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# New Oscillation Results for Forced Second Order Differential Equations with Mixed Nonlinearities

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## ABSTRACT

Some new oscillation criteria are given for forced second order differential equations with mixed nonlinearities by using the generalized variational principle and Riccati technique. Our results generalize and extend some known oscillation results in the literature.

**Keywords:** Generalized Variational Principle; Variational Principle; Second Order Differential Equations; Riccati Transformation; Oscillation

## 1. Introduction

The oscillatory behavior of second order differential equations has a major role in the theory of differential equations. It has been shown that many real world problems can be modelled, in particular, by half linear differential equations which can be regarded as a natural generalization of linear differential equations [1-14]. A considerable amount of research has also been done on quasi-linear [15-18] and nonlinear second order differential equations [19-23].

In this paper, we investigate the oscillatory behavior of second order forced differential equation with mixed nonlinearities.

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)'+p(t)\left|x(t)\right|^{\alpha-1}x(t)+\sum_{j=1}^{m}q_j(t)\left|x(t)\right|^{\beta_j-1}x(t)=e(t),\ t\ge t_0, \tag{1}$$

where $r\in C\left(\left[t_0,\infty\right),\left(0,\infty\right)\right)$, $p,q_j\left(1\le j\le m\right),e\in C\left(\left[t_0,\infty\right),\left(-\infty,\infty\right)\right)$ and $0<\alpha<\beta_1<\beta_2<\cdots<\beta_m$ are real numbers, $p,q_j\ \left(1\le j\le m\right)$ and $e$ might alternate signs.

By a solution of Equation (1), we mean a function $x(t)\in C^1\left(\left[T_x,\infty\right),\left(-\infty,\infty\right)\right)$, where $T_x\ge t_0$ depends on the particular solution, which has the property that $r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\in C^1\left[T_x,\infty\right)$ and satisfies Equation (1). We restrict our attention to the nontrivial solutions $x(t)$ of Equation (1) only, *i.e.*, to solutions $x(t)$ such that $\sup\left\{\left|x(t)\right|:t\ge T\right\}>0$ for all $T\ge T_x$. A nontrivial solution of (1) is oscillatory if it has arbitrarily large zeros, otherwise, it is called non-oscillatory. Equation (1) is said to be oscillatory if all its nontrivial solutions are oscillatory.

Equation (1) and its special cases such as the linear differential equation

$$\left(r(t)x'(t)\right)'+q(t)x(t)=e(t), \tag{2}$$

the half-linear differential equation

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)'+q(t)\left|x(t)\right|^{\alpha-1}x(t)=e(t) \tag{3}$$

and the quasi-linear differential equation

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)'+q(t)\left|x(t)\right|^{\beta-1}x(t)=e(t) \tag{4}$$

have been extensively studied by numerous authors with different methods (see, for example, [1-5,15-19] and the references quoted therein).

In 1999, Wong [1] proved the following theorem by making use of the "oscillatory intervals" of $e(t)$ and Leighton's variational principle (see [10]) for (2).

**Theorem 1.1.** Suppose that for any $T\ge t_0$, there exist $T\le s_1<t_1\le s_2<t_2$ such that

$$e(t)\begin{cases}\le 0,\ t\in\left[s_1,t_1\right]\\ \ge 0,\ t\in\left[s_2,t_2\right].\end{cases} \tag{5}$$

Denote

$$D\left(s_i,t_i\right)=\left\{u\in C^1\left[s_i,t_i\right]:u(t)\not\equiv 0,u\left(s_i\right)=u\left(t_i\right)=0\right\},$$
$i=1,2.$

If there exist $u\in D\left(s_i,t_i\right)$ such that

 

$$Q_i(u)=\int_{s_i}^{t_i}\left[q(t)u^2(t)-r(t)\left(u'(t)\right)^2\right]\mathrm{d}t\geq 0,\ i=1,2 \tag{6}$$

then Equation (2) is oscillatory.

Afterwards, in 2002, the authors of [2] extended Wong's results, using a similar method, to Equation (3) as follows.

**Theorem 1.2.** Suppose that for any $T\geq t_0$, there exist $T\leq s_1<t_1\leq s_2<t_2$ such that (5) holds. Let

$D(s_i,t_i)=\left\{u\in C^1[s_i,t_i]:u(t)\not\equiv 0,u(s_i)=u(t_i)=0\right\}$, $i=1,2$.

If there exist $H\in D(s_i,t_i)$ and a positive, nondecreasing function $\phi\in C^1([t_0,\infty))$ such that

$$\int_{s_i}^{t_i}H^2(t)\phi(t)q(t)\mathrm{d}t > K\int_{s_i}^{t_i}\frac{r(t)\phi(t)}{|H(t)|^{\alpha-1}}\left(2|H'(t)|+|H(t)|\frac{\phi'(t)}{\phi(t)}\right)^{\alpha+1}\mathrm{d}t, \tag{7}$$

for $i = 1, 2$, where $K=\left(1/(\alpha+1)\right)^{\alpha+1}$, then (3) is oscillatory.

Later, in 2007, Zheng and Meng [16], considering a more general Equation (4), improved the paper [2] and showed that the results obtained in [2] for Equation (3) can not be applied to the case $\alpha>1$. The main result of Zheng and Meng [16] is the following.

**Theorem 1.3.** Assume that for any $T\geq t_0$, there exist $T\leq s_1<t_1\leq s_2<t_2$ such that (5) holds. Let

$$D(s_i,t_i)=\left\{u\in C^1[s_i,t_i]:u^{\alpha+1}(t)>0,\ t\in(s_i,t_i)\text{ and }u(s_i)=u(t_i)=0\right\}\text{ for }i=1,2.$$

Suppose that there exist $H\in D(s_i,t_i)$ and a positive, nondecreasing function $\phi\in C^1([t_0,\infty),(-\infty,\infty))$ such that

$$Q_i^{\phi}(H):=\int_{s_i}^{t_i}\phi(t)\left[Q(t)H^{\alpha+1}(t)-r(t)\left(|H'(t)|+\frac{H(t)\phi'(t)}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right]\mathrm{d}t>0 \tag{8}$$

for $i=1, 2$. Then Equation (4) is oscillatory, where

$$Q(t)=\alpha^{-\alpha/\beta}\beta\left((\beta-\alpha)\right)^{(\alpha-\beta)/\beta}\left[q(t)\right]^{\alpha/\beta}|e(t)|^{(\beta-\alpha)/\beta},\ 0<\alpha\leq\beta \tag{9}$$

with the convention that $0^0=1$.

Also, in [2009], Zheng *et al.* [17] extended the results obtained for Equation (4) to Equation (1) as follows.

**Theorem 1.4.** Assume that for any $T\geq t_0$, there exist $T\leq s_1<t_1\leq s_2<t_2$ such that $q_j(t)\geq 0\,(1\leq j\leq m)$ for $t\in[s_1,t_1]\cup[s_2,t_2]$ and (5) holds. Let

$$D(s_i,t_i)=\left\{u\in C^1[s_i,t_i]:u^{\alpha+1}(t)>0,t\in(s_i,t_i)\text{ and }u(s_i)=u(t_i)=0\right\}\text{ for }i=1,2.$$

If there exist $H\in D(s_i,t_i)$ and a positive function $\phi\in C^1([t_0,\infty),R)$ such that

$$\int_{s_i}^{t_i}\phi(t)\left[\left(p(t)+\sum_{j=1}^{m}Q_j(t)\right)H^{\alpha+1}(t)-r(t)\left(|H'(t)|+\frac{H(t)\phi'(t)}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right]\mathrm{d}t>0 \tag{10}$$

for $i = 1, 2$. Then Equation (1) is oscillatory, where

$$Q_j(t)=\alpha^{-\alpha/\beta_j}\beta_j\left[m(\beta_j-\alpha)\right]^{(\alpha-\beta_j)/\beta_j}\left[q_j(t)\right]^{\alpha/\beta_j}|e(t)|^{(\beta_j-\alpha)/\beta_j},\ \ 1\leq j\leq m \tag{11}$$

with the convention that $0^0=1$.

Recently, Shao [15] generalized the results of Zheng and Meng [16] by using the generalized variational principle due to Komkov [24] and gave the following result for Equation (4).

**Theorem 1.5.** Assume that, for any $T\geq t_0$, there exist $T\leq s_1<t_1\leq s_2<t_2$ such that (1.5) holds. Let $u\in C^1[s_i,t_i]$, and nonnegative functions $G_1,G_2$ satisfying $G_i(u(s_i))=G_i(u(t_i))=0$, $g_i(u)=G_i'(u)$ are continuous and $\left(g_i(u(t))\right)^{\alpha+1}\leq(\alpha+1)^{\alpha+1}G_i^{\alpha}(u(t))$ for $t\in[s_i,t_i]$, $i = 1, 2$. If there exists a positive function $\phi\in C^1([t_0,\infty),R)$ such that

$$Q_i^{\phi}(u):=\int_{s_1}^{t_1}\phi\left[QG_i(u)-r\left(|u'|+\frac{G_i^{1/(\alpha+1)}(u)|\phi'|}{(\alpha+1)\phi}\right)^{\alpha+1}\right](t)\mathrm{d}t>0 \tag{12}$$

for $i = 1, 2$, then Equation (4) is oscillatory, where $Q(t)$ is the same as (9).

Motivated by the above theorems we propose some new oscillation results by employing the generalized variational principle and Riccati technique for Equation (1). Our results extend and generalize some known results in the literature. We now state our main results and several remarks.

## 2. New Oscillation Results

In order to prove our results we use the following well-known inequality which is presented by Hardy *et al.* [25].

**Lemma 2.1.** (see [25]). If $X$ and $Y$ are non-negative, then

$$X^{\gamma}+(\gamma-1)Y^{\gamma}\ge\gamma XY^{\gamma-1},\ \gamma>1, \tag{13}$$

where equality holds if and only if $X=Y$.

**Theorem 2.1.** Assume that, for any $T\ge t_0$, there exist $T\le s_1<t_1\le s_2<t_2$ such that $q_j(t)\ge 0(1\le j\le m)$ for $t\in[s_1,t_1]\cup[s_2,t_2]$ and (5) holds. Let $u\in C^1[s_i,t_i]$ and nonnegative functions $G_1,G_2$ satisfying

$$G_i(u(s_i))=G_i(u(t_i))=0,\ g_i(u)=G_i'(u)$$

are continuous and

$$\left(g_i(u(t))\right)^{\alpha+1}\le(\alpha+1)^{\alpha+1}G_i^{\alpha}(u(t))$$

for $t\in[s_i,t_i]$, $i$ = 1, 2. If there exists a positive function $\phi\in C^1([t_0,\infty),R)$ such that

$$\int_{s_i}^{t_i}\phi(t)\left[\left(p(t)+\sum_{j=1}^{m}Q_j(t)\right)G_i(u(t))-r(t)\left(|u'(t)|+\frac{G_i^{1/(\alpha+1)}(u(t))|\phi'(t)|}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right]dt>0 \tag{14}$$

for $i$ =1, 2, then Equation (1) is oscillatory, where $Q_j(t)$ is the same as (11).

**Proof.** Suppose that $x=x(t)$ is a nonoscillatory solution of Equation (1). Then, there exists a $T_0\ge t_0$ such that $x(t)\ne 0$ for all $t\ge T_0$. Without loss of generality, we may assume that $x(t)>0$ for all $t\ge T_0$. We introduce the Ricccati transformation

$$w(t)=\phi(t)\frac{r(t)|x'(t)|^{\alpha-1}x'(t)}{|x(t)|^{\alpha-1}x(t)},\ t\ge T_0. \tag{15}$$

Differentiating (15) and using (1), we obtain, for all $t\ge T_0$,

$$\begin{aligned}w'(t)&=\frac{\phi'(t)}{\phi(t)}w(t)\\&+\phi(t)\left[\frac{\left(-p(t)|x(t)|^{\alpha-1}x(t)-\sum_{j=1}^{m}q_j(t)|x(t)|^{\beta_j-1}x(t)+e(t)\right)|x(t)|^{\alpha-1}x(t)-r(t)|x'(t)|^{\alpha-1}x'(t)\left(|x(t)|^{\alpha-1}x(t)\right)'}{\left(|x(t)|^{\alpha-1}x(t)\right)^2}\right]\\&=\frac{\phi'(t)}{\phi(t)}w(t)-\phi(t)p(t)-\phi(t)\left[\sum_{j=1}^{m}q_j(t)|x(t)|^{\beta_j-\alpha}-\frac{e(t)}{|x(t)|^{\alpha-1}x(t)}\right]-\alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\phi(t)r(t))^{1/\alpha}}.\end{aligned} \tag{16}$$

By the assumption, we can choose $s_1,t_1\ge T_0$ so that $e(t)\le 0$ on the interval $I_1=[s_1,t_1]$ with $s_1<t_1$. As in [18], for given $t\in I_1$, set

$$F_j(s)=q_j(t)s^{\beta_j-\alpha}-\frac{e(t)}{ms^{\alpha}},\ 1\le j\le m,\ s>0.$$

It is easy to verify that

$$F_j'(s_j^*)=0\text{ and }F_j''(s_j^*)>0,$$

$$\text{where }s_j^*=\left[\frac{-\alpha e(t)}{m(\beta_j-\alpha)q_j(t)}\right]^{1/\beta_j}.$$

So $F_j(s)$ obtains its minimum on $s_j^*$ and

$$F_j(s)\ge F_j(s_j^*)=\alpha^{-\alpha/\beta_j}\beta_j\left[m(\beta_j-\alpha)\right]^{(\alpha-\beta_j)/\beta_j}\left[q_j(t)\right]^{\alpha/\beta_j}|e(t)|^{(\beta_j-\alpha)/\beta_j}=Q_j(t). \tag{17}$$

Then, by using (17) in (16), we get

 

$$\phi(t)\left(p(t)+\sum_{j=1}^{m}Q_j(t)\right)\le -w'(t)+\frac{\phi'(t)}{\phi(t)}w(t)-\alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\phi(t)r(t))^{1/\alpha}}. \tag{18}$$

Multiplying $G_1(u(t))$ through (18) and integrating over $I_1$, we have

$$\int_{s_1}^{t_1}\phi(t)\left(p(t)+\sum_{j=1}^{m}Q_j(t)\right)G_1(u(t))\mathrm{d}t\le -\int_{s_1}^{t_1}G_1(u(t))w'(t)\mathrm{d}t+\int_{s_1}^{t_1}G_1(u(t))\frac{\phi'(t)}{\phi(t)}w(t)\mathrm{d}t-\alpha\int_{s_1}^{t_1}G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\phi(t)r(t))^{1/\alpha}}\mathrm{d}t. \tag{19}$$

By integration by parts and using the fact that $G_1(u(s_1))=G_1(u(t_1))=0,$ we have

$$\int_{s_1}^{t_1}G_1(u(t))w'(t)\mathrm{d}t=G_1(u(t))w(t)\Big|_{s_1}^{t_1}-\int_{s_1}^{t_1}G_1'(u(t))u'(t)w(t)\mathrm{d}t=-\int_{s_1}^{t_1}g_1(u(t))u'(t)w(t)\mathrm{d}t. \tag{20}$$

In view of (19) and (20), we conclude that

$$\begin{aligned}
&\int_{s_1}^{t_1}\phi(t)\left(p(t)+\sum_{j=1}^{m}Q_j(t)\right)G_1(u(t))\mathrm{d}t\\
&\le\int_{s_1}^{t_1}\left[g_1(u(t))u'(t)+G_1(u(t))\frac{\phi'(t)}{\phi(t)}\right]w(t)\mathrm{d}t-\alpha\int_{s_1}^{t_1}G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\phi(t)r(t))^{1/\alpha}}\mathrm{d}t\\
&\le\int_{s_1}^{t_1}\left[|g_1(u(t))||u'(t)|+G_1(u(t))\frac{|\phi'(t)|}{\phi(t)}\right]|w(t)|dt-\alpha\int_{s_1}^{t_1}G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\phi(t)r(t))^{1/\alpha}}\mathrm{d}t\\
&\le(\alpha+1)\int_{s_1}^{t_1}\left[G_1^{\alpha/(\alpha+1)}(u(t))|u'(t)|+G_1(u(t))\frac{|\phi'(t)|}{(\alpha+1)\phi(t)}\right]|w(t)|\mathrm{d}t-\alpha\int_{s_1}^{t_1}G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\phi(t)r(t))^{1/\alpha}}\mathrm{d}t.
\end{aligned} \tag{21}$$

Let

$$X=\left[\frac{\alpha}{(\phi(t)r(t))^{1/\alpha}}\right]^{\alpha/(\alpha+1)}G_1^{\alpha/(\alpha+1)}(u(t))|w(t)|,\ \gamma=\frac{\alpha+1}{\alpha},\ Y=(\alpha\phi(t)r(t))^{\alpha/(\alpha+1)}\left[|u'(t)|+\frac{G_1^{1/(\alpha+1)}(u(t))|\phi'(t)|}{(\alpha+1)\phi(t)}\right]^{\alpha}.$$

According to Lemma 2.1, we obtain for $t\in[s_1,t_1]$

$$\begin{aligned}
&(\alpha+1)\left[G_1^{\alpha/(\alpha+1)}(u(t))|u'(t)|+G_1(u(t))\frac{|\phi'(t)|}{(\alpha+1)\phi(t)}\right]|w(t)|-\alpha G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\phi(t)r(t))^{1/\alpha}}\\
&\le\frac{1}{\alpha}(\alpha\phi(t)r(t))\left[|u'(t)|+\frac{G_1^{1/(\alpha+1)}(u(t))|\phi'(t)|}{(\alpha+1)\phi(t)}\right]^{\alpha+1}=\phi(t)r(t)\left[|u'(t)|+\frac{G_1^{1/(\alpha+1)}(u(t))|\phi'(t)|}{(\alpha+1)\phi(t)}\right]^{\alpha+1}.
\end{aligned}$$

Therefore, (21) yields

$$\int_{s_1}^{t_1}\phi(t)\left(p(t)+\sum_{j=1}^{m}Q_j(t)\right)G_1(u(t))\mathrm{d}t\le\int_{s_1}^{t_1}\phi(t)r(t)\left[|u'(t)|+\frac{G_1^{1/(\alpha+1)}(u(t))|\phi'(t)|}{(\alpha+1)\phi(t)}\right]^{\alpha+1}\mathrm{d}t,$$

which contradicts the assumption (14) for $i=1$.

When $x(t)$ is a negative solution for $t\ge T_0>t_0$, we may employ the fact that $e(t)\ge 0$ on $I_2=[s_2,t_2]$ to reach a similar contradiction. Therefore, any solution $x(t)$ can be neither eventually positive nor eventually negative. Hence, any solution is oscillatory. This completes the proof of Theorem 2.1.

If $p(t)\equiv 0$ and $m=1$, then Equation (1) reduces to Equation (4). Thus by Theorem 2.1, we have the following oscillation result:

**Corollary 2.1.** Assume that, for any $T \ge t_0$, there exist $T \le s_1 < t_1 \le s_2 < t_2$ such that (5) holds. Let $u \in C^1[s_i, t_i]$, and nonnegative functions $G_1, G_2$ satisfying $G_i(u(s_i)) = G_i(u(t_i)) = 0$, $g_i(u) = G_i'(u)$ are continuous and $(g_i(u(t)))^{\alpha+1} \le (\alpha+1)^{\alpha+1} G_i^{\alpha}(u(t))$ for $t \in [s_i, t_i]$ for $i$ =1, 2. If there exists a positive function $\phi \in C^1([t_0,\infty), R)$ such that

$$\int_{s_i}^{t_i} \phi(t)\left[Q(t)G_i(u(t)) - r(t)\left(|u'(t)| + \frac{G_i^{1/(\alpha+1)}(u(t))|\phi'(t)|}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right]\mathrm{d}t > 0 \tag{22}$$

for $i$ = 1, 2, then Equation (4) is oscillatory, where $Q(t)$ is the same as (9).

**Remark 1.** Corollary 2.1 shows that Theorem 2.1 is a generalization of Theorem 1.5.

**Remark 2.** Let $G_1(u) = G_2(u) = u^{\alpha+1}$ in Corollary 2.1, then our main Theorem 2.1 reduces to Theorem 1.3.

**Remark 3.** If we choose $G_1(u) = G_2(u) = u^{\alpha+1}$ in Theorem 2.1, then we obtain Theorem 1.4.

**Remark 4.** If we choose $G_1(u) = G_2(u) = u^{\alpha+1}$ and $\phi(t) \equiv 1$ in Theorem 2.1, then we obtain Corollary 2.3 of Paper [17].

**Remark 5.** If we choose $G_1(u) = G_2(u) = u^{\alpha+1}$ and $\phi(t) \equiv 1$ in Corollary 2.1, then we obtain Corollary 2.3 of paper [16].

**Remark 6.** Let

$$G_1(u) = G_2(u) = u^{\alpha+1},\ q_j(t) \equiv 0, 1 \le j \le m$$

and $\phi(t) \equiv 1$ in Theorem 2.1, then Theorem 2.1 is a generalization of Theorem 1.1.

**Remark 7.** Let $q_j(t) \equiv 0, (1 \le j \le m)$. If we choose $G_1(u) = G_2(u) = u^{\alpha+1}$ in Theorem 2.1, then Theorem 2.1 improves Theorem 1.2, since the positive constant $\alpha$ in Theorem 2.1 can be chosen as any number lying in $(0,\infty)$.

**Remark 8.** If the condition (5) in Theorem 2.1 and Corollary 2.1 is replaced by

$$e(t)\begin{cases} \ge 0, t \in [s_1, t_1] \\ \le 0, t \in [s_2, t_2] \end{cases}$$

then the results given in this paper are still valid.

## 3. Examples

**Example 3.1.** Consider

$$\left(\gamma t^{\lambda/3}|x'(t)|^{\alpha-1}x'(t)\right)' + t^{\lambda/3}|x(t)|^{\alpha-1}x(t) + t^{\lambda}|x(t)|^{\beta-1}x(t) = -\sin^3 t, \tag{23}$$

for $t \ge 1$, where $\gamma, \lambda > 0$ are constants. Let $\alpha = 1$ and $\beta = 3$, so $Q(t) = (3/\sqrt[3]{4})t^{\lambda/3}\sin^2 t$. The zeros of forcing term $-\sin^3 t$ are $n\pi$. For any $T \ge 1$, we choose $n$ sufficiently large so that $n\pi = 2k\pi \ge T$, $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$. Letting $u(t) = \sin t \ge 0$, $G_1(u) = u^2\exp(-u)$ (it is easy to verify that $(G_1'(u))^2 \le 4G_1(u)$ for $u \ge 0$), $\phi(t) = t^{-\lambda/3}$, then we obtain

$$\int_{s_1}^{t_1}\phi(t)(p(t)+Q(t))G_i(u(t))\mathrm{d}t = \int_{2k\pi}^{(2k+1)\pi}\left(\sin^2 t + \left(3/\sqrt[3]{4}\right)\sin^4 t\right)\exp(-\sin t)\mathrm{d}t \ge \frac{\pi}{2e}\left(1+\frac{9}{4\sqrt[3]{4}}\right),$$

and

$$\int_{s_1}^{t_1}\phi(t)r(t)\left(|u'(t)| + \frac{G_i^{1/(\alpha+1)}(u(t))|\phi'(t)|}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\mathrm{d}t = \int_{2k\pi}^{(2k+1)\pi}\gamma\left(|\cos t| + \frac{\lambda\sin t\exp(-\sin t/2)}{6t}\right)^2\mathrm{d}t < \gamma\left(1+\frac{\lambda}{6}\right)^2\pi.$$

Therefore, Equation (14) is satisfied for $i$ = 1 provided that $0 < \gamma < \frac{1}{2e}\left(1+\frac{9}{4\sqrt[3]{4}}\right)\Big/\left(1+\frac{\lambda}{6}\right)^2$. In a similar way, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we choose $u(t) = \sin t \le 0$, $G_2(u) = u^2\exp(u)$ (it is easy to verify that $(G_2'(u))^2 \le 4G_2(u)$ for $u \le 0$) so that that (14) is valid for $i$ = 2. Thus (23) is oscillatory for

$$0 < \gamma < \frac{1}{2e}\left(1+\frac{9}{4\sqrt[3]{4}}\right)\Big/\left(1+\frac{\lambda}{6}\right)^2$$

by Theorem 2.1.

**Example 3.2.** Consider the following forced quasi-linear differential equation

$$\left[\gamma(2+\cos t)t^{-\lambda/5}|x'|^{\alpha-1}x'\right]'+t^{-\lambda/5}\exp(\sin t/5)|x|^{\alpha-1}x+t^{-\lambda}\exp(\sin t)|x|^{\beta-1}x=-\sin^5 t \tag{24}$$

for $t\geq 1$, where $\gamma,\lambda>0$ are constants. Let $\alpha=1$ and $\beta=5$, so $Q(t)=\frac{5}{\sqrt[5]{4^4}}t^{-\lambda/5}\exp(\sin t/5)\sin^4 t$. The zeros of forcing term $-\sin^5 t$ are $n\pi$. For any $T\geq 1$, we choose $n$ sufficiently large so that $n\pi=2k\pi\geq T$, $s_1=2k\pi$ and $t_1=(2k+1)\pi$. Letting $u(t)=\sin t\geq 0$, $G_1(u)=u^2\exp(-u)$, $\phi(t)=t^{\lambda/5}$, then we obtain

$$\int_{s_1}^{t_1}\phi(t)(p(t)+Q(t))G_i(u(t))\mathrm{d}t=\int_{2k\pi}^{(2k+1)\pi}\left(\sin^2 t+\frac{5}{\sqrt[5]{4^4}}\sin^6 t\right)\exp\left(\frac{-4\sin t}{5}\right)\mathrm{d}t$$
$$\geq\frac{1}{e^{4/5}}\int_0^{\pi}\left(\sin^2 t+\frac{5}{\sqrt[5]{4^4}}\sin^6 t\right)=\frac{\pi}{16e^{4/5}}\left(8+\frac{25}{\sqrt[5]{4^4}}\right)$$

and

$$\int_{s_1}^{t_1}\phi(t)r(t)\left(|u'(t)|+\frac{G_i^{1/(\alpha+1)}(u(t))|\phi'(t)|}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\mathrm{d}t=\int_{2k\pi}^{(2k+1)\pi}\gamma(2+\cos t)\left(|\cos t|+\frac{\lambda\sin t\exp(-\sin t/2)}{10t}\right)^2\mathrm{d}t$$
$$<\int_0^{\pi}3\gamma(1+\lambda/10)^2\mathrm{d}t=3\gamma(1+\lambda/10)^2\pi.$$

Therefore, Equation (14) is satisfied for $i$ = 1 provided that $0<\gamma<\frac{\delta}{3(1+\lambda/10)^2}$, where $\delta=\frac{1}{16e^{4/5}}\left(8+\frac{25}{\sqrt[5]{4^4}}\right)$.

In a similar way, for $s_2=(2k+1)\pi$ and $t_2=(2k+2)\pi$, we choose $u(t)=\sin t\leq 0$, $G_2(u)=u^2\exp(u)$ so that (14) is valid for $i$ = 2. Thus (24) is oscillatory for $0<\gamma<\delta/3(1+\lambda/10)^2$ by Theorem 2.1.

## 4. Conclusion

The oscillatory behavior of many different kinds of differential equations has been investigated and a great deal of results has been obtained in the literature. In this article, we generalized the results obtained in [16,17] and extended the results of Shao [15] by using the generalized variational principle and Riccati tecnique. In a similar way, the results obtained for Equation (1) can be extended to a more general class of differential equations.

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