa_1}{m^3} - 3\frac{\left((a_1y-a_2x)^2+(b_2x-b_1y)^2\right)a_1^2}{m^4} + \frac{(2v-3)a_1^2}{m^2},$$

$$\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_2 \partial v} = \frac{x}{b_2x-b_1y} - 2\frac{a_1}{m}, \quad \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial v^2} = -\frac{1}{2}\Psi'\left(\frac{v}{2}\right),$$

where $\Psi'(x)$ is the derivative of digamma function.

For computing the elements of FIM, following [7], let $X = a_1U + b_1V$ and $Y = a_2U + b_2V$, then, $a_1Y - a_2X = (a_1b_2 - a_2b_1)V = mV$, and $b_2X - b_1Y = mU$.

We have found these identities take the computations easy.

At first note that if U has chi-squred distribution with $v(v > 4)$ degree of freedom then,

$$E(U) = v, E(U^2) = v(v+2), E\left(\frac{1}{U}\right) = \frac{1}{(v-2)},$$

$$E\left(\frac{1}{U^2}\right) = \frac{1}{(v-4)(v-2)}.$$

$$E\left(\frac{Y^2}{(a_1Y - a_2X)^2}\right) = \frac{a_2^2}{m^2}E(U^2)E\left(\frac{1}{V^2}\right) + \frac{b_2^2}{m^2} + \frac{2a_2b_2}{m^2}E(U)E\left(\frac{1}{V}\right) = \frac{a_2^2(v(v+2))}{m^2(v-4)(v-2)} + \frac{b_2^2}{m^2} + \frac{2a_2b_2v}{m^2(v-2)},$$

$$E(Y^2) = a_2^2E(U^2) + 2a_2b_2E(UV) + b_2^2E(V^2) = (a_2^2 + b_2^2)v(v+2) + 2a_2b_2v^2,$$

$$E((a_1Y - a_2X)Y) = E(mV(a_2U + b_2V)) = ma_2v^2 + mb_2v(v+2),$$

$$E((a_1Y - a_2X)^2) = E((mV)^2) = m^2v(v+2),$$

$$E((b_2X - b_1Y)^2) = E((mU)^2) = m^2v(v+2),$$

$$E\left(\frac{XY}{(a_1Y - a_2X)^2}\right) = E\left(\frac{a_1a_2U^2 + b_1b_2V^2 + nUV}{m^2V^2}\right) = \frac{a_1a_2v(v+2)}{m^2(v-4)(v-2)} + \frac{b_1b_2}{m^2} + \frac{nv}{m^2(v-2)},$$

$$E((a_1Y - a_2X)X) = E(mV(a_1U + b_1V)) = ma_1v^2 + mb_1v(v+2).$$

Similarly, we have

$$E((a_1Y - a_2X)Y) = ma_2v^2 + mb_2v(v+2),$$

$$E((b_2X - b_1Y)Y) = ma_2v(v+2) + mb_2v^2,$$

$$E\left(\frac{Y}{a_1Y - a_2X}\right) = \frac{a_2v}{m(v-2)} + \frac{b_2}{m},$$

$$E\left(\frac{X^2}{(a_1Y - a_2X)^2}\right) = \frac{a_1^2v(v+2)}{m^2(v-4)(v-2)} + \frac{2a_1b_1v}{m^2(v-2)} + \frac{b_1^2}{m^2},$$

$$E(X^2) = (a_1^2 + b_1^2)v(v+2) + 2a_1b_1v^2,$$

$$E\left(\frac{X}{a_1Y - a_2X}\right) = \frac{a_1v}{m(v-2)} + \frac{b_1}{m},$$

$$E\left(\frac{Y^2}{(b_2X - b_1Y)^2}\right) = \frac{a_2^2}{m^2} + \frac{2a_2b_2v}{m^2(v-2)} + \frac{b_2^2v(v+2)}{m^2(v-4)(v-2)},$$

$$E(XY) = (a_1a_2 + b_1b_2)v(v+2) + nv^2,$$

$$E\left(\frac{XY}{(b_2X - b_1Y)^2}\right) = \frac{a_1a_2}{m^2} + \frac{b_1b_2v(v+2)}{m^2(v-4)(v-2)} + \frac{nv}{m^2(v-2)},$$

$$E\left(\frac{Y}{b_2X - b_1Y}\right) = \frac{a_2}{m} + \frac{b_2v}{m(v-2)},$$

$$E\left(\frac{X^2}{(b_2X - b_1Y)^2}\right) = \frac{a_1^2}{m^2} + \frac{2a_1b_1v}{m^2(v-2)} + \frac{b_1^2v(v+2)}{m^2(v-4)(v-2)},$$

and

$$E\left(\frac{X}{b_2X - b_1Y}\right) = \frac{a_1}{m} + \frac{b_1v}{m(v-2)}.$$

Therefore, the elements of FIM can be calculated as below.

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1^2}\right) = \frac{(v-2)}{m^2}\left(\frac{a_2^{\,2}v(v+2)}{v(v-2)-4} + 2\frac{a_2b_2v}{v-2} + b_2^2\right) + \frac{(a_2^2 + b_2^2)v(v+2) + 2a_2b_2v^2}{m^2} - 4\frac{b_2(a_2v^2 + b_2v(v+2))}{m^2} + 6\frac{b_2^2v(v+2)}{m^2} - \frac{(2v-3)b_2^2}{m^2}$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1 \partial a_2}\right) = -\frac{(v-2)}{m^2}\left(\frac{a_1a_2v(v+2)}{(v-4)(v-2)} + \frac{nv}{v-2} + b_1b_2\right) - \frac{(a_1a_2 + b_1b_2)v(v+2) + nv^2}{m^2} + 2\frac{b_1(a_2v^2 + b_2v(v+2))}{m^2} + 2\frac{b_2(a_1v^2 + b_1v(v+2))}{m^2} - 6\frac{b_1v(v+2)b_2}{m^2} + \frac{(2v-3)b_2b_1}{m^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1 \partial b_1}\right) = +2\frac{a_2\left(a_2 v^2 + b_2 v(v+2)\right)}{m^2} + 2\frac{b_2\left(a_2 v(v+2) + b_2 v^2\right)}{m^2} - 6\frac{a_2 b_2 v(v+2)}{m^2} + \frac{(2v-3) b_2 a_2}{m^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1 \partial b_2}\right) = -2\frac{a_1\left(a_2 v^2 + b_2 v(v+2)\right)}{m^2} - 2\frac{b_2\left(a_1 v(v+2) + b_1 v^2\right)}{m^2} + 6\frac{a_1 b_2 v(v+2)}{m^2} - 2\frac{v(v+2)}{m} - \frac{(2v-3) b_2 a_1}{m^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1 \partial v}\right) = -\frac{a_2 v}{(v-2)m} - \frac{b_2}{m} + 2\frac{b_2}{m} = \frac{b_2}{m} - \frac{a_2 v}{(v-2)m},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_2^2}\right) = \frac{(v-2)}{m^2}\left(\frac{a_1^2 v(v+2)}{(v-4)(v-2)} + 2\frac{a_1 b_1 v}{v-2} + b_1^2\right) + \frac{\left(a_1^2 + b_1^2\right) v(v+2) + 2 a_1 b_1 v^2}{m^2} - 4\frac{b_1\left(a_1 v^2 + b_1 v(v+2)\right)}{m^2}$$
$$+ 6\frac{b_1^2 v(v+2)}{m^2} - \frac{(2v-3) b_1^2}{m^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_2 \partial b_1}\right) = -2\frac{a_2\left(a_1 v^2 + b_1 v(v+2)\right)}{m^2} - 2\frac{b_1\left(a_2 v(v+2) + b_2 v^2\right)}{m^2} + 6\frac{a_2 b_1 v(v+2)}{m^2} + 2\frac{v(v+2)}{m} - \frac{(2v-3) b_1 a_2}{m^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_2 \partial b_2}\right) = 2\frac{a_1\left(a_1 v^2 + b_1 v(v+2)\right)}{m^2} + 2\frac{b_1\left(a_1 v(v+2) + b_1 v^2\right)}{m^2} - 6\frac{b_1 a_1 v(v+2)}{m^2} + \frac{(2v-3) b_1 a_1}{m^2}$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_2 \partial v}\right) = \frac{a_1 v}{m(v-2)} + \frac{b_1}{m} - 2\frac{b_1}{m} = \frac{a_1 v}{m(v-2)} - \frac{b_1}{m},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_1^2}\right) = \frac{(v-2)}{m^2}\left(a_2^2 + \frac{b_2^2 v(v+2)}{(v-4)(v-2)} - 2\frac{a_2 b_2 v}{v-2}\right) + \frac{\left(a_2^2 + b_2^2\right) v(v+2) + 2 a_2 b_2 v^2}{m^2} - 4\frac{a_2\left(a_2 v(v+2) + b_2 v^2\right)}{m^2}$$
$$+ 6\frac{a_2^2 v(v+2)}{m^2} - \frac{(2v-3) a_2^2}{m^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_1 \partial b_2}\right) = -\frac{(v-2)}{m^2}\left(a_1 a_2 + \frac{b_1 b_2 v(v+2)}{(v-4)(v-2)} + \frac{nv}{v-2}\right) - \frac{\left(a_1 a_2 + b_1 b_2\right) v(v+2) + n v^2}{m^2} + 2\frac{a_1\left(a_2 v(v+2) + b_2 v^2\right)}{m^2}$$
$$+ 2\frac{a_2\left(a_1 v(v+2) + b_1 v^2\right)}{m^2} - 6\frac{a_1 a_2 v(v+2)}{m^2} + \frac{(2v-3) a_2 a_1}{m^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_1 \partial v}\right) = \frac{a_2}{m} + \frac{b_2 v}{m(v-2)} - 2\frac{a_2}{m} = \frac{b_2 v}{m(v-2)} - \frac{a_2}{m},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_2^2}\right) = \frac{(v-2)}{m^2}\left(a_1^2 + \frac{b_1^2 v(v+2)}{(v-4)(v-2)} + 2\frac{a_1 b_1 v}{v-2}\right) + \frac{\left(a_1^2 + b_1^2\right) v(v+2) + 2 a_1 b_1 v^2}{m^2} - 4\frac{a_1\left(a_1 v(v+2) + b_1 v^2\right)}{m^2}$$
$$+ 6\frac{a_1^2 v(v+2)}{m^2} - \frac{(2v-3) a_1^2}{m^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial b_2 \partial v}\right) = -\frac{a_1}{m} - \frac{b_1 v}{m(v-2)} + 2\frac{a_1}{m} = \frac{a_1}{m} - \frac{b_1 v}{m(v-2)},$$

and finally

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial v^2}\right) = \frac{1}{2}\Psi'\left(\frac{v}{2}\right).$$

3. Special Case

Since above experessions are very tedios, we also compute FIM for some assumtions as below. We assume that, $a_1 = b_1 + k$ and $a_2 = b_2 + k$ for some fixed *k*. Then the elements of FIM can be computed as below. As we see the experessions are easier than the former experessoins and the dimension of the parameters is shorter than the peremier parameters.

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial a_1^2}\right)=\frac{\left(a_2^2+b_2^2\right)\nu(\nu+2)(2\nu-7)}{k^2\left(b_2-b_1\right)^2(\nu-4)}+\frac{\left(a_2^2+b_2^2\right)(\nu-2)}{k^2\left(b_2-b_1\right)^2}+\frac{4a_2b_2\nu(\nu+1)}{k^2\left(b_2-b_1\right)^2}+\frac{6\nu^2+10\nu+3}{k^2\left(b_2-b_1\right)^2}-\frac{8\nu}{k\left(b_2-b_1\right)^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \beta_2^2}\right)=\frac{\left(a_1^2+b_1^2\right)\nu(\nu+2)(2\nu-7)}{k^2\left(b_2-b_1\right)^2(\nu-4)}+\frac{\left(a_1^2+b_1^2\right)(\nu-2)}{k^2\left(b_2-b_1\right)^2}+\frac{4a_1b_1\nu(\nu+1)}{k^2\left(b_2-b_1\right)^2}+\frac{6\nu^2+10\nu+3}{k^2\left(b_2-b_1\right)^2}-\frac{8\nu}{k\left(b_2-b_1\right)^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \beta_1\partial \beta_2}\right)=-\frac{\left(a_1a_2+b_1b_2\right)\nu(\nu+2)(2\nu-7)}{k^2\left(b_2-b_1\right)^2(\nu-4)}-\frac{\left(a_1a_2+b_1b_2\right)(\nu-2)}{k^2\left(b_2-b_1\right)^2}-\frac{2\left(2b_1b_2+k\left(b_2+b_1\right)\right)\nu(\nu+1)}{k^2\left(b_2-b_1\right)^2}$$
$$-\frac{6\nu^2+10\nu+3}{k^2\left(b_2-b_1\right)^2}+\frac{8\nu}{k\left(b_2-b_1\right)^2},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \beta_1\partial \nu}\right)=-\frac{\left(a_2-b_2\right)\nu}{k\left(b_2-b_1\right)(\nu-2)}+\frac{a_2-b_2}{k\left(b_2-b_1\right)}+\frac{2}{b_1-b_2}=\frac{2(\nu-1)}{\left(b_1-b_2\right)(\nu-2)},$$

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \beta_2\partial \nu}\right)=\frac{\left(a_1-b_1\right)\nu}{k\left(b_2-b_1\right)(\nu-2)}-\frac{a_1-b_1}{k\left(b_2-b_1\right)}+\frac{2}{b_2-b_1}=\frac{2(\nu-1)}{\left(b_2-b_1\right)(\nu-2)},$$

and finally,

$$E\left(-\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \nu^2}\right)=\frac{1}{2}\Psi'\left(\frac{\nu}{2}\right)$$

4. Numerical Computation and Tables of the FIM

In this section, some numerical tabulations of the FIM are given to illustrate the computations in the last section.

Take,

$$E_1=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial a_1^2}\right), E_2=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial a_2^2}\right),$$

$$E_3=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial b_1^2}\right), E_4=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial b_2^2}\right),$$

$$E_5=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial a_1\partial a_2}\right), E_6=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial a_1\partial b_1}\right),$$

$$E_7=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial a_1\partial b_2}\right), E_8=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial a_2\partial b_1}\right),$$

and

$$E_9=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial a_2\partial b_2}\right), E_{10}=E\left(-\frac{\partial^2 \ln f(x,y)}{\partial b_1\partial b_2}\right).$$

In the **Tables 1-6**, we have computed the elements of FIM for the Bivariate Bessel distribution of type I, for

Table 1. Elements of the FIM for (a_1, b_1, a_2, b_2) = (2, 1, 3, 1).

ν	5	7	8	9	10
E_1	3131	5223	7097	9479.8	12411
E_2	301	351	409	476.2	551
E_3	859	1479	1861	2287.8	2759
E_4	394	648	808	986.8	1184
E_5	–421	–477	–553	–641.8	–741
E_6	311	635	839	1071	1331
E_7	–161	–341	–455	–585	–731
E_8	–224	–456	–602	–768	–954
E_9	124	260	346	444	554
E_{10}	–576	–972	–1218	–1492.8	–1796

Table 2. Elements of the FIM for (a_1, b_1, a_2, b_2) = (2, 1, 3, 2).

ν	5	7	8	9	10
E_1	3314	5520	7460	9914.8	12924
E_2	301	351	409	476.2	551
E_3	949	1479	1825	2212.2	2639
E_4	394	648	808	986.8	1184
E_5	–482	–576	–674	–786.8	–912
E_6	272	584	782	1008	12621
E_7	–211	–439	–583	–747	–931
E_8	–148	–324	–436	–564	–708
E_9	124	260	346	444	554
E_{10}	–606	–972	–1206	–1467.6	–1756

Table 3. Elements of the FIM for (a_1, b_1, a_2, b_2) = (3, 1, 3, 2).

v	5	7	8	9	10
E_1	368.2	613.3	828.8	1101.6	1436.1
E_2	67.8	76.3	88.5	102.9	119
E_3	105.4	164.3	202.7	245.8	293.2
E_4	95.4	164.3	206.7	254.2	306.5
E_5	72.4	–82.6	–95.7	–111	–128
E_6	30.2	64.8	86.8	112	140.2
E_7	–40.1	–81.4	–107.4	–137	–170.1
E_8	–19.1	–43.1	–58.4	–76	–95.7
E_9	34.5	70.5	93.2	119	147.8
E_{10}	–96.5	–159.6	–199.2	–243.4	–292.1

Table 4. Elements of the FIM for (a_1, b_1, a_2, b_2) = (3, 2, 3, 1).

v	5	7	8	9	10
E_1	347.8	347.8	788.5	1053.1	1379
E_2	88.2	88.2	128.8	151.2	176
E_3	95.4	95.4	206.7	254.8	306.5
E_4	105.4	105.4	202.7	254.8	293.2
E_5	–72.4	–72.4	–95.7	–111	–128.1
E_6	34.5	34.5	93.2	119	147.8
E_7	–19.1	–19.1	–58.4	–76	–95.7
E_8	–40.1	–40.1	–107.4	–137	–170.1
E_9	30.2	30.2	86.8	112	140.2
E_{10}	–96.5	–96.5	–199.2	–243.4	–292.1

Table 5. Elements of the FIM for (a_1, b_1, a_2, b_2) = (3, 2, 2, 1).

v	5	7	8	9	10
E_1	1421	2367	3209	4277.8	5591
E_2	794	984	1160	1361.2	1584
E_3	394	648	808	986.8	1184
E_4	949	1479	1825	2212.2	2639
E_5	–482	–576	–674	–786.8	–912
E_6	124	260	346	444	554
E_7	–148	–324	–436	–564	–708
E_8	–211	–439	–583	–747	–931
E_9	272	584	782	1008	1262
E_{10}	–606	–972	–1206	–1467.6	–1756

Table 6. Elements of the FIM for (a_1, b_1, a_2, b_2) = (3, 1, 2, 1).

v	5	7	8	9	10
E_1	1421	2367	3209	4277.8	5591
E_2	611	687	797	926.2	1071
E_3	394	648	808	986.8	1184
E_4	859	1479	1861	2287.8	2759
E_5	–421	–477	–553	–641.8	–741
E_6	124	260	346	444	554
E_7	–224	–456	–602	–768	–954
E_8	–161	–341	–455	–585	–731
E_9	311	635	839	1071	1331
E_{10}	–576	–972	–1218	–1492.8	–1796

different values of

$$(a_1,b_1,a_2,b_2)\in\left\{\begin{matrix}(2,1,3,1),(2,1,3,2),(3,1,3,2),\\(3,2,3,1),(3,2,2,1),(3,1,2,1)\end{matrix}\right\}$$

and $v\in\{5,7,8,9,10\}$.

5. Acknowledgements

The authors would like to thank respectful editor in chief and a referee for their helpful comments.

REFERENCES

[1] A. T. McKay, “A Bessel Function Distribution,” *Biometrika*, Vol. 24, No. 1-2, 1932, pp. 39-44. doi:10.1093/biomet/24.1-2.39

[2] R. G. Laha, “On Some Properties of the Bessel Function Distributions,” *Bulletin of the Calcutta Mathematical Society*, Vol. 46, 1954, pp. 59-72.

[3] D. L. McLeish, “A Robust Alternative to the Normal Distribution,” *Canadian Journal of Statistics*, Vol. 10, No. 2, 1982, pp. 89-102. doi:10.2307/3314901

[4] F. McNolty, “Applications of Bessel Function Distributions,” *Sankhya B*, Vol. 29, 1967, pp. 235-248.

[5] L. Yuan and J. D. Kalbfleisch, “On the Bessel Distribution and Related Problems,” *Annals of the Institute of Statistical Mathematics*, Vol. 52, No. 3, 2000, pp. 438-447. doi:10.1023/A:1004152916478

[6] L. Devrore, “Non-Uniform Random Variate Generation,” Springer, New York, 1986.

[7] S. Nadarajah and S. Kotz, “Some Bivariate Bessel Distributions,” *Applied Mathematics and Computation*, Vol. 187, No. 1, 2007, pp. 332-339. doi:10.1016/j.amc.2006.08.130

[8] B. C. Bhattacharyya, “The Use of Mackay’s Bessel Function Curves for Graduating Frequency Distributions,”

Sankhya, Vol. 6, 1942, pp. 175-182.

[9] S. Nadarajah, "FIM for Arnold and Strauss's Bivariate Gamma Distribution," *Computational Statistics and Data Analysis*, Vol. 51, No. 3, 2006, pp. 1584-1590. doi:10.1016/j.csda.2006.05.009

