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Further Results on Pair Sum Graphs

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ABSTRACT

Let G be a (p,q) graph. An injective map $f:V(G)\to\{\pm 1,\pm 2,\cdots,\pm p\}$ is called a pair sum labeling if the induced edge function, $f_e:E(G)\to Z-\{0\}$ defined by $f_e(uv)=f(u)+f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1,\pm k_2,\cdots,\pm k_{q/2}\}$ or $\{\pm k_1,\pm k_2,\cdots,\pm k_{(q-1)/2}\}\cup\{k_{(q+1)/2}\}$ according as q is even or odd. A graph with a pair sum labeling is called a pair sum graph. In this paper we investigate the pair sum labeling behavior of subdivision of some standard graphs.

Keywords: Path; Cycle; Ladder; Triangular Snake; Quadrilateral Snake

1. Introduction

The graphs considered here will be finite, undirected and simple. $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The cardinality of the vertex set of a graph G is denoted by p and the cardinality of its edge set is denoted by q. The corona G_1G_2 of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 (with p_1 vertices) and p_1 copies of G_2 and then joining the ith vertex of G_1 to all the vertices in the ith copy of G_2. If $e=uv$ is an edge of G and w is a vertex not in G then e is said to be subdivided when it is replaced by the edges uw and wv. The graph obtained by subdividing each edge of a graph G is called the subdivision graph of G and it is denoted by $S(G)$. The graph $P_n\times P_2$ is called the ladder. A dragon is a graph formed by joining an end vertex of a path P_m to a vertex of the cycle C_n. It is denoted as $C_n@P_m$. The triangular snake T_n is obtained from the path P_n by replacing every edge of a path by a triangle C_3. The quadrilateral snake Q_n is obtained from the path P_n by every edge of a path is replaced by a cycle C_4. The concept of pair sum labeling has been introduced in [1]. The Pair sum labeling behavior of some standard graphs like complete graph, cycle, path, bistar, and some more standard graphs are investigated in [1-3]. That all the trees of order ≤9 are pair sum have been proved in [4]. Terms not defined here are used in the sense of Harary [5]. Let x be any real number. Then $\lfloor x\rfloor$ stands for the largest integer less than or equal to x and $\lceil x\rceil$ stands for the smallest integer greater than or equal to x. Here we investigate the pair sum labeling behavior of $S(G)$, for some standard graphs G.

2. Pair Sum Labeling

Definition 2.1. Let G be a (p,q) graph. An injective map $f:V(G)\to\{\pm 1,\pm 2,\cdots,\pm p\}$ is called a pair sum labeling if the induced edge function, $f_e:E(G)\to Z-\{0\}$ defined by $f_e(uv)=f(u)+f(v)$ is one-one and $f_e(E(G))$ is either of the form

$$\{\pm k_1,\pm k_2,\cdots,\pm k_{q/2}\}$$

or

$$\{\pm k_1,\pm k_2,\cdots,\pm k_{(q-1)/2}\}\cup\{k_{(q+1)/2}\}$$

according as q is even or odd. A graph with a pair sum labeling defined on it is called a pair sum graph.

Theorem 2.2 [1]. Any path is a pair sum graph.

Theorem 2.3 [1]. Any cycle is a pair sum graph.

3. On Standard Graphs

Here we investigate pair sum labeling behavior of $C_n@P_m$ and $K_n^c+2K_2$.

Theorem 3.1. If n is even, $C_n@P_m$ is a pair sum graph.

Proof. Let C_n be the cycle $u_1u_2u_3\cdots u_nu_1$ and let P_m be the path $v_1v_2\cdots v_m$.

Case 1. $m\equiv 0\pmod 4$

Define

$$f: V(C_n @ P_m) \to \{\pm 1, \pm 2, \cdots, \pm(m+n)\}$$

by

$$f\left(v_{\lfloor m/2 \rfloor - i + 1}\right) = i, 1 \le i \le \lfloor m/2 \rfloor$$

$$f\left(v_{\lfloor m/2 \rfloor + 2i - 1}\right) = -2i - 2, 1 \le i \le \lfloor m/4 \rfloor$$

$$f\left(v_{\lfloor m/2 \rfloor + 2i}\right) = -2i + 1, 1 \le i \le \lfloor m/4 \rfloor$$

$$f(u_i) = m/2 + 2i - 1, 1 \le i \le n/2$$

$$f\left(u_{n/2+i}\right) = -m/2 - 2i + 1, 1 \le i \le n/2 .$$

Here

$$\begin{aligned} &f_e\left(E(C_n @ P_m)\right) \\ &= \{3, 5, 7, \cdots, (m+1)\} \cup \{-3, -5, \cdots, -(m+1)\} \\ &\cup \{m+4, m+8, \cdots, (m+2n-4)\} \\ &\cup \{-(m+4), -(m+8), \cdots, -(m+2n-4)\} \\ &\cup \{n-2, -(n-2)\}. \end{aligned}$$

Therefore f is a pair sum labeling.

Case 2. $m \equiv 2 (\text{mod}\, 4)$

Define

$$f: V(C_n @ P_m) \to \{\pm 1, \pm 2, \cdots, \pm(m+n)\}$$

by

$$f\left(v_{\lfloor m/2 \rfloor - 2i}\right) = 1 - 2i, 1 \le i \le \lfloor (m+2)/4 \rfloor$$

$$f\left(v_{\lfloor m/2 \rfloor - 2i + 1}\right) = -2i - 2, 1 \le i \le \lfloor (m-2)/4 \rfloor$$

$$f\left(v_{\lfloor m/2 \rfloor + i - 1}\right) = i, 1 \le i \le \lceil m/2 \rceil + 1$$

$$f(u_i) = -\lfloor m/2 \rfloor - 2i - 1, 1 \le i \le n/2$$

$$f\left(u_{n/2+i}\right) = \lfloor m/2 \rfloor + 2i + 1, 1 \le i \le n/2$$

Here

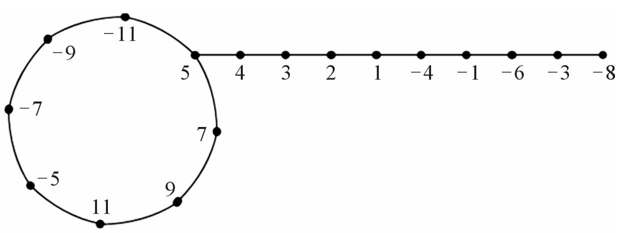

Figure 1. A pair sum labeling of $C_8 @ P_9$.

$$\begin{aligned} &f_e\left(E(C_n @ P_m)\right) \\ &= \{-3, -5, -7, \cdots, -m, -(m+1)\} \\ &\cup \{3, 5, 7, \cdots, (m+1)\} \\ &\cup \{m+2, m+10, \cdots, (m+2n)\} \\ &\cup \{-(m+2), -(m+10), \cdots, -(m+2n)\} \\ &\cup \{(n-2), -(n-2)\}. \end{aligned}$$

Hence f is a pair sum labeling.

Case 3. $m \equiv 1 (\text{mod}\, 4)$

Label the vertex $u_i\,(1 \le i \le n)$, $v_i\,(1 \le i \le m-1)$ as in Case 1. Then label $-m-2$ to v_m.

Case 4. $m \equiv 3 (\text{mod}\, 4)$

Assign the label $m+2$ to v_m and assign the label to the remaining vertices as in Case 2.

Illustration 1. A pair sum labeling of $C_8 @ P_9$ is shown in **Figure 1**.

Theorem 3.2. $K_n^c + 2K_2$ is pair sum graph if n is even.

Proof: Let $u_1, u_2, \cdots, u_n$ be the vertices of K_n and u, v, w, z. be the vertices in $2K_2$. Let

$$V\left(K_n^c + 2K_2\right) = V\left(K_n^c\right) \cup V(2K_2)$$

and

$$E\left(K_n^c + 2K_2\right) = \{uv, wz, uu_i, vu_i, wu_i, zu_i : 1 \le i \le n\}.$$

Define

$$f: V\left(K_n^c + 2K_2\right) \to \{\pm 1, \pm 2, \cdots, \pm(n+4)\}$$

by

$$f(u_i) = 2i - 1, 1 \le i \le n/2$$

$$f\left(u_{n/2+i}\right) = -(2i-1), 1 \le i \le n/2$$

$$f(u) = n,\ f(v) = n+3$$

$$f(w) = -n,\ f(z) = -(n+3)$$

Here

$$\begin{aligned} &f_e\left(E\left(K_n^c + 2K_2\right)\right) \\ &= \{n+1, n+3, n+5, \cdots, 2n\} \\ &\cup \{-(n+1), -(n+3), -(n+5), \cdots, -2n\} \\ &\cup \{n-1, n-3, n-5, \cdots, 1\} \\ &\cup \{-(n-1), -(n-3), -(n-5), \cdots, -1\} \\ &\cup \{n+4, n+8, n+12, \cdots, 2n+2\} \\ &\cup \{-(n+4), -(n+8), -(n+12), \cdots, -(2n+2)\} \\ &\cup \{n+2, n, n-2, \cdots, 2\} \\ &\cup \{-(n+2), -n, -(n-2), \cdots, -2\} \\ &\cup \{2n+3, -(2n+3)\}. \end{aligned}$$

Therefore f is a pair sum labeling.

Illustration 2. A pair sum labeling of $K_8^c + 2K_2$ is shown in **Figure 2**.

4. On Subdivision Graph

Here we investigate the pair sum labeling behavior of $S(G)$ for some standard graphs G.

Theorem 4.1. $S(L_n)$ is a pair sum graph, where L_n is a ladder on n vertices.

Proof. Let

$$V(S(L_n)) = \{u_i, v_i, w_i, a_j, b_j : 1 \le i \le n, 1 \le j \le n-1\}$$

Let

$$E(S(L_n)) = \{u_i w_i, w_i v_i : 1 \le i \le n\} \cup \{u_i a_i, a_i u_{i+1}, v_i b_i, b_i v_{i+1} : 1 \le i \le n-1\}.$$

Case 1: n is even.

When $n = 2$, the proof follows from the Theorem 2.3. For $n > 2$,

Define

$$f : V(S(L_n)) \to \{\pm 1, \pm 2, \pm 3, \cdots, \pm(5n-2)\}$$

by

$$f(u_{n/2}) = -1,\ f(u_{n/2+1}) = -3$$

$$f(u_{n/2-i}) = 10i + 3, 1 \le i \le (n-2)/2$$

$$f(u_{n/2+i+1}) = -10i + 1, 1 \le i \le (n-2)/2$$

$$f(w_{n/2}) = 5,\ f(w_{n/2+1}) = -5$$

$$f(w_{n/2-i}) = 10i + 1, 1 \le i \le (n-2)/2$$

$$f(w_{n/2+1+i}) = -(10i + 1), 1 \le i \le (n-2)/2$$

$$f(v_{n/2}) = 3,\ f(v_{n/2+1}) = 1$$

$$f(v_{n/2-i}) = 10i - 1, 1 \le i \le (n-2)/2$$

$$f(v_{n/2+1+i}) = -10i - 3, 1 \le i \le (n-2)/2$$

$$f(a_{n/2}) = -2$$

$$f(a_{n/2-i}) = 10i + 5, 1 \le i \le (n-2)/2$$

$$f(a_{n/2+i}) = -10i + 3, 1 \le i \le (n-2)/2$$

$$f(b_{n/2}) = 2$$

$$f(b_{n/2-i}) = 10i - 3, 1 \le i \le (n-2)/2$$

$$f(b_{n/2+i+1}) = -10i - 5, 1 \le i \le (n-2)/2.$$

When $n = 4$,

$$f_e(E(S(L_n))) = \{3,4,5,8,10,16,20,24,28\} \cup \{-3,-4,-5,-8,-10,-16,-20,-24,-28\}.$$

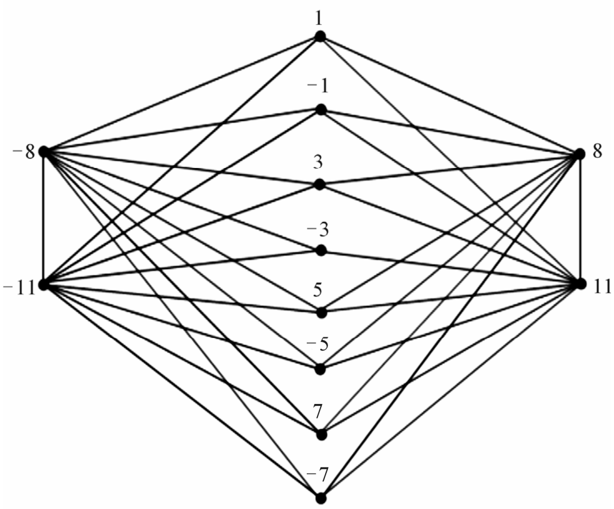

Figure 2. A pair sum labeling of $K_8^c + 2K_2$.

For $n > 4$,

$$f_e(E(S(L_n))) = f_e(E(S(L_4))) \cup \{(26,36,40,44,48,38), (-26,-36,-40,-44,-48,-38), (46,56,60,64,68,58), (-46,-56,-60,-64,-68,-58), \cdots, (10n-34, 10n-24, 10n-20, 10n-16, 10n-12, 10n-22), (-10n+34, -10n+24, -10n+20, -10n+16, -10n+12, -10n+22)\}.$$

Therefore f is a pair sum labeling.

Case 2. n is odd.

Clearly $S(L_1) \cong P_3$ and hence $S(L_n)$ is a pair sum graph by Theorem 2.2. For $n > 1$,

Define

$$f : V(S(L_n)) \to \{\pm 1, \pm 2, \cdots, \pm(5n-2)\}$$

by

$$f(u_{(n+1)/2}) = 6,\ f(u_{(n-1)/2}) = 12$$

$$f(u_{(n+3)/2}) = -12,\ f(a_{(n-1)/2}) = -9$$

$$f(a_{(n+1)/2}) = 3$$

$$f(u_{(n+3)/2+i}) = 10i + 10, 1 \le i \le (n-3)/2$$

$$f(u_{(n-1)/2-i}) = -(10i + 10), 1 \le i \le (n-3)/2$$

$$f(v_{(n+3)/2+i}) = -(6 + 10i), 1 \le i \le (n-3)/2$$

$f\left(v_{(n-1)/2-i}\right)=6+10i, 1\le i\le (n-3)/2$

$f\left(w_{(n+3)/2+i}\right)=-10i+2, 1\le i\le (n-3)/2$

$f\left(w_{(n-1)/2-i}\right)=10i-2, 1\le i\le (n-3)/2$

$f\left(v_{(n+1)/2}\right)=2,\ f\left(v_{(n-1)/2}\right)=10$

$f\left(v_{(n+3)/2}\right)=-10,\ f\left(b_{(n-1)/2}\right)=-6$

$f\left(b_{(n+1)/2}\right)=4,\ f\left(w_{(n+1)/2}\right)=-4$

$f\left(w_{(n-1)/2}\right)=8,\ f\left(w_{(n+1)/2}\right)=-8$

$f\left(a_{(n+1)/2+i}\right)=-(10i+12), 1\le i\le (n-3)/2$

$f\left(a_{(n-1)/2-i}\right)=10i+12, 1\le i\le (n-3)/2$

$f\left(b_{(n+1)/2+i}\right)=-(10i+4), 1\le i\le (n-3)/2$

$f\left(b_{(n-1)/2-i}\right)=10i+4, 1\le i\le (n-3)/2.$

Therefore

$$f_e\left(E\left(S\left(L_3\right)\right)\right)=\{2,3,4,6,9,18,20,-2,-3,-4,-6,-9,-18,-20\}$$

and

$$f_e\left(E\left(S\left(L_5\right)\right)\right)=f_e\left(E\left(S\left(L_3\right)\right)\right)\cup\{24,30,34,38,42,36,-24,-30,-34,-38,-42,-36\}$$

when $n>5$,

$$\begin{aligned}f_e\left(E\left(S\left(L_n\right)\right)\right)&=f_e\left(E\left(S\left(L_5\right)\right)\right)\cup\{(40,50,54,58,62,52),\\&(-40,-50,-54,-58,-62,-52),\\&(60,70,74,78,82,72),\\&(-60,-70,-74,-78,-82,-72),\ldots,\\&(10n-30,10n-20,10n-16,\\&10n-12,10n-8,10n-18),\\&(-10n+30,-10n+20,-10n+16,\\&-10n+12,-10n+8,-10n+18)\}.\end{aligned}$$

Then f is a pair sum labeling.

Illustration 3. A pair sum labeling of $S(L_7)$ is shown in **Figure 3**.

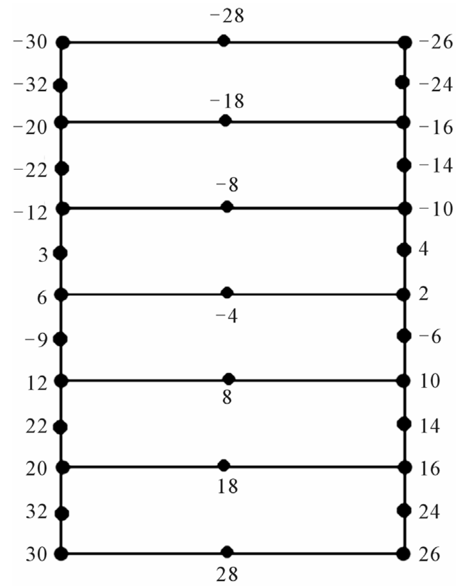

Figure 3. A pair sum labeling of $S(L_7)$.

Theorem 4.2. $S(C_nK_1)$ is a pair sum graph

Proof. Let

$$V\left(S\left(C_nK_1\right)\right)=\{u_i:1\le i\le 2n\}\cup\{w_i,v_i:1\le i\le n\}$$

Let

$$E\left(S\left(C_nK_1\right)\right)=\{u_iu_{i+1}:1\le i\le 2n-1\}\cup\{u_{2i-1}w_i:1\le i\le n\}\cup\{v_iw_i:1\le i\le n\}.$$

Case 1. n is even.

Define

$$f:S\left(C_nK_1\right)\to\{\pm1,\pm2,\cdots,\pm4n\}$$

by

$f\left(u_i\right)=2i-1, 1\le i\le n$

$f\left(u_{n+i}\right)=-(2i-1), 1\le i\le n$

$f\left(w_i\right)=2n-1+2i, 1\le i\le n/2$

$f\left(w_{n/2+i}\right)=-2n+1-2i, 1\le i\le n/2$

$f\left(v_i\right)=3n-1-2i, 1\le i\le n/2$

$f\left(w_{n/2+i}\right)=-3n+1+2i, 1\le i\le n/2$

Here

$$\begin{aligned}f_e(E)&=\{4,8,12,\cdots,(4n-4)\}\\&\cup\{-4,-8,-12,\cdots,-(4n-4)\}\\&\cup\{2n+2,2n+8,2n+14,\cdots,5n-4\}\\&\cup\{-(2n+2),-(2n+8),-(2n+14),\cdots,-(5n-4)\}\\&\cup\{5n+2,5n+6,5n+10,\cdots,7n-2\}\\&\cup\{-(5n+2),-(5n+6),-(5n+10),\cdots,-(7n-2)\}.\end{aligned}$$

Then f is pair sum labeling.

Case 2. n is odd.

Define

$$f : V\left(S\left(C_nK_1\right)\right) \to \left\{\pm 1, \pm 2, \cdots, \pm 4n\right\}$$

by

$$f\left(u_i\right) = 4n - 2i + 2, 1 \le i \le n$$

$$f\left(u_{n/2+i}\right) = -4n + 2i - 2, 1 \le i \le n$$

$$f\left(w_i\right) = -n - 1 + i, 1 \le i \le \lceil n/2 \rceil$$

$$f\left(w_{\lceil n/2 \rceil + i}\right) = n - i, 1 \le i \le \lfloor n/2 \rfloor$$

$$f\left(v_i\right) = -2n - 2 + 2i, 1 \le i \le \lceil n/2 \rceil$$

$$f\left(v_{\lceil n/2 \rceil + i}\right) = 2n + 2 - 2i, 1 \le i \le \lfloor n/2 \rfloor$$

Here

$$f_e\left(E\left(S\left(C_nK_1\right)\right)\right)$$
$$= \left\{8n-2, 8n-6, \cdots 4n+10, 40+6\right\}$$
$$\cup\left\{-\left(8n-2\right), -\left(8n-6\right), \cdots, -\left(4n+10\right), -\left(4n+6\right)\right\}$$
$$\cup\left\{2n-2, -2n+2\right\}$$
$$\cup\left\{3n, 3n-3, 3n-6, \cdots, 3\left(n+1\right)/2\right\}$$
$$\cup\left\{-3n, -\left(3n-3\right), -\left(3n-6\right), \cdots, -3\left(n+1\right)/2\right\}$$
$$\cup\left\{3n-1, 3n-4, \cdots, 3\left(n+7\right)/2\right\}$$
$$\cup\left\{-\left(3n-1\right), -\left(3n-4\right), \cdots, -3\left(n+7\right)/2\right\}.$$

Then *f* is pair sum labeling.

Illustration 4. A pair sum labeling of $S\left(C_7K_1\right)$ is shown in **Figure 4**.

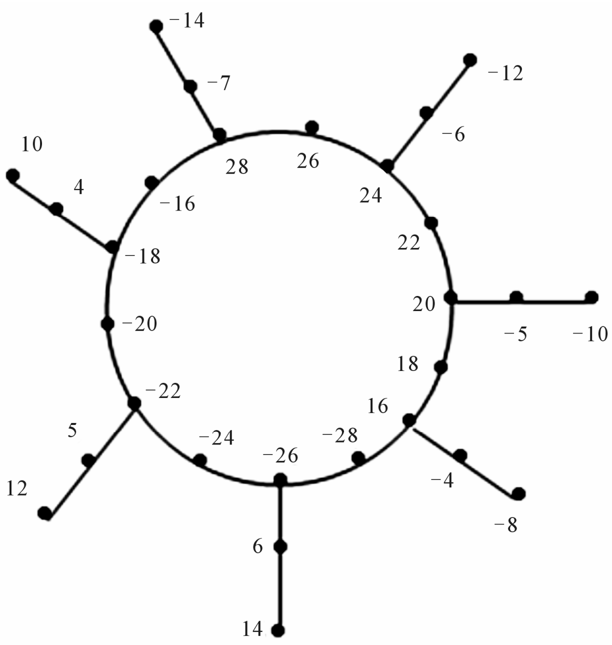

Figure 4. A pair sum labeling of $S\left(C_7K_1\right)$.

Theorem 4.3. $S\left(P_nK_1\right)$ is a pair sum graph.

Proof. Let

$$V\left(S\left(P_nK_1\right)\right) = \left\{u_i : 1 \le i \le 2n-1\right\} \cup \left\{w_{i,} v_i : 1 \le i \le n\right\}$$

Let

$$E\left(S\left(P_nK_1\right)\right) = \left\{u_iu_{i+1} : 1 \le i \le 2n-2\right\}$$
$$\cup\left\{u_{2i-1}w_i : 1 \le i \le n\right\} \cup \left\{v_iw_i : 1 \le i \le n\right\}.$$

Case 1. *n* is even.

When $n = 2$, the proof follows from Theorem 2.2. For $n > 2$, Define

$$f : V\left(S\left(P_nK_1\right)\right) \to \left\{\pm 1, \pm 2, \cdots, \pm\left(4n-1\right)\right\}$$

by

$$f\left(u_n\right) = 1,\ f\left(u_{n-1}\right) = -2$$

$$f\left(u_{n+1}\right) = 2$$

$$f\left(u_{n-1-2i}\right) = -5i - 2, 1 \le i \le n/2 - 1$$

$$f\left(u_{n-2i}\right) = -(5i+3), 1 \le i \le n/2 - 1$$

$$f\left(u_{n+1+2i}\right) = 5i + 3, 1 \le i \le n/2 - 1$$

$$f\left(u_{n+2i}\right) = 5i + 2, 1 \le i \le n/2 - 1$$

$$f\left(w_{n/2}\right) = 4,\ f\left(w_{n/2+1}\right) = -5$$

$$f\left(w_{n/2-i}\right) = -5i - 4, 1 \le i \le \left(n-2\right)/2$$

$$f\left(w_{n/2+i+1}\right) = 5i + 4, 1 \le i \le \left(n-2\right)/2$$

$$f\left(v_{n/2}\right) = -6,\ f\left(v_{n/2+1}\right) = 6$$

$$f\left(v_{n/2-i}\right) = -5i - 5, 1 \le i \le \left(n-2\right)/2$$

$$f\left(v_{n/2+i+1}\right) = 5i + 5, 1 \le i \le \left(n-2\right)/2$$

Here

$$f_e\left(E\left(S\left(P_4K_1\right)\right)\right) = \left\{1, 2, 3, 9, 15, 17, 19\right\}$$
$$\cup\left\{-1, -2, -3, -9, -15, -17, -19\right\}.$$

For $n > 4$,

$$f_e\left(E\left(S\left(P_nK_1\right)\right)\right) = f_e\left(E\left(S\left(P_4K_1\right)\right)\right)$$
$$\cup\left\{20, 25, 27, 29\right\} \cup \left\{-20, -25, -27, -29\right\}$$
$$\cup\left\{30, 35, 37, 39\right\} \cup \left\{-30, -35, -37, -39\right\} \bigcup, \cdots,$$
$$\cup\left\{5n-10, 5n-5, 5n-3, 5n-1\right\}$$
$$\cup\left\{-\left(5n-10\right), -\left(5n-5\right), -\left(5n-3\right), -\left(5n-1\right)\right\}.$$

Then *f* is pair sum labeling.

Case 2. *n* is odd.

Since $S\left(P_1K_1\right) \cong P_3$, which is a pair sum graph by Theorem 2.3. For $n > 1$, Define

$$f: V\left(S\left(P_n K_1\right)\right) \to \left\{\pm 1, \pm 2, \cdots, \pm(4n-1)\right\}$$

by

$$f\left(u_{(n+1)/2}\right) = 1,\ f\left(u_{(n-1)/2}\right) = 8$$

$$f\left(u_{(n+3)/2}\right) = -8$$

$$f\left(u_{(n-1)/2-2i}\right) = -10i+1, 1 \le i \le \lfloor n/2 \rfloor - 1$$

$$f\left(u_{(n+1)/2-2i}\right) = 5i+5, 1 \le i \le \lfloor n/2 \rfloor - 1$$

$$f\left(u_{(n+3)/2+2i}\right) = -10i-1, 1 \le i \le \lfloor n/2 \rfloor - 1$$

$$f\left(u_{(n+1)/2+2i}\right) = -(5i+5), 1 \le i \le \lfloor n/2 \rfloor - 1$$

$$f\left(w_{(n+1)/2}\right) = -2,\ f\left(w_{(n-1)/2}\right) = -5$$

$$f\left(w_{(n+3)/2}\right) = 6$$

$$f\left(w_{(n-1)/2-i}\right) = 5i+7, 1 \le i \le \lfloor n/2 \rfloor - 1$$

$$f\left(w_{(n+3)/2+i}\right) = -(5i+7), 1 \le i \le \lfloor n/2 \rfloor - 1$$

$$f\left(v_{(n+1)/2}\right) = 3,\ f\left(v_{(n-1)/2}\right) = 9$$

$$f\left(v_{(n+3)/2}\right) = -9$$

$$f\left(v_{(n-1)/2-i}\right) = 5i+8, 1 \le i \le \lfloor n/2 \rfloor - 1$$

$$f\left(v_{(n+3)/2+i}\right) = -(5i+8), 1 \le i \le \lfloor n/2 \rfloor - 1$$

Here

$$f_e\left(E\left(S\left(P_3 K_1\right)\right)\right) = \{1,-1,2,-2,3,-3,4,-4,5,-5\}$$

$$f_e\left(E\left(S\left(P_5 K_1\right)\right)\right) = f_e\left(E\left(S\left(P_3 K_1\right)\right)\right) \cup \{18,21,23,25,-18,-21,-23,-25\}$$

When $n > 5$,

$$f_e\left(E\left(S\left(P_n K_1\right)\right)\right) = \{26,31,33,35\} \cup \{-26,-31,-33,-35\} \cup \{36,41,43,45\} \cup \{-36,-41,-43,-45\} \cup, \cdots, \cup \{5n-9, 5n-4, 5n-2, 5n\} \cup \{-(5n-9), -(5n-4), -(5n-2), -5n\}.$$

Then f is pair sum labeling.

Illustration 5. A pair sum labeling of $S\left(P_8 K_1\right)$ is shown in **Figure 5**.

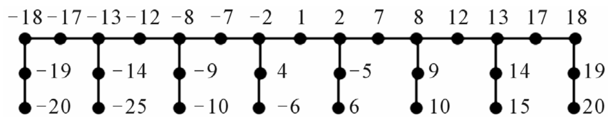

Figure 5. A pair sum labeling of $S\left(P_8 K_1\right)$.

Theorem 4.4. $S\left(T_n\right)$ is a pair sum graph where T_n is a triangular snake with n triangle.

Proof. Let

$$V\left(S\left(T_n\right)\right) = \{u_i : 1 \le i \le 2n+1\} \cup \{v_i : 1 \le i \le 2n\} \subset \{w_i : 1 \le i \le n\}$$

and

$$E\left(S\left(T_n\right)\right) = \{u_i u_{i+1} : 1 \le i \le 2n\} \cup \{v_{2i-1} w_i, v_{2i} w_i : 1 \le i \le n\} \cup \{u_{2i-1} v_{2i-1}, u_{2i+1} v_{2i} : 1 \le i \le n\}.$$

Case 1. n is even.

When $n = 2$, Define $f(u_1) = 7$, $f(u_2) = 6$, $f(u_3) = 1$, $f(u_4) = -6$, $f(u_5) = -7$, $f(v_1) = 5$, $f(v_2) = 2$, $f(v_3) = -4$, $f(v_4) = -5$, $f(w_1) = 3$, $f(w_2) = -3$. When $n > 2$, Define

$$f: V\left(S\left(T_n\right)\right) \to \left\{\pm 1, \pm 2, \cdots, \pm(5n+1)\right\}$$

by

$$f\left(u_{n+1}\right) = 1,\ f\left(u_n\right) = 6$$

$$f\left(u_{n-1}\right) = 7,\ f\left(u_{n+2}\right) = 6$$

$$f\left(u_{n+3}\right) = -7$$

$$f\left(u_{n-2i}\right) = 10i+6, 1 \le i \le (n-2)/2$$

$$f\left(u_{n-1-2i}\right) = 10i+4, 1 \le i \le (n-2)/2$$

$$f\left(u_{n+2+2i}\right) = -10i-6, 1 \le i \le (n-2)/2$$

$$f\left(u_{n+3+2i}\right) = -10i-4, 1 \le i \le (n-2)/2$$

$$f\left(v_n\right) = 2,\ f\left(v_{n-1}\right) = 5$$

$$f\left(v_{n+1}\right) = -4,\ f\left(v_{n+2}\right) = -5$$

$$f\left(v_{n-2i}\right) = 10i-2, 1 \le i \le (n-2)/2$$

$$f\left(v_{n-1-2i}\right) = 10i+2, 1 \le i \le (n-2)/2$$

$$f\left(v_{n+1+2i}\right) = -10i+2, 1 \le i \le (n-2)/2$$

$$f\left(v_{n+2+2i}\right) = -10i-2, 1 \le i \le (n-2)/2$$

$$f\left(w_{n/2}\right) = 3,\ f\left(w_{(n+2)/2}\right) = -3$$

$$f\left(w_{(n-2i)/2}\right) = 10i, 1 \le i \le (n-2)/2$$

$$f\left(w_{n/2+i+1}\right) = -10i, 1 \le i \le (n-2)/2$$

For $n = 4$,

$$f_e\left(E\left(S\left(T_n\right)\right)\right)=\{3,5,7,8,12,13,15,18,22,23,26,30\}$$
$$\cup\{-3,-5,-7,-8,-12,-13,-15,-18,-22,-23,-26,-30\}.$$

For $n > 4$

$$f_e\left(E\left(S\left(T_n\right)\right)\right)= f_e\left(E\left(S\left(T_4\right)\right)\right)$$
$$\cup\{(32,38,40,42,46,50),(-32,-38,-40,-42,-46,-50),$$
$$(52,58,60,62,66,70),(-52,-58,-60,-62,-66,-70),\cdots,$$
$$(10n-28,10n-22,10n-20,10n-18,10n-14,10n-10),$$
$$(-10n+28,-10n+22,-10n+20,$$
$$-10n+18,-10n+14,-10n+10)\}.$$

Then f is pair sum labeling.

Case 2. n is odd.

Clearly $S(T_1)\cong C_6$, and hence $S(T_1)$ is a pair sum graph by Theorem 2.3.

For > 1, Define

$$f:V\left(S\left(T_n\right)\right)\to\{\pm1,\pm2,\cdots,\pm(5n+1)\}\cong C_6$$

by

$$f(u_{n+1})=-8,\ f(u_n)=3$$
$$f(u_{n-1})=-2,\ f(u_{n-2})=-15$$
$$f(u_{n+2})=-5,\ f(u_{n+3})=4$$
$$f(u_{n+4})=15$$
$$f(u_{n-1-2i})=-14-5i, 1\le i\le(n-3)/2$$
$$f(u_{n-2-2i})=-10i-4, 1\le i\le(n-3)/2$$
$$f(u_{n+3+2i})=14+5i, 1\le i\le(n-3)/2$$
$$f(u_{n+4+2i})=15+5i, 1\le i\le(n-3)/2$$
$$f(v_n)=5,\ f(v_{n-1})=-7$$
$$f(v_{n-2})=-4,\ f(v_{n+1})=-3$$
$$f(v_{n+2})=7,\ f(v_{n+3})=2$$
$$f(v_{n-1-2i})=-11-5i, 1\le i\le(n-3)/2$$
$$f(v_{n-2-2i})=-5i-13, 1\le i\le(n-3)/2$$
$$f(v_{n+1+2i})=11+5i, 1\le i\le(n-3)/2$$
$$f(v_{n+2+2i})=13+5i, 1\le i\le(n-3)/2$$
$$f\left(w_{(n+1)/2}\right)=8,\ f\left(w_{(n-1)/2}\right)=11$$
$$f\left(w_{(n+3)/2}\right)=-9,$$
$$f\left(w_{(n-1)/2-i}\right)=-12-5i, 1\le i\le(n-3)/2$$
$$f\left(w_{(n+3)/2+i}\right)=12+5i, 1\le i\le(n-3)/2$$

Here $n = 3$,

$$f_e\left(E\left(S\left(T_n\right)\right)\right)=\{1,2,4,5,7,8,13,17,19\}$$
$$\cup\{-1,-2,-4,-5,-7-8,-13,-17,-19\}.$$

For $n > 3$,

$$f_e\left(E\left(S\left(T_n\right)\right)\right)= f_e\left(E\left(S\left(T_3\right)\right)\right)$$
$$\cup\{(31,33,34,35,38,39),(-31,-33,-34,-35,-38,-39),$$
$$(41,43,44,45,48,49),(-41,-43,-44,-45,-48,-49),\cdots,$$
$$(5n+6,5n+8,5n+9,5n+10,5n+13,5n+14),$$
$$(-5n-6,-5n-8,-5n-9,-5n-10,-5n-13,-5n-14)\}.$$

Then f is pair sum labeling.

Illustration 6. A pair sum labeling of $S(T_5)$ is shown in **Figure 6**.

Theorem 4.5. $S(Q_n)$ is a pair sum graph.

Proof. Let

$$V\left(S\left(Q_n\right)\right)=\{u_i:1\le i\le 2n+1\}$$
$$\cup\{v_i:1\le i\le 3n\}\cup\{w_i:1\le i\le 2n\}$$

and

$$E\left(S(Q_n)\right)\{u_iu_{i+1}:1\le i\le 2n\}$$
$$\cup\{u_{2i+1}w_{2i},v_{3i}w_{2i}:1\le i\le n\}$$
$$\cup\{u_{2i-1}w_{2i-1},w_{2i-1}v_{3i-2}:1\le i\le n\}$$
$$\cup\{v_iv_{i+1}:1\le i\le 3n-1\}-\{v_{3i}v_{3i+1}:1\le i\le n\}.$$

Case 1. n is even.

When $n = 2$, Define $f(u_1) = 11$, $f(u_2) = 6$, $f(u_3) = 1$, $f(u_4) = -6$, $f(u_5) = -11$, $f(w_1) = 9$, $f(w_2) = 2$, $f(w_3) = -4$, $f(w_4) = -9$, $f(v_1) = 7$, $f(v_2) = 5$, $f(v_3) = 3$, $f(v_4) = -3$, $f(v_5) = -5$, $f(v_6) = -7$. When > 2, Define

$$f:V\left(S\left(Q_n\right)\right)\to\{\pm1,\pm2,\cdots,\pm(7n+1)\}$$

by

$$f(u_{n+1})=1,\ f(u_{n-1})=6$$
$$f(u_{n-2})=8,\ f(u_{n+1})=-6,\ f(u_{n+2})=-8$$
$$f(u_{n-2-2i})=14i+8, 1\le i\le(n-2)/2$$
$$f(u_{n-1-2i})=14i+6, 1\le i\le(n-2)/2$$

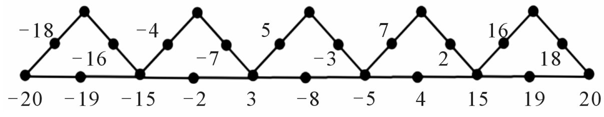

Figure 6. A pair sum labeling of $S(T_5)$.

$$f\left(u_{n+1+2i}\right) = -14i - 6, 1 \le i \le (n-2)/2$$
$$f\left(u_{n+2+2i}\right) = -14i - 8, 1 \le i \le (n-2)/2$$
$$f\left(w_n\right) = 2,\ f\left(w_{n-1}\right) = 9$$
$$f\left(w_{n+1}\right) = -4,\ f\left(w_{n+2}\right) = -9$$
$$f\left(w_{n-2i}\right) = 14i - 4, 1 \le i \le (n-2)/2$$
$$f\left(w_{n-1-2i}\right) = 14i + 4, 1 \le i \le (n-2)/2$$
$$f\left(w_{n+1+2i}\right) = -14i + 4, 1 \le i \le (n-2)/2$$
$$f\left(w_{n+2+2i}\right) = -14i - 4, 1 \le i \le (n-2)/2$$
$$f\left(v_{3n/2}\right) = 3,\ f\left(v_{3n-1/2}\right) = 5$$
$$f\left(v_{3n-2/2}\right) = 7,\ f\left(v_{3n+1/2}\right) = -3$$
$$f\left(v_{3n+2/2}\right) = -5,\ f\left(v_{3n+3/2}\right) = -7$$
$$f\left(v_{3n/2-3i}\right) = 14i - 2, 1 \le i \le (n-2)/2$$
$$f\left(v_{3n/2-2-3i}\right) = 14i, 1 \le i \le (n-2)/2$$
$$f\left(v_{3n/2-1-3i}\right) = 14i + 2, 1 \le i \le (n-2)/2$$
$$f\left(v_{3n/2+1+3i}\right) = -14i + 2, 1 \le i \le (n-2)/2$$
$$f\left(v_{3n/2+2+3i}\right) = -14i, 1 \le i \le (n-2)/2$$
$$f\left(v_{3n/2+3+3i}\right) = -14i - 2, 1 \le i \le (n-2)/2$$

Here

$$\begin{aligned} f_e\left(E\left(S\left(Q_n\right)\right)\right) = \{&3,-3,5,-5,7,-7,8,-8,\\ &12,-12,14,-1416,-16,17,-17\}\\ &\cup\{18,22,26,28,30,34,40,42\}\\ &\cup\{-18,-22,-26,-28,-30,-34,-40,-42\}\\ &\cup\{46,50,54,56,58,62,68,70\}\\ &\cup\{-46,-50,-54,-56,-58,-62,-68,-70\}\\ &\cup\{74,78,82,86,84,90,96,98\}\\ &\cup\{-74,-78,-82,-86,-84,-90,-96,-98\}\bigcup,\cdots,\\ &\cup\{14n-38,14n-34,14n-30,14n-28,\\ &14n-26,14n-22,14n-16,14n-14\}\\ &\cup\{-14n+38,-14n+34,-14n+30,-14n+28,\\ &-14n+26,-14n+22,-14n+16,-14n+14\}. \end{aligned}$$

Then f is pair sum labeling.

Case 2. n is odd.

$S\left(Q_1\right)$ is a pair sum graph follows from Theorem 2.3.When $n > 1$. Define

$$f : V\left(S\left(Q_n\right)\right) \to \{\pm 1, \pm 2, \cdots, \pm(7n+1)\}$$

by

$$f\left(u_{n+1}\right) = -6,\ f\left(u_n\right) = 7,\ f\left(u_{n-1}\right) = 8$$
$$f\left(u_{n-2}\right) = 22,\ f\left(u_{n+2}\right) = -3,\ f\left(u_{n+3}\right) = -8$$
$$f\left(u_{n+4}\right) = -22$$
$$f\left(u_{n-1-2i}\right) = 14i + 20, 1 \le i \le (n-3)/2$$
$$f\left(u_{n-2-2i}\right) = 14i + 22, 1 \le i \le (n-3)/2$$
$$f\left(u_{n+3+2i}\right) = -14i - 20, 1 \le i \le (n-3)/2$$
$$f\left(u_{n+4+2i}\right) = -14i - 22, 1 \le i \le (n-3)/2$$
$$f\left(w_n\right) = 5,\ f\left(w_{n-1}\right) = 4,\ f\left(w_{n-2}\right) = 20$$
$$f\left(w_{n+1}\right) = -5,\ f\left(w_{n+2}\right) = -12,\ f\left(w_{n+3}\right) = -20$$
$$f\left(w_{n-1-2i}\right) = 14i + 10, 1 \le i \le (n-3)/2$$
$$f\left(w_{n-2-2i}\right) = 18 + 14i, 1 \le i \le (n-3)/2$$
$$f\left(w_{n+2+2i}\right) = -14i - 10, 1 \le i \le (n-3)/2$$
$$f\left(w_{n+3+2i}\right) = -18 - 14i, 1 \le i \le (n-3)/2$$
$$f\left(v_{(3n+1)/2}\right) = 6,\ f\left(v_{(3n-1)/2}\right) = 3$$
$$f\left(v_{(3n-3)/2}\right) = -10,\ f\left(v_{(3n-5)/2}\right) = 16$$
$$f\left(v_{(3n-7)/2}\right) = 18,\ f\left(v_{(3n+3)/2}\right) = -7$$
$$f\left(v_{(3n+5)/2}\right) = 14,\ f\left(v_{(3n+7)/2}\right) = -16$$
$$f\left(v_{(3n+9)/2}\right) = -18$$
$$f\left(v_{(3n-3)/2-3i}\right) = 14i + 12, 1 \le i \le (n-3)/2$$
$$f\left(v_{(3n-5)/2-3i}\right) = 14i + 14, 1 \le i \le (n-3)/2$$
$$f\left(v_{(3n-7)/2-3i}\right) = 14i + 6, 1 \le i \le (n-3)/2$$
$$f\left(v_{(3n+5)/2+3i}\right) = -14i - 12, 1 \le i \le (n-3)/2$$
$$f\left(v_{(3n+7)/2+3i}\right) = -14i - 14, 1 \le i \le (n-3)/2$$
$$f\left(v_{(3n+9)/2+3i}\right) = -14i - 6, 1 \le i \le (n-3)/2$$

For $n = 3$,

$$\begin{aligned} f_e\left(E\left(S\left(Q_n\right)\right)\right) &= \{1,2,6,8,9,11,12,15,30,34,38,42\}\\ &\cup\{-1,-2,-6,-8,-9,-11,-12,-15,-30,-34,-38,-42\}. \end{aligned}$$

$n > 3$,

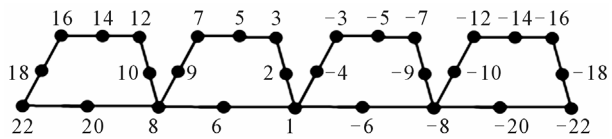

Figure 7. A pair sum labeling of $S(Q_4)$.

$$f_e(E(S(Q_n))) = f_e(E(S(Q_3))) \cup \{(46,50,54,56,58,62,68,70), (-46,-50,-54,-56,-58,-62,-68,-70), (74,78,82,84,86,90,96,98), (-74,-78,-82,-84,-86,-90,-96,-98),\cdots, (14n-24,14n-20,14n-16,14n-14, 14n-12,14n-8,14n-2,14n) (-14n+24,-14n+20,-14n+16,-14n+14, -14n+12,-14n+8,-14n+2,-14n)\}.$$

Then *f* is pair sum labeling

Illustration 7. A pair sum labeling of $S(Q_4)$ is shown in **Figure 7**.

5. Acknwledgements

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