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On a Population Model of Systems $\begin{cases} x_{n+1} = \alpha x_n e^{-y_n} + \beta \\ y_{n+1} = \alpha x_n \left(1 - e^{-y_n}\right) \end{cases}$*

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ABSTRACT

In this paper, we investigate the global character of all positive solutions of a population model of systems. Some interesting convergence properties of the solution are given, and lastly, we obtain that the solution is permanent under some conditions.

Keywords: Population Model; Global Attractor; Difference Equations

1. Introduction

In the recent monograph [1, p.129], Kulenovic and Glass give an open problem as follows:

Open problem 6.10.16 (A population model).

Assume that $\alpha \in (0,1)$ and $\beta \in (1,\infty)$. Investigate the global character of all positive solutions of the systems:

$$\begin{cases} x_{n+1} = \alpha x_n e^{-y_n} + \beta \\ y_{n+1} = \alpha x_n \left(1 - e^{-y_n}\right) \end{cases} \quad (1)$$

where $n = 0,1,\cdots$, which may be viewed as a population model.

To this end, we consider Equation (1) and obtain some interesting results about the positive solutions of Equation (1).

2. Basic Lemma

Lemma 1 *Assume that* $\alpha \in (0,1)$, $\beta \in (1,\infty)$. *Then the following statements are true*:

1) If $1 < \beta \le \frac{1-\alpha}{\alpha}$, then Equation (1) has a unique no-negative equilibrium solution as follows:

$$(\bar{x}_1, \bar{y}_1) = \left(\frac{\beta}{1-\alpha}, 0\right)$$

2) If $\beta > \frac{1-\alpha}{\alpha}$, then Equation (1) has two no-negative equilibrium solutions as follows:

$$(\bar{x}_1, \bar{y}_1) = \left(\frac{\beta}{1-\alpha}, 0\right) \quad \text{or} \quad (\bar{x}_2, \bar{y}_2)$$

where $0 < \bar{y}_2 < \beta$, $\beta < \bar{x}_2 < \frac{\beta}{1-\alpha}$ such that

$$\begin{cases} 1 - e^{-\bar{y}_2} = \frac{1-\alpha}{\alpha}\left(\frac{\bar{y}_2}{\beta - \bar{y}_2}\right) \\ \bar{x}_2 = \frac{1}{1-\alpha}(\beta - \bar{y}_2) \end{cases} \quad (2)$$

Proof: The equilibrium equations about Equation (1) can be written as follows:

$$\begin{cases} \bar{x} = \alpha \bar{x} e^{-\bar{y}} + \beta \\ \bar{y} = \alpha \bar{x}\left(1 - e^{-\bar{y}}\right) \end{cases} \quad (3)$$

It is easy to see that $\bar{x}_1 = \frac{\beta}{1-\alpha}$, $\bar{y}_1 = 0$ is a group solutions of Equation (3).

By (3) we obtain

$$\begin{aligned} \bar{x} + \bar{y} &= \alpha \bar{x} + \beta \\ \bar{x} &= \frac{1}{1-\alpha}(\beta - \bar{y}) \end{aligned} \quad (4)$$

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Thus

$$\bar{y} = \frac{\alpha}{1-\alpha}(\beta - \bar{y})\left(1-e^{-\bar{y}}\right) \quad (5)$$

Noting that (3) and (4) we get:

$$0 < \bar{y}_2 < \beta \quad \text{and} \quad \beta < \bar{x}_2 < \frac{\beta}{1-\alpha}$$

Changing (5) to (6)

$$1-e^{-\bar{y}} = \frac{1-\alpha}{\alpha}\left(\frac{\bar{y}}{\beta-\bar{y}}\right) \quad (6)$$

Set

$$f(x) = \frac{\alpha}{1-\alpha}[\beta - x]\left(1-e^{-x}\right) - x,$$

for $0 < \alpha < \beta, 0 < x < \beta$

Observing that

$$f(0) = 0, \quad f(\beta) = -\beta$$

$$f'(x) = \frac{\alpha}{1-\alpha}\left[-1+(\beta+1-x)e^{-x}\right]-1$$

$$f''(x) = \frac{\alpha}{1-\alpha}\left[-e^{-x}-(\beta+1-x)e^{x}\right] < 0$$

So, by the convex functions properties, if $\lim_{x\to 0^+} f'(x) > 0$, then we can obtain Equation (6) has a unique positive solution $\bar{y}_2$.

In fact, by the continuous of f, we can get

$$\lim_{x\to 0^+} f'(x) = f'(0) = \frac{\alpha}{1-\alpha}\beta - 1 > 0.$$

Hence, we complete the proof.

3. Main Results

Theorem 3.1 *Assume that* $\alpha \in (0,1)$ *and* $\beta \in (1,\infty)$. *Then every positive solutions* $\{x_n\}_{n=0}^{\infty}$ *and* $\{y_n\}_{n=0}^{\infty}$ *of Equation* (1) *have the following properties*:

1) $\lim_{n\to\infty} sup\{x_n\} \le \frac{\beta}{1-\alpha}, \lim_{n\to\infty} inf\{x_n\} > \beta$;

2) $\lim_{n\to\infty} sup\{y_n\} < \frac{\alpha\beta}{1-\alpha}, \lim_{n\to\infty} inf\{y_n\} \ge 0$.

Proof: By Equation (1) we have

$$\beta < x_{n+1} \le \alpha x_n + \beta \le \alpha\left[\alpha x_{n-k}\right] + \beta$$
$$\le \cdots \le \beta + \alpha\beta + \cdots + \alpha^{n-1}\beta + \alpha^{n+1}x_0$$

It is to say that $\lim_{n\to\infty} sup\{x_n\} \le \frac{\beta}{1-\alpha}, \quad \lim_{n\to\infty} inf\{x_n\} > \beta$.

By Equation (1) we also get

$$y_{n+1} < \alpha x_n$$

Thus $sup\{y_n\} < \frac{\alpha\beta}{1-\alpha}, \quad inf\{y_n\} \ge 0$.

This completes the proof.

Theorem 3.2 *Assume that* $\alpha \in (0,\infty)$, $\beta \in (1,\infty)$ *and* $\beta \le \frac{1-\alpha}{\alpha}$. *Then every positive solutions of Equation* (1) *convergences to the unique no-negative equilibrium solution* $\left(\frac{\beta}{1-\alpha}, 0\right)$.

Proof: By Theorem 3.1, we have that there exists a nature number n_0 such that $x_n \le \frac{\beta}{1-\alpha}$ for $n > n_0$.

Hence, by Equation (1) we get

$$y_{n+1} = \alpha x_n\left(1-e^{-y_n}\right) \le \alpha x_n y_n \le \frac{\alpha\beta}{1-\alpha} y_n \le y_n$$

Thus $\{y_n\}_{n=n_0+1}^{\infty}$ is decreasing.

Suppose that

$$\lim_{n\to\infty} y_{n+1} = l_0 > 0 \quad (7)$$

Then by Equation (1) we have

$$x_{n+1} \le \alpha e^{-l_0} x_n + \beta \quad \text{for} \quad n \ge n_0 + 1$$

By induction we obtain

$$x_{n+1} \le \alpha e^{-l_0} x_n + \beta \le \alpha e^{-l_0}\left[\alpha e^{-l_0} + \beta\right] + \beta$$
$$\le \cdots \le \left(\alpha e^{-l_0}\right)^{n-n_0+1}\beta + \cdots + \beta + \left(\alpha e^{-l_0}\right)^{n-n_0} x_{n_0+1}$$

Thus $sup\{x_{n+1}\} \le \frac{\beta}{1-\alpha e^{-l_0}}$. Hence there exists a $n_0' \in N^+$ such that $x_n \le \frac{\beta}{1-\alpha e^{-l_0}}$ for $n > n_0'$.

Noting that Equation (1)

$$y_{n+1} = \alpha x_n\left(1-e^{-y_n}\right) \le \frac{\alpha\beta}{1-\alpha e^{l_0}} y_n \quad \text{for} \quad n > n_0'$$

By induction,

$$y_{n+1} \le \left[\frac{\alpha\beta}{1-\alpha e^{-l_0}}\right]^{n-n_0+1} y_{n_0+1}$$

It is to see that $\lim_{n\to\infty} y_n = 0$. This is a contradiction with (7), then $\lim_{n\to\infty} y_n = 0$.

Noting that Equation (1) we have

$$x_{n+1} + y_{n+1} = \alpha x_n + \beta$$

i.e.

$$x_{n+1} = \alpha x_n + \beta - y_{n+1}$$

Let $sup\{y_{n+1}\} = \mu_1$, $inf\{y_{n+1}\} = \lambda_1$. Then

$$\alpha x_n + \beta - \mu_1 \le x_{n+1} < \alpha x_n + \beta - \lambda_1$$

By induction we obtain

$$\frac{\beta - \mu_1}{1-\alpha}\left(1-\alpha^{n+1}\right) + \alpha^{n+1} x_0 \le x_{n+1} \le \frac{\beta - \lambda_1}{1-\alpha}\left(1-\alpha^{n+1}\right) + \alpha^{n+1} x_0$$

as $0 < \alpha < 1$, then

$$\lim_{n\to\infty} sup\{x_n\} \le \frac{\beta-\lambda_1}{1-\alpha} \quad \text{and} \quad \lim_{n\to\infty} inf\{x_n\} \ge \frac{\beta-\mu_1}{1-\alpha} \qquad (8)$$

Because of $\lim_{n\to\infty} y_n = 0$, we obtain that $\lambda_1 = \mu_1 = 0$.

Hence

$$\lim_{n\to\infty} sup\{x_n\} \le \frac{\beta}{1-\alpha} \quad \text{and} \quad \lim_{n\to\infty} inf\{x_n\} \ge \frac{\beta}{1-\alpha} \qquad (9)$$

By (9) we get $\lim_{n\to\infty} x_n = \frac{\beta}{1-\alpha}$.

We complete the proof.

Theorem 3.3 *Assume that* $\alpha \in (0,\infty)$, $\beta \in (1,\infty)$ *and* $\alpha\beta > 1$. *Then Equation* (1) *is permanent.*

Proof: By Equation (1) we obtain

$$\begin{aligned} y_{n+1} &> \alpha x_n\left(1-e^{-y_n}\right) > \alpha\beta\left(1-e^{-y_n}\right) \\ &= \alpha\beta\left[y_n - \frac{y_n^2}{2!} + \frac{y_n^3}{3!} - \frac{y_n^4}{4!} + \cdots\right] \\ &= \alpha\beta\left[1 - \frac{y_n}{2!} + \frac{y_n^2}{3!} - \frac{y_n^3}{4!} + \cdots\right] y_n \end{aligned}$$

There exists two positive constants δ_1 and δ_2 such that

$$y_{n+1} \ge y_n \quad \text{for} \quad \delta_2 < y_n < \delta_1 < 1$$

Hence $\lim_{n\to\infty} inf\{y_n\} > 0$.

Using Theorem 3.1, we complete the proof.

REFERENCES

[1] M. R. S. Kulenovic and G. Ladas, "Dynamics of Second Order Rational Difference Equations," Chapman & Hall/CRC, Boca Raton, 2002.