Applied Mathematics, 2012, 3, 169-171
http://dx.doi.org/10.4236/am.2012.32026 Published Online February 2012 (http://www.SciRP.org/journal/am)

A Nonstationary Halley's Iteration Method by Using Divided Differences Formula

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Received July 22, 2011; revised August 13, 2011; accepted August 21, 2011

ABSTRACT

This paper presents a new nonstationary iterative method for solving non linear algebraic equations that does not require the use of any derivative. The study uses only the Newton's divided differences of first and second orders instead of the derivatives of (1).

Keywords: Nonstationary; Iterative Method; Hally's Formula; Divided Differences

1. Introduction

In order to solve the nonlinear algebraic equations of the form.

$$f(x)=0 \qquad (1)$$

where $f(x)$ is a known function. Let α be a simple real root of the Equation (1) and let x_0 an initial approximation to α. An iterative method [1] for solving the Equation (1), in form of sequence of approximations by using the formula:

$$x_n = F(x_{n-1}), \qquad n = 1,2,3\cdots \qquad (2)$$

The most popular iterative process, is the Newton's one-point process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n \geq 1 \qquad (3)$$

We know that Newton's method is quadratically convergent, this method requires two operations at each iteration, evaluation of $f(x_n)$ and $f'(x_n)$. Tamara Kogan, Luba Sapir and Amir Sapir [1] illustrate that the secant method is classical 2-point iterative process which does not require use of any derivative, and they defined this method as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f_{n,n-1}}, \qquad n \geq 1 \qquad (4)$$

where x_{n+1}, x_n are two successive approximations to a and $f_{n,n-1} = \dfrac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}$ is the first-order divided difference. The construction of many iterative processes is based on Newton's divided difference formula, where

$$f_{n,n-k} = f(x_n, x_{n-1}, \cdots, x_{n-k}) = \begin{cases} f(x_n), & k=0 \\ \dfrac{f_{n,n-k+1} - f_{n-1,n-k}}{x_n - x_{n-k}}, & 1 \leq k \leq n \end{cases} \qquad (5)$$

Is a divided difference of order K. Kogan [2] extended the secant method as follows:

$$x_{n+1} = x_n - \frac{f_{n,n}}{f_{n,n-1} \cdot f_{n-1,n-2} - f_{n-1,n-1} \cdot f_{n,n-2}}, \qquad n \geq 2 \quad (6)$$

where x_n, x_{n-1}, x_{n-2} are three successive approximations to the simple root α of (1), Many iterative processes of third-order have been studied, Tamara Kogan, Luba Sapir and Amir Sapir in [1] suggest the following iterative method for approximation of a simple root α of (1):

$$x_{n+1} = x_n - \frac{f_{n,n}}{f_{n,n-1} + \sum_{i=2}^{n} f_{n,n-i} \cdot \left(\prod_{j=n-i+1}^{n-1} (x_n - x_j)\right)}, \quad n \geq 1 \qquad (7)$$

As a nonstationary iterative process, (*i.e.* the function F depends on the number of iteration):

$$x_2 = x_1 - \frac{f_{1,1}}{f_{1,0}},$$

$$x_3 = x_2 - \frac{f_{2,2}}{f_{2,1} + f_{2,0}(x_2 - x_1)},$$

$$x_4 = x_3 - \frac{f_{3,3}}{f_{3,2} + f_{3,1}(x_3 - x_2) + f_{3,0}(x_3 - x_2)(x_3 - x_1)}$$

In our study, we suggest a new iterative method for approximation of a simple root α of (1) by using only the Newton's divided differences of first and second orders instead of the derivatives of the first and second order.

2. The Principle of the Nonstationary Halley's Iteration Method by Using Divided Differences Formula

We suggest the following iterative method for approximation of a simple root α of (1):

$$x_{n+1} = x_n - \frac{2 \cdot f_{n,n} \cdot f_{n,n-1}}{2 \cdot \left[f_{n,n-1} \right]^2 + f_{n,n} \cdot f_{n,n-2}}, \quad n \geq 2$$

$$x_2 = x_1 - \frac{f(x_0)}{f_{1,0}} \qquad (n = 1) \tag{8}$$

It is clear that (8) is a nonstationary iterative process,

$$x_3 = x_2 - \frac{2 \cdot f_{1,1} \cdot f_{2,1}}{2 \cdot \left[f_{2,1} \right]^2 + f_{2,2} \cdot f_{2,0}}$$

$$x_4 = x_3 - \frac{2 \cdot f_{3,3} \cdot f_{3,2}}{2 \cdot \left[f_{3,2} \right]^2 + f_{3,3} \cdot f_{3,1}}$$

$$x_5 = x_4 - \frac{2 \cdot f_{4,4} \cdot f_{4,3}}{2 \cdot \left[f_{4,3} \right]^2 + f_{4,4} \cdot f_{4,2}}$$

The iterative method (8) is the Hally's formula, see [3], but we take instead of $f'(x_n)$ and $f''(x_n)$, successively, the divided differences $f_{n,n-1}$ and $f_{n,n-2}$ (of first and second orders only), hence we have two types of errors:

1) Cubic error comes from the Hally's iterative processes (of third-order).

2) Error comes from the approximation of divided differences.

The following example (given by [1] also), illustrates the suggested method

3. The Convergence of the Method

Let e_i be the error at the iteration, then $e_i = x_i - \alpha$. Let the function :

$$F(x) = x - \frac{2f(x) \cdot f_{n.n-1}}{2f_{n.n-1}^2 + f(x).f_{n,n-1}} \tag{9}$$

Then,

$$F(x_i) = x_{i+1}, \; F(\alpha) = \alpha \tag{10}$$

and

$$F'(\alpha) = 1 - 2f_{n,n-1} \cdot f'(\alpha) \tag{11}$$

Expending $F(x)$ about α and using (9) and (10) we obtain

$$F(x) = F(\alpha) + \left(1 - 2f_{n,n-1} \cdot f'(\alpha) \cdot (x - \alpha) + O\left(e_i^2\right)\right) \tag{12}$$

$$\left| F(x) - F(\alpha) \right| \leq K \cdot |x - \alpha| \tag{13}$$

where,

$$K = \left| 1 - 2f_{n,n-1} \cdot f'(\alpha) \right| < 1 \tag{14}$$

For $x = x_i$, and using (13) we obtain,

$$\left| x_{i+1} - \alpha \right| \leq K \cdot \left| x_i - \alpha \right| \tag{15}$$

This yields that $e_{i+1} = O\left(e_i^2\right)$ and this proves the required.

4. Example 1

Consider the equation given in [1]:

$$f(x) = \left(x^2 - 3\right) \cdot \ln\left(x^2 + \sqrt{x^2 - 1}\right) - \frac{x^2}{x^2 + 5} - 2 \cdot \ln 7 + 0.5 = 0,$$

which has $a = \sqrt{5}$ as a simple root, the following **Table 1** illustrates the computation by formula (8) started by $x_0 = 3$ and $x_1 = 2$. The correct value of the root α to 9 decimal places is 2.236067978.

Table 2 illustrates the computation by formula (6), given in [1] started by the same values $x_0 = 3$ and $x_1 = 2$.

5. Discussion

Example 1 shows a comparison of convergence for the

Table 1. Suggested iteration for solving $f(x) = \left(x^2 - 3\right) \cdot \ln\left(x^2 + \sqrt{x^2 - 1}\right) - \frac{x^2}{x^2 + 5} - 2 \cdot \ln 7 + 0.5 = 0$.

n	x_n	$f_{n,n}$	$f_{n,n-1}$	$f_{n,n-2}$
0	3	10.78835684	12.8785482	
1	2	–2.090191369	8.472925162	5.259234325
2	2.162307142	–0.714975111	9.760078132	5.170119908
3	2.248959985	0.130763408	10.1405983	5.191098574
4	2.235609576	–0.004617738	10.07354888	5.199651523
5	2.236065001	–0.000029992	10.07692308	7.360788129
6	2.236067978	0.0000000072		

Table 2. The iteration by formula (6), given in [1] for solving $f(x)=(x^2-3)\cdot\ln\left(x^2+\sqrt{x^2-1}\right)-\dfrac{x^2}{x^2+5}-2\cdot\ln 7+0.5=0$.

n	x_n	$f_{n,n}$	$f_{n,n-1}$	$f_{n,n-2}$	$f_{n,n-3}$	$f_{n,n-4}$
0	3	10.78835687	12.8785482			
1	2	–2.090191369	8.472889464	5.25923357	0.118269587	
2	2.162300233	–0.715039434	9.708169348	5.169226632		1.21095655
3	2.238968026	0.029264484	10.07546357	4.978760191	–0.806813475	
4	2.236072459	0.000045151				
5	2.236067971					

suggested iteration method and the iteration given by Tamara Kogan, Luba Sapir and Amir Sapir [1], the result reveals that the correct value of the root α to 9 decimal places 2.236067978 takes one step more, in addition our suggested method used only the divided differences of first and second order.

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