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Asymptotic Behaviour to a Von Kármán System with Internal Damping

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ABSTRACT

In this work we consider the Von Kármán system with internal damping acting on the displacement of the plate and using the Theorem due to Nakao [1] we prove the exponential decay of the solution.

Keywords: Von Kármán System; Internal Damping; Exponential Decay; Theorem of Nakao

1. Introduction

Theodor von Kármán (1910) [2] started the nonlinear system of partial differential for great deflections and for the Airy stress function of a thin elastic plate. For several years this system was studied in different situations. Using frictional dissipation at boundary, I. Lasiecka *et al.* [3-5] proved the uniform decay of the solution. G. P. Menzala and E. Zuazua [6] by semigroup properties gave the exponential decay when thermal damping was considered. For Viscoelastic plates with memory, J. E. M. Rivera *et al.* [7,8] proved that the energy decays uniformly, exponentially or algebraically with the same rate of decay of the relaxation function. C. A. Raposo and M. L. Santos [9] gave a General Decay of solution for the memory case. In [10-13] the authors consider the von Kármán system with frictional dissipations effective in the whole plate, in a part of the plate or at the boundary. It is shown in these works that these dissipations produce uniform rate of decay of the solution when t goes to infinity. In this work we also consider the system with internal damping, which is the natural problem. A distinctive feature of our paper is to use Nakao's method to show that the energy decays exponentially to zero.

2. Existence of Solution

We use the standard Lebesgue space and Sobolev space with their usual properties as in [14] and in this sense $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in L^2 and H_0^1 respectively and by $|\cdot|$ we denote the usual norm in L^2. Let Ω be a bounded domain of the plane with regular boundary Γ. For a real number $T>0$ we denote $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$. Here $u=u(x,t)$ is the displacement, $v=v(x,t)$ the Airy stress function and η is the unit normal external in Ω. With this notation we have the following system

$$u_{tt}-\Delta^2 u+u_t=[u,v] \quad \text{in} \quad Q \tag{1}$$

$$-\Delta^2 v=[u,u] \quad \text{in} \quad Q \tag{2}$$

$$u(0)=u_0,\ u_t(0)=u_1 \quad \text{in} \quad \Omega \tag{3}$$

$$u=\partial u/\partial\eta=v=\partial v/\partial\eta=0 \quad \text{in} \quad \Sigma \tag{4}$$

where

$$[u,v]=\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2}-2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}$$

Now using the same idea of [6] we have the following result of existence of solution.

Theorem 2.1. *For* $u_0\in H_0^2(\Omega)$, $u_1\in L^2(\Omega)$ *there exists* $u,v:Q\to\mathbb{R}$ *such that*

$$u,v\in L^\infty\left(0,T;H_0^2(\Omega)\right)$$
$$u_t\in L^\infty\left(0,T;L^2(\Omega)\right)$$

u,v *weak solution of* (1)-(4).

Proof. We defining the energy $E(t)$ of the system (1)-(4) by

$$E(t)=|u_t(t)|^2+|\Delta u(t)|^2+\frac{1}{2}|\Delta v(t)|^2.$$

This system is well posed in the energy space (see [15]) and we have and $E'(t)<0$. Galerkin's method together with the dissipative properties of the energy give us

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global existence of solution in the energy space. Finally using the results from [5] on the regularity properties of von Kármám bracket the uniqueness follows.

3. Asymptotic Behaviour

In this section, we will use the Theorem of Nakao to prove the exponential decay of the solution.

Theorem 3.1. (*Theorem of Nakao*) *Let* $E(t)$ *be a nonnegative function on* $[0,\infty)$ *satisfying*

$$\sup_{s\in[t,t+1]} E(s) \le C_0\left(E(t)-E(t+1)\right)$$

where C_0 *is a positive constant. Then we have*

$$E(t) \le C_1 e^{-wt} \quad \text{with} \quad w=\frac{1}{C_0+1}.$$

Proof. See page 748 of [1].

In the sequel we have two lemmas,

Lemma 3.1. *The functional* $F^2(t)=E(t)-E(t+1)$ *satisfies*

$$\int_t^{t+1} \left|u_t(s)\right|^2 \mathrm{d}s \le F^2(t).$$

Proof. Multiplying (1) by u_t and integrating in Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left|u_t(t)\right|^2+\left|\Delta u(t)\right|^2\right] - \left\langle\left[u(t),v(t)\right],u_t(t)\right\rangle + \left|u_t(t)\right|^2 = 0.$$

Using (2) we obtain

$$\begin{aligned}\left\langle\left[u(t),v(t)\right],u_t(t)\right\rangle &= \left\langle\left[u(t),u_t(t)\right],v(t)\right\rangle \\ &= \frac{1}{2}\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\left[u(t),u(t)\right],v(t)\right\rangle - \frac{1}{2}\left\langle\Delta^2 v_t(t),v(t)\right\rangle \\ &= -\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left|\Delta v(t)\right|^2,\end{aligned}$$

from where follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left|u_t(t)\right|^2+\left|\Delta u(t)\right|^2+\frac{1}{2}\left|\Delta v(t)\right|^2\right]+2\left|u_t(t)\right|^2=0 \quad (5)$$

Performing integration in $0<t<t+1$, we have

$$E(t+1)+2\int_t^{t+1}\left|u_t(s)\right|^2 \mathrm{d}s = E(t) \quad (6)$$

then

$$\int_t^{t+1}\left|u_t(s)\right|^2 \mathrm{d}s \le E(t)-E(t+1)=F^2(t). \quad (7)$$

Lemma 3.2. *The functional*

$$G^2(t)=8C\sup_{s\in[t,t+1]}\left|\Delta u(s)\right|F(t)+2\left(1+C^2\right)\int_{t_1}^{t_2}\left|u_t(t)\right|^2 \mathrm{d}t$$

satisfies

$$\int_{t_1}^{t_2}\left(\left|\Delta u(t)\right|^2+\frac{1}{2}\left|\Delta v(t)\right|^2\right)\mathrm{d}t \le G^2(t).$$

Proof. First we note that

$$\begin{aligned}\left\langle\left[u(t),v(t)\right],u(t)\right\rangle &= \left\langle\left[u(t),u(t)\right],v(t)\right\rangle \\ &= -\left\langle\Delta^2 v(t),v(t)\right\rangle = -\left|\Delta v(t)\right|^2\end{aligned} \quad (8)$$

From (7) we get $t_1\in[t,t+1/4]$ and $t_2\in[t+3/4,t+1]$ such that

$$u_t(t_i) \le 2F(t), \quad i=1,2. \quad (9)$$

Multiplying (1) by u and integrating in Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u_t(t),u(t)\right)-\left|u_t(t)\right|^2+\left|\Delta u(t)\right|^2 - \left\langle\left[u(t),v(t)\right],u(t)\right\rangle+\left(u_t(t),u(t)\right)=0.$$

Integrating from t_1 to t_2 and using (8) we have

$$\begin{aligned}&\int_{t_1}^{t_2}\left(\left|\Delta u(t)\right|^2+\left|\Delta v(t)\right|^2\right)\mathrm{d}t \\ &= \left(u_t(t_1),u(t_1)\right)-\left(u_t(t_2),u(t_2)\right) \\ &\quad + \int_{t_1}^{t_2}\left(\left|u_t(t)\right|^2-\left(u_t(t),u(t)\right)\right)\mathrm{d}t\end{aligned}$$

Now, choosing C such that $|u|\le C|\Delta u|$ and applying Cauchy-Schuwarz inequality we get

$$\begin{aligned}&\int_{t_1}^{t_2}\left(\frac{1}{2}\left|\Delta u(t)\right|^2+\left|\Delta v(t)\right|^2\right)\mathrm{d}t \\ &\le C\sup_{s\in[t,t+1]}\left\{\left|\Delta u(s)\right|\left(\left|u_t(t_1)\right|+\left|u_t(t_2)\right|\right)\right\} \\ &\quad +\left(1+C^2\right)\int_{t_1}^{t_2}\left|u_t(t)\right|^2 \mathrm{d}t,\end{aligned}$$

and using (9),

$$\begin{aligned}&\int_{t_1}^{t_2}\left(\frac{1}{2}\left|\Delta u(t)\right|^2+\left|\Delta v(t)\right|^2\right)\mathrm{d}t \\ &\le 8C\sup_{s\in[t,t+1]}\left|\Delta u(s)\right|F(t)+2\left(1+C^2\right)\int_{t_1}^{t_2}\left|u_t(t)\right|^2 \mathrm{d}t,\end{aligned}$$

from where follows

$$\int_{t_1}^{t_2}\left(\left|\Delta u(t)\right|^2+\frac{1}{2}\left|\Delta v(t)\right|^2\right)\mathrm{d}t \le G^2(t). \quad (10)$$

Now we are in position of to prove our principal result.

Theorem 3.2. *The solution* (u,v) *satisfies*

$$\left|u_t(t)\right|^2+\left|\Delta u(t)\right|^2+\frac{1}{2}\left|\Delta v(t)\right|^2+\int_t^{t+1}\left|u_t(s)\right|^2 \mathrm{d}s \le C_1 e^{-wt}, \quad (11)$$

for almost every $t\ge 1$, *with* $C_1,w>0$, *constants independents from t.*

Proof. From (7) and (10) we obtain

$$\int_{t_1}^{t_2}\left(\left|u_t(t)\right|^2+\left|\Delta u(t)\right|^2+\frac{1}{2}\left|\Delta v(t)\right|^2\right)dt\le F^2(t)+G^2(t).$$

There exists $t^*\in[t_1,t_2]$ such that

$$\begin{aligned}E(t^*)&=\left|u_t(t^*)\right|^2+\left|\Delta u(t^*)\right|^2+\frac{1}{2}\left|\Delta v(t^*)\right|^2\\&\le 2\left(F^2(t)+G^2(t)\right).\end{aligned}\tag{12}$$

From (6) we get

$$E(t_1)=E(t^*)+2\int_{t_1}^{t^*}\left|u_t(s)\right|^2 ds.$$

Then

$$E(t)<E(t^*)+2\int_{t_1}^{t^*}\left|u_t(s)\right|^2 ds,$$

and

$$\sup_{s\in[t,t+1]}E(s)\le E(t^*)+3\int_t^{t+1}\left|u_t(s)\right|^2 ds$$

Now using (11) and (12) we obtain

$$\begin{aligned}\sup_{s\in[t,t+1]}E(s)&\le 2\left(F^2(t)+G^2(t)\right)+3F^2(t)\\&\le 5F^2(t)+16C\sup_{s\in[t,t+1]}\left|\Delta u(s)\right|F(t)\\&\quad+4\left(1+C^2\right)\int_t^{t+1}\left|u_t(s)\right|^2 ds\\&\le\left(9+4C^2\right)F^2(t)+\frac{1}{2}\sup_{s\in[t,t+1]}E(s)\\&\quad+128C^2F^2(t),\end{aligned}$$

then

$$\sup_{s\in[t,t+1]}E(s)\le\left(274+8C^2\right)F^2(t),$$

and finally by Theorem of Nakao follows

$$E(t)\le C_1e^{-wt}$$

with $w=\dfrac{1}{275+8C^2}$.

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